



An efficient technique for a fractional-order system of equations describing the unsteady flow of a polytropic gas

P VEERESHA¹ , D G PRAKASHA² * and HACI MEHMET BASKONUS³

¹Department of Mathematics, Karnatak University, Dharwad 580 003, India

²Department of Mathematics, Faculty of Science, Davangere University, Shivagangothri, Davangere 577 007, India

³Department of Mathematics and Science Education, Faculty of Education, Harran University, Şanlıurfa, Turkey

*Corresponding author. E-mail: prakashadg@gmail.com

MS received 2 February 2019; revised 26 April 2019; accepted 22 May 2019

Abstract. In the present investigation, the q -homotopy analysis transform method (q -HATM) is applied to find approximated analytical solution for the system of fractional differential equations describing the unsteady flow of a polytropic gas. Numerical simulation has been conducted to prove that the proposed technique is reliable and accurate, and the outcomes are revealed using plots and tables. The comparison between the obtained solutions and the exact solutions shows that the proposed method is efficient and effective in solving nonlinear complex problems. Moreover, the proposed algorithm controls and manipulates the obtained series solution in a huge acceptable region in an extreme manner and it provides us a simple procedure to control and adjust the convergence region of the series solution.

Keywords. q -Homotopy analysis transform method; polytropic gas; Laplace transform.

PACS Nos 02.60.–Cb; 02.60.–x; 05.45.Df

1. Introduction

The concept of fractional calculus (FC) was debated at the end of the 17th century. Systems with an arbitrary order have been lately attracting significant attention and gaining more acceptance as generalisation to the classical order system. The fundamental foundation of FC was laid nearly 324 years ago and since then it has proved deeply rooted mathematical concepts. Every real-world problem can be effectively described by the system of integer- and fractional-order differential equation [1–5]. In practice we can encounter such systems in control theory, medicine, biology, thermodynamics, signal processing, electronics etc. [6–10].

The basic and essential results related to the solutions of fractional differential equations are found in [11–14]. The integer-order derivatives are local in nature, whereas the fractional derivatives are non-local. By using the integer-order derivative, we can analyse the variations in the neighbourhood of a point but by employing the fractional derivative, we can study the changes in the complete interval. The problems relating to applications of FC are present in various connected branches

of science and engineering such as fluid and continuum mechanics [15], electrodynamics [16], chaos [17], optics [18], cosmology [19] and many other branches [20–23].

In the present investigation, we consider the fractional-order system of gas dynamic equations describing the evolution of the two-dimensional unsteady flow of a perfect gas. The polytropic gas in astrophysics is given by [24]

$$P = K\rho^{1+(1/n)},$$

where $\rho = U/V$ is the energy density, U is the total energy of the gas, V is the container volume, K is a constant and n is the polytropic index. Degenerate electron gas and adiabatic gas are two examples of such gases. The study of polytropic gases plays a vital role in cosmology and astrophysics [25] and these gases can behave like dark energy [26]. Now consider the system of gas dynamic equations describing the evolution of unsteady flow of a perfect gas having an arbitrary order [27,28]:

$$D_t^\mu u + u_x u + v u_y + \frac{P_x}{\rho} = 0,$$

$$\begin{aligned}
 D_t^\mu v + uv_x + vv_y + \frac{P_y}{\rho} &= 0, \\
 D_t^\mu \rho + u\rho_x + v\rho_y + \rho u_x + \rho v_y &= 0, \\
 D_t^\mu p + up_x + vp_y + \tau pu_x + \tau pv_y &= 0,
 \end{aligned} \tag{1}$$

under initial conditions

$$\begin{aligned}
 u(x, y, 0) &= a(x, y), \quad v(x, y, 0) = b(x, y), \\
 \rho(x, y, 0) &= c(x, y) \text{ and } p(x, y, 0) = d(x, y),
 \end{aligned} \tag{2}$$

where $u(x, y, t)$ and $v(x, y, t)$ are the velocity components, $\rho(x, y, t)$ is the density, $p(x, y, 0)$ is the pressure and τ is the ratio of the specific heat and it represents the adiabatic index.

The analytical and numerical solutions for the nonlinear fractional differential equations have fundamental importance as most of the complex phenomena are modelled mathematically by nonlinear fractional differential equations. The last three decades have seen the discovery of a number of new techniques to elucidate a nonlinear differential system having a fractional order, in parallel with the development of new symbolic programming and computational algorithms. In connection with this, in 1992, Liao [29,30] introduced the homotopy analysis method (HAM), and it has been effectively employed to find solution to problems arising in science and technology. It does not require any discretisation, linearisation and perturbation. But, it requires more computation and computer memory to solve nonlinear problems that arise in complex phenomena. Hence, it necessitates a mixture of transformation algorithm to overcome these limitations.

In the present framework, we consider q -homotopy analysis transform method (q -HATM) to find analytically approximated solution for the system of equations describing a polytropic gas with an unsteady flow. The proposed technique is a modified technique which is an elegant blend of q -HAM with Laplace transform (LT) [31,32]. As q -HATM is a modified technique of HAM, it does not require linearisation, discretisation or perturbation, and additionally, it will decrease huge mathematical computations, needs less computer memory and is free from difficult polynomials, integrations and physical parameters. It provides us exceptional freedom to pick the equation type of linear subproblems, physical parameters and initial assumption. Due to this, complicated nonlinear differential equations can often be solved in a simple way. The novelty of the proposed technique is that it offers a simple solution procedure, large convergence region and non-local effect in the obtained solution. The future scheme controls and manipulates the series solution, which quickly converges to the analytical solution in a small acceptable region. Recently, many researchers like Srivastava *et al*

[33] studied the model of vibration equation of arbitrary order, Singh *et al* [34] found the solution for the fractional Drinfeld–Sokolov–Wilson equation, Bulut *et al* [35] analysed the fractional model of HIV infection of CD4⁺T cells, Kumar *et al* [36] analysed the model of Lienard’s equation and many others [37–39] with the help of q -HATM.

On the other hand, the solution for the considered system of equations was analysed using distinct techniques, such as the Adomian decomposition technique [40], variational iteration technique [27], fractional natural decomposition scheme [41] and others [28,42]. But in the above cited papers, the researchers do not present numerical simulation and behaviour of system (1) with fractional order. Hence, the authors in the present investigation present the behaviour of the obtained solution and the numerical simulation of the future problem.

2. Preliminaries

We recall the definitions and notations of FC and LT, which shall be employed in the present framework:

DEFINITION 1

The fractional integral of a function $f(t) \in C_\delta$ ($\delta \geq -1$) and of order $\mu > 0$, initially defined by Riemann–Liouville, is presented as

$$\begin{aligned}
 J^\mu f(t) &= \frac{1}{\Gamma(\mu)} \int_0^t (t - \vartheta)^{\mu-1} f(\vartheta) d\vartheta, \\
 J^0 f(t) &= f(t).
 \end{aligned} \tag{3}$$

DEFINITION 2

The fractional derivative of $f \in C_{-1}^n$ in the Caputo sense is defined as [8]

$$\begin{aligned}
 D_t^\mu f(x, y, t) &= \frac{\partial^\mu f(x, y, t)}{\partial t^\mu} \\
 &= \begin{cases} \frac{d^n f(x, y, t)}{dt^n} & \mu = n \in \mathbb{N}, \\ I^{n-\mu} \left[\frac{\partial^n f(x, y, t)}{\partial t^n} \right] & n - 1 < \mu < n, \quad n \in \mathbb{N}. \end{cases}
 \end{aligned} \tag{4}$$

DEFINITION 3

The LT of a Caputo fractional derivative $D_t^\mu f(t)$ is represented as

$$L[D_t^\mu f(t)] = s^\mu F(s)$$

$$-\sum_{r=0}^{n-1} s^{\mu-r-1} f^{(r)}(0^+) (n-1 < \mu \leq n), \tag{5}$$

where $F(s)$ symbolises the LT of the function $f(t)$.

3. Fundamental idea of q -HATM

In this section, we present the fundamental solution procedure of the proposed method, the fractional partial differential equation of the form

$$D_t^\mu u(x, y, t) + Ru(x, y, t) + Nu(x, y, t) = f(x, y, t), \quad n-1 < \mu \leq n, \tag{6}$$

where $D_t^\mu v(x, y, t)$ denotes the Caputo’s fractional derivative of the function $u(x, y, t)$, R is the bounded linear differential operator in x, y and t (i.e. for a number $\varepsilon > 0$ we have $\|Ru\| \leq \varepsilon\|u\|$), N specifies the nonlinear differential operator and Lipschitz continuous with $\sigma > 0$ satisfying $|Nu - Nv| \leq \sigma|u - v|$, and $f(x, y, t)$ represents the source term.

Now, by employing the LT on eq. (6), we get

$$\begin{aligned} s^\mu L[u(x, y, t)] & - \sum_{k=0}^{n-1} s^{\mu-k-1} u^{(k)}(x, y, 0) \\ & + L[Ru(x, y, t)] + L[Nu(x, y, t)] \\ & = L[f(x, y, t)]. \end{aligned} \tag{7}$$

On simplifying eq. (7), we have

$$\begin{aligned} L[u(x, y, t)] - \frac{1}{s^\mu} \sum_{k=0}^{n-1} s^{\mu-k-1} u^{(k)}(x, y, 0) \\ + \frac{1}{s^\mu} \{L[Ru(x, y, t)] \\ + L[Nu(x, y, t)] - L[f(x, y, t)]\} = 0. \end{aligned} \tag{8}$$

According to the HAM, the nonlinear operator is defined as

$$\begin{aligned} N[\varphi(x, y, t; q)] & = L[\varphi(x, y, t; q)] \\ & - \frac{1}{s^\mu} \sum_{k=0}^{n-1} s^{\mu-k-1} \varphi^{(k)}(x, y, t; q)(0^+) \\ & + \frac{1}{s^\mu} \{L[R\varphi(x, y, t; q)] \\ & + L[N\varphi(x, y, t; q)] - L[f(x, y, t)]\}, \end{aligned} \tag{9}$$

where $q \in [0, \frac{1}{n}]$ and $\varphi(x, y, t; q)$ is the real function of x, y, t and q . For a non-zero auxiliary function, we construct a homotopy as follows:

$$\begin{aligned} (1 - nq)L[\varphi(x, y, t; q) - u_0(x, y, t)] \\ = \hbar q N[\varphi(x, y, t; q)], \end{aligned} \tag{10}$$

where L is a symbol of the LT, $\hbar \neq 0$ is an auxiliary parameter, $q \in [0, 1/n]$ ($n \geq 1$) is the embedding parameter, $u_0(x, y, t)$ is an initial condition of $u(x, y, t)$ and $\varphi(x, y, t; q)$ is an unknown function. The following results hold, respectively, for $q = 0$ and $1/n$:

$$\begin{aligned} \varphi(x, y, t; 0) & = u_0(x, y, t), \\ \varphi\left(x, y, t; \frac{1}{n}\right) & = u(x, y, t). \end{aligned} \tag{11}$$

Thus, by amplifying q from 0 to $1/n$, the solution $\varphi(x, y, t; q)$ converges from $u_0(x, y, t)$ to the solution $u(x, y, t)$. On expanding the function $\varphi(x, y, t; q)$ in series form by employing Taylor’s theorem near q , one can get

$$\varphi(x, y, t; q) = u_0(x, y, t) + \sum_{m=1}^{\infty} u_m(x, y, t) q^m, \tag{12}$$

where

$$u_m(x, y, t) = \frac{1}{m!} \left. \frac{\partial^m \varphi(x, y, t; q)}{\partial q^m} \right|_{q=0}. \tag{13}$$

On choosing the auxiliary linear operator, $u_0(x, y, t)$, n and \hbar , series (12) converges at $q = 1/n$ and then it yields one of the solutions for eq. (6):

$$u(x, y, t) = u_0(x, y, t) + \sum_{m=1}^{\infty} u_m(x, y, t) \left(\frac{1}{n}\right)^m. \tag{14}$$

Now, differentiating the zeroth-order deformation equation. (10) m times with respect to q and then dividing by $m!$ and finally taking $q = 0$, gives

$$L[u_m(x, y, t) - k_m u_{m-1}(x, y, t)] = \hbar \mathfrak{R}_m(\vec{u}_{m-1}), \tag{15}$$

where

$$\vec{u}_m = \{u_0(x, y, t), u_1(x, y, t), \dots, u_m(x, y, t)\}. \tag{16}$$

Employing the inverse LT on eq. (15), it yields

$$u_m(x, y, t) = k_m u_{m-1}(x, y, t) + \hbar L^{-1}[\mathfrak{R}_m(\vec{u}_{m-1})], \tag{17}$$

where

$$\begin{aligned} \mathfrak{R}_m(\vec{u}_{m-1}) \\ = L[u_{m-1}(x, y, t)] \end{aligned}$$

$$\begin{aligned}
 & -\left(1 - \frac{k_m}{n}\right) \left(\sum_{k=0}^{n-1} s^{\mu-k-1} u^{(k)}(x, y, 0)\right. \\
 & \left. + \frac{1}{s^\mu} L[f(x, y, t)]\right) + \frac{1}{s^\mu} L[Ru_{m-1} + \mathcal{H}_{m-1}]
 \end{aligned} \tag{18}$$

and

$$k_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \tag{19}$$

In eq. (18), \mathcal{H}_m denotes the homotopy polynomial and is defined as

$$\mathcal{H}_m = \frac{1}{m!} \left[\frac{\partial^m \varphi(x, y, t; q)}{\partial q^m} \right]_{q=0}$$

and

$$\varphi(x, y, t; q) = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \dots \tag{20}$$

By eqs (17) and (18), we have

$$\begin{aligned}
 u_m(x, y, t) &= (k_m + \hbar)u_{m-1}(x, y, t) \\
 & - \left(1 - \frac{k_m}{n}\right) L^{-1} \left(\sum_{k=0}^{n-1} s^{\mu-k-1} u^{(k)}(x, y, 0)\right. \\
 & \left. + \frac{1}{s^\mu} L[f(x, y, t)]\right) \\
 & + \hbar L^{-1} \left[\frac{1}{s^\mu} L[Ru_{m-1} + \mathcal{H}_{m-1}] \right].
 \end{aligned} \tag{21}$$

Finally, on solving eq. (21), we get the iterative terms of $u_m(x, y, t)$. The q -HATM series solution is presented as

$$u(x, y, t) = \sum_{m=0}^{\infty} u_m(x, y, t). \tag{22}$$

4. Solution for a fractional system of nonlinear equations of unsteady flow of a polytropic gas

To validate the applicability and the accuracy of the proposed technique, in this section, we consider a system of equations which describes the unsteady flow of a polytropic gas of fractional order.

Example 1. Consider the fractional system of equations of the form [27,40]:

$$\begin{aligned}
 D_t^\mu u + u_x u + v u_y + \frac{p_x}{\rho} &= 0, \\
 D_t^\mu v + u v_x + v v_y + \frac{p_y}{\rho} &= 0, \\
 D_t^\mu \rho + u \rho_x + v \rho_y + \rho u_x + \rho v_y &= 0,
 \end{aligned}$$

$$D_t^\mu p + u p_x + v p_y + \tau p u_x + \tau p v_y = 0, \tag{23}$$

under initial conditions:

$$\begin{aligned}
 u(x, y, 0) &= e^{x+y}, \\
 v(x, y, 0) &= -1 - e^{x+y}, \\
 \rho(x, y, 0) &= e^{x+y}, \\
 p(x, y, 0) &= \eta,
 \end{aligned} \tag{24}$$

where η is the real constant.

By applying LT on system (23) and then employing the condition given in system (24), we have

$$\begin{aligned}
 L[u(x, y, t)] - \frac{1}{s} \{e^{x+y}\} \\
 + \frac{1}{s^\mu} L \left\{ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} \right\} &= 0, \\
 L[v(x, y, t)] + \frac{1}{s} \{1 + e^{x+y}\} \\
 + \frac{1}{s^\mu} L \left\{ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} \right\} &= 0, \\
 L[\rho(x, y, t)] - \frac{1}{s} \{e^{x+y}\} \\
 + \frac{1}{s^\mu} L \left\{ u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} \right\} &= 0, \\
 L[p(x, y, t)] - \frac{1}{s} \{\eta\} \\
 + \frac{1}{s^\mu} L \left\{ u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + \tau p \frac{\partial u}{\partial x} + p \tau \frac{\partial v}{\partial y} \right\} &= 0.
 \end{aligned} \tag{25}$$

By using the proposed algorithm, the nonlinear operator N is defined as

$$\begin{aligned}
 N^1[\varphi_1(x, y, t; q), \varphi_2(x, y, t; q), \varphi_3(x, y, t; q), \\
 \varphi_4(x, y, t; q)] \\
 = L[\varphi_1(x, y, t; q)] - \frac{1}{s} \{e^{x+y}\} \\
 + \frac{1}{s^\mu} L \left\{ \varphi_1(x, y, t; q) \frac{\partial \varphi_1(x, y, t; q)}{\partial x} \right. \\
 + \varphi_2(x, y, t; q) \frac{\partial \varphi_1(x, y, t; q)}{\partial y} \\
 \left. + \frac{1}{\varphi_3(x, y, t; q)} \frac{\partial \varphi_4(x, y, t; q)}{\partial x} \right\}, \\
 N^2[\varphi_1(x, y, t; q), \varphi_2(x, y, t; q), \varphi_3(x, y, t; q), \\
 \varphi_4(x, y, t; q)] \\
 = L[\varphi_2(x, y, t; q)] + \frac{1}{s} \{1 + e^{x+y}\} \\
 + \frac{1}{s^\mu} L \left\{ \varphi_1(x, y, t; q) \frac{\partial \varphi_2(x, y, t; q)}{\partial x} \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \varphi_2(x, y, t; q) \frac{\partial \varphi_2(x, y, t; q)}{\partial y} \\
 & + \frac{1}{\varphi_3(x, y, t; q)} \frac{\partial \varphi_4(x, y, t; q)}{\partial y} \Bigg\}, \\
 N^3[\varphi_1(x, y, t; q), \varphi_2(x, y, t; q), \varphi_3(x, y, t; q), \\
 & \varphi_4(x, y, t; q)] \\
 & = L[\varphi_3(x, y, t; q)] - \frac{1}{s} \{e^{x+y}\} \\
 & + \frac{1}{s^\mu} L \left\{ \varphi_1(x, y, t; q) \frac{\partial \varphi_3(x, y, t; q)}{\partial x} \right. \\
 & + \varphi_2(x, y, t; q) \frac{\partial \varphi_3(x, y, t; q)}{\partial y} \\
 & + \varphi_3(x, y, t; q) \frac{\partial \varphi_1(x, y, t; q)}{\partial x} \\
 & \left. + \varphi_3(x, y, t; q) \frac{\partial \varphi_2(x, y, t; q)}{\partial y} \right\}, \\
 N^4[\varphi_1(x, y, t; q), \varphi_2(x, y, t; q), \varphi_3(x, y, t; q), \\
 & \varphi_4(x, y, t; q)] \\
 & = L[\varphi_4(x, y, t; q)] - \frac{1}{s} \{\eta\} \\
 & + \frac{1}{s^\mu} L \left\{ \varphi_1(x, y, t; q) \frac{\partial \varphi_4(x, y, t; q)}{\partial x} \right. \\
 & + \varphi_2(x, y, t; q) \frac{\partial \varphi_4(x, y, t; q)}{\partial y} \\
 & + \tau \varphi_4(x, y, t; q) \frac{\partial \varphi_1(x, y, t; q)}{\partial x} \\
 & \left. + \tau \varphi_4(x, y, t; q) \frac{\partial \varphi_2(x, y, t; q)}{\partial y} \right\}. \tag{26}
 \end{aligned}$$

By adopting the foregoing procedure of q -HATM, the deformation equation of m th order at $\mathcal{H}(x, y, t) = 1$, is given as

$$\begin{aligned}
 & L[u_m(x, y, t) - k_m u_{m-1}(x, y, t)] \\
 & = \hbar \mathfrak{R}_{1,m}[\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}], \\
 & L[v_m(x, y, t) - k_m v_{m-1}(x, y, t)] \\
 & = \hbar \mathfrak{R}_{2,m}[\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}], \\
 & L[\rho_m(x, y, t) - k_m \rho_{m-1}(x, y, t)] \\
 & = \hbar \mathfrak{R}_{3,m}[\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}], \\
 & L[p_m(x, y, t) - k_m p_{m-1}(x, y, t)] \\
 & = \hbar \mathfrak{R}_{4,m}[\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}],
 \end{aligned} \tag{27}$$

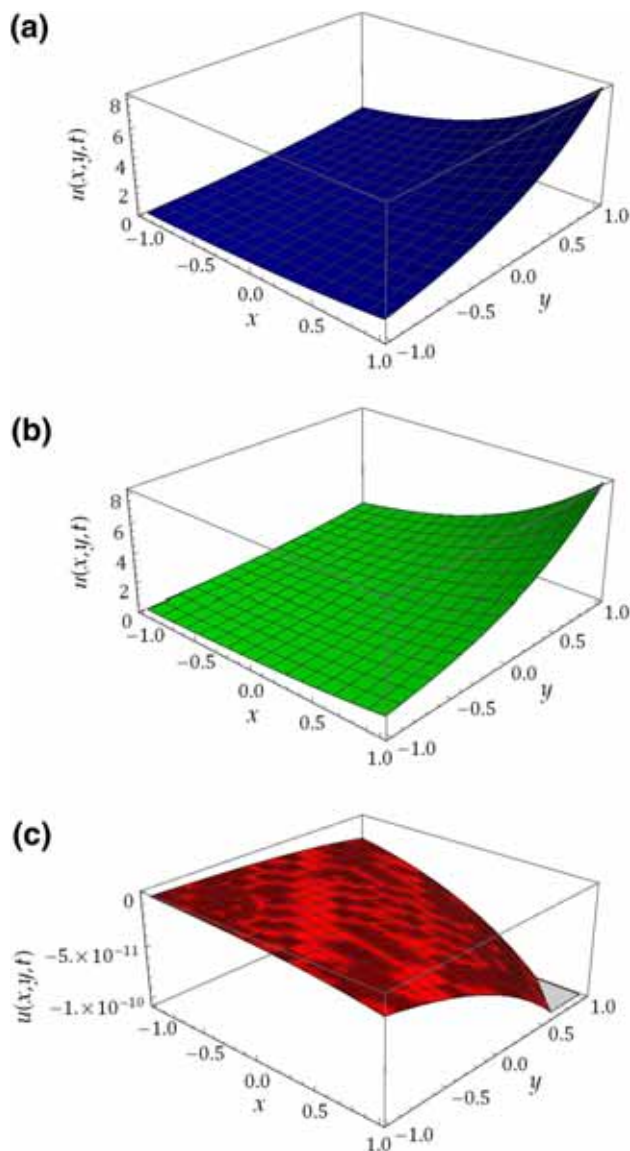


Figure 1. Surfaces of (a) q -HATM solution, (b) exact solution and (c) absolute error = $|u_{\text{Exact}} - u_{q\text{-HATM}}|$ when $\hbar = -1$, $t = 0.1$, $n = 1$ and $\mu = 1$.

where

$$\begin{aligned}
 & \mathfrak{R}_{1,m}[\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}] \\
 & = L[u_{m-1}(x, y, t)] - \left(1 - \frac{k_m}{n}\right) \frac{1}{s} (e^{x+y}) \\
 & + \frac{1}{s^\mu} L \left\{ \sum_{i=0}^{m-1} u_i \frac{\partial u_{m-1-i}}{\partial x} \right. \\
 & \left. + \sum_{i=0}^{m-1} v_i \frac{\partial u_{m-1-i}}{\partial y} + \chi_{m-1}(\rho, p_x) \right\}, \\
 & \mathfrak{R}_{2,m}[\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}]
 \end{aligned}$$

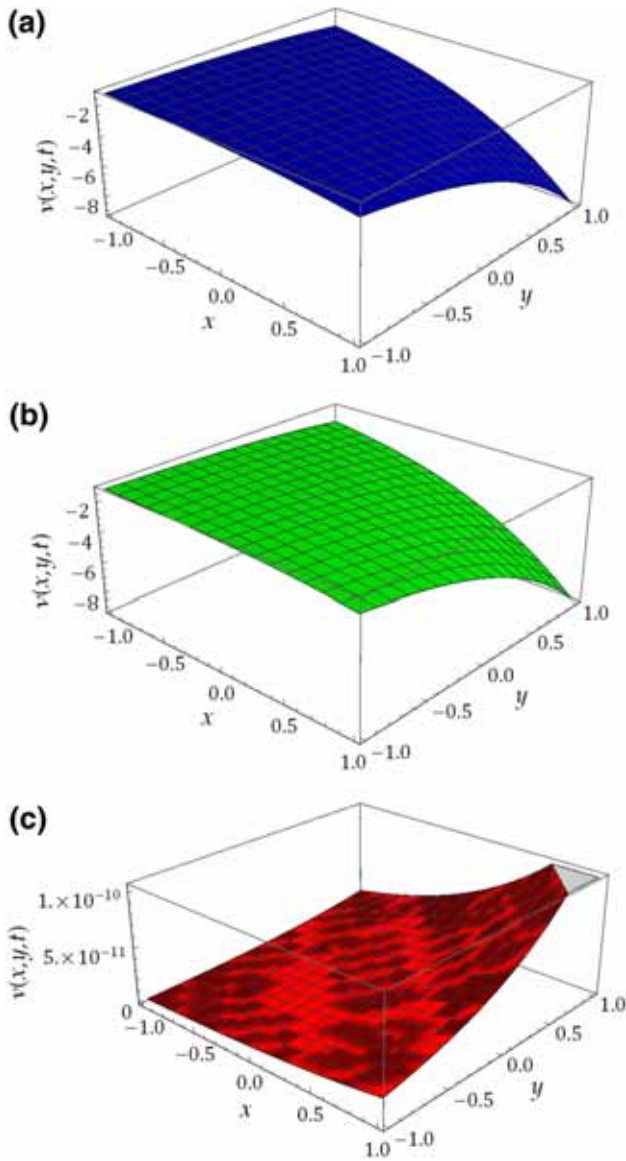


Figure 2. Nature of (a) q -HATM solution, (b) exact solution and (c) absolute error = $|v_{\text{Exact}} - v_{q\text{-HATM}}|$ when $t = 0.1$, $\hbar = -1$, $n = 1$ and $\mu = 1$.

$$\begin{aligned}
 &= L[v_{m-1}(x, y, t)] + \left(1 - \frac{k_m}{n}\right) \frac{1}{s} (1 + e^{x+y}) \\
 &+ \frac{1}{s^\mu} L \left\{ \sum_{i=0}^{m-1} u_i \frac{\partial v_{m-1-i}}{\partial x} \right. \\
 &\left. + \sum_{i=0}^{m-1} v_i \frac{\partial v_{m-1-i}}{\partial y} + \psi_{m-1}(\rho, p_y) \right\}, \\
 \mathfrak{R}_{3,m}[\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}] \\
 &= L[\rho_{m-1}(x, y, t)] - \left(1 - \frac{k_m}{n}\right) \frac{1}{s} (e^{x+y})
 \end{aligned}$$

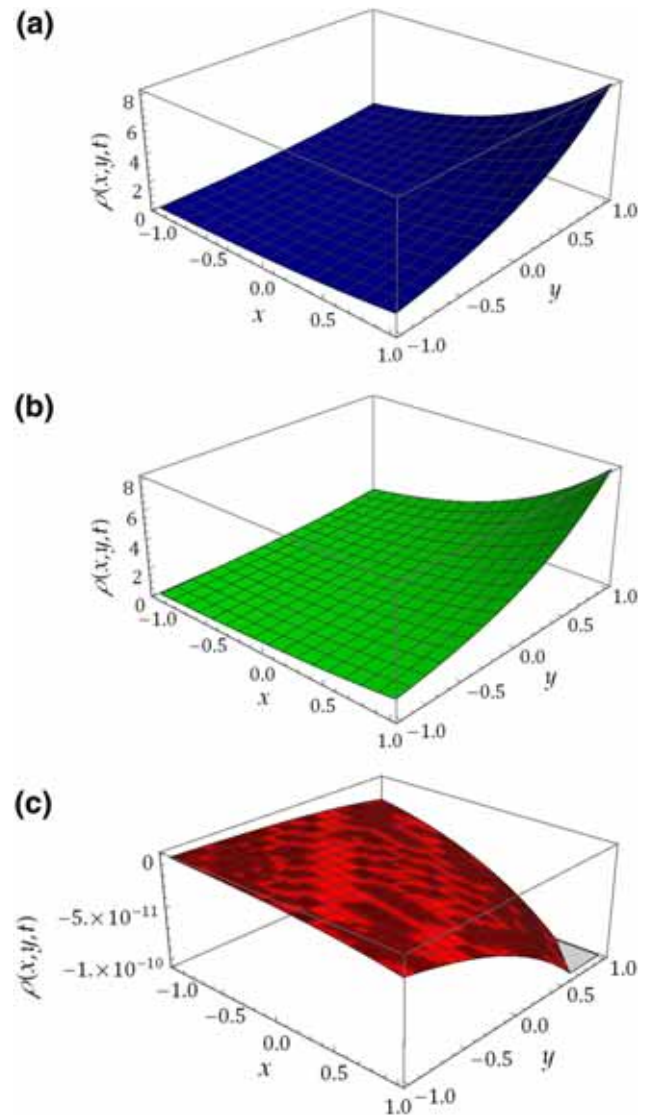


Figure 3. Behaviour of (a) q -HATM solution, (b) exact solution and (c) absolute error = $|\rho_{\text{Exact}} - \rho_{q\text{-HATM}}|$ when $\hbar = -1$, $n = 1$, $t = 0.1$ and $\mu = 1$.

$$\begin{aligned}
 &+ \frac{1}{s^\mu} L \left\{ \sum_{i=0}^{m-1} u_i \frac{\partial \rho_{m-1-i}}{\partial x} + \sum_{i=0}^{m-1} v_i \frac{\partial \rho_{m-1-i}}{\partial y} \right. \\
 &\left. + \sum_{i=0}^{m-1} \rho_i \frac{\partial u_{m-1-i}}{\partial x} + \sum_{i=0}^{m-1} \rho_i \frac{\partial v_{m-1-i}}{\partial y} \right\}, \\
 \mathfrak{R}_{4,m}[\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}] \\
 &= L[p_{m-1}(x, y, t)] - \left(1 - \frac{k_m}{n}\right) \frac{1}{s} (\eta) \\
 &+ \frac{1}{s^\mu} L \left\{ \sum_{i=0}^{m-1} u_i \frac{\partial p_{m-1-i}}{\partial x} + \sum_{i=0}^{m-1} v_i \frac{\partial p_{m-1-i}}{\partial y} \right.
 \end{aligned}$$

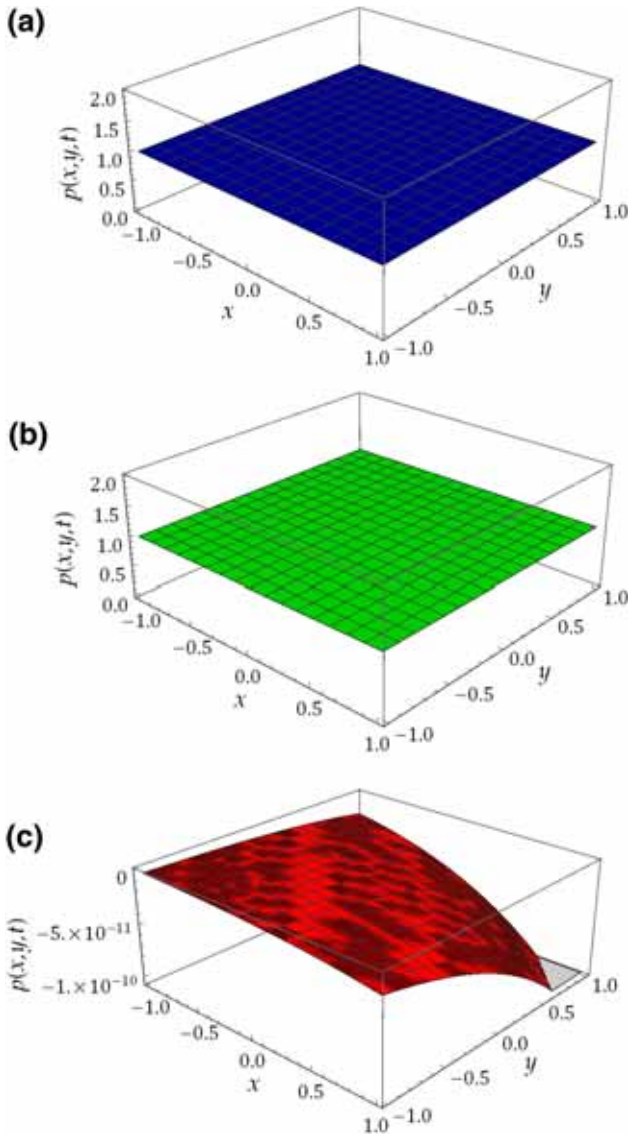


Figure 4. Nature of (a) q -HATM solution, (b) exact solution and (c) absolute error = $|p_{\text{Exact}} - p_{q\text{-HATM}}|$ when $t = 0.1$, $\hbar = -1$, $n = 1$ and $\mu = 1$.

$$+ \tau \sum_{i=0}^{m-1} p_i \frac{\partial u_{m-1-i}}{\partial x} + \tau \sum_{i=0}^{m-1} p_i \frac{\partial v_{m-1-i}}{\partial y} \Bigg\}, \quad (28)$$

where

$$\chi_{m-1}(\rho, p_x) = \frac{1}{\rho_0} \frac{\partial p_0}{\partial x} + \frac{1}{(\rho_0)^2} \left(\rho_0 \frac{\partial p_1}{\partial x} - \rho_1 \frac{\partial p_0}{\partial x} \right) + \dots$$

and

$$\psi_{m-1}(\rho, p_y) = \frac{1}{\rho_0} \frac{\partial p_0}{\partial y} + \frac{1}{(\rho_0)^2} \left(\rho_0 \frac{\partial p_1}{\partial y} - \rho_1 \frac{\partial p_0}{\partial y} \right) + \dots$$

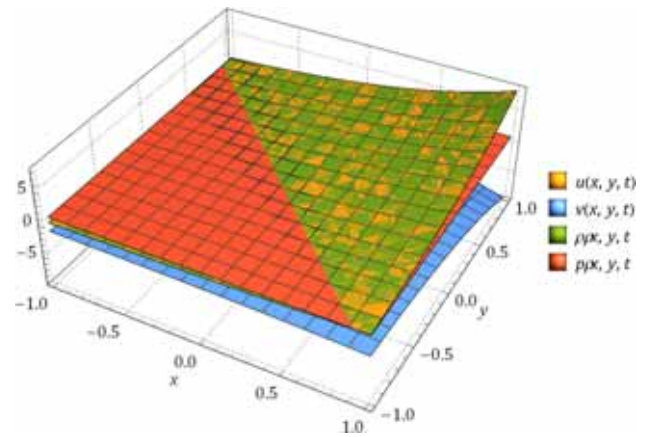


Figure 5. Surface of q -HATM solution for the system of equations describing the unsteady flow of a polytropic gas when $\hbar = -1$, $t = 0.1$, $n = 1$ and $\mu = 1$.

By applying inverse LT on both sides of system (27), we get

$$\begin{aligned} u_m(x, y, t) &= k_m u_{m-1}(x, y, t) \\ &\quad + \hbar L^{-1} \{ \mathfrak{A}_{1,m} [\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}] \}, \\ v_m(x, y, t) &= k_m v_{m-1}(x, y, t) \\ &\quad + \hbar L^{-1} \{ \mathfrak{A}_{2,m} [\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}] \}, \\ \rho_m(x, y, t) &= k_m \rho_{m-1}(x, y, t) \\ &\quad + \hbar L^{-1} \{ \mathfrak{A}_{3,m} [\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}] \}, \\ p_m(x, y, t) &= k_m p_{m-1}(x, y, t) \\ &\quad + \hbar L^{-1} \{ \mathfrak{A}_{3,m} [\vec{u}_{m-1}, \vec{v}_{m-1}, \vec{\rho}_{m-1}, \vec{p}_{m-1}] \}. \quad (29) \end{aligned}$$

On solving the forgoing system of equations systematically, we obtain

$$\begin{aligned} u_0(x, y, t) &= e^{x+y}, \\ v_0(x, y, t) &= -1 - e^{x+y}, \\ \rho_0(x, y, t) &= e^{x+y}, \\ p_0(x, y, t) &= \eta, \\ u_1(x, y, t) &= \frac{-\hbar e^{x+y} t^\mu}{\Gamma[\mu + 1]}, \\ v_1(x, y, t) &= \frac{\hbar e^{x+y} t^\mu}{\Gamma[\mu + 1]}, \\ \rho_1(x, y, t) &= \frac{-\hbar e^{x+y} t^\mu}{\Gamma[\mu + 1]}, \end{aligned}$$

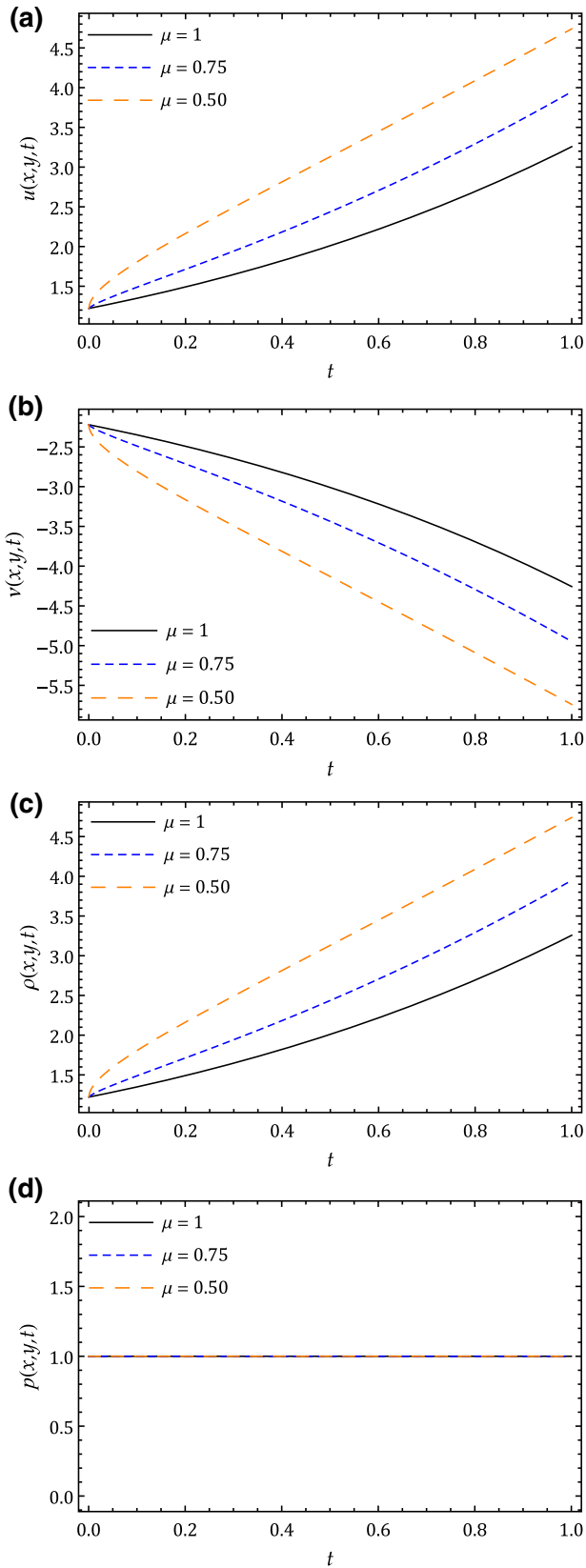


Figure 6. Plot of the q -HATM solution: (a) $u(x, y, t)$, (b) $v(x, y, t)$, (c) $\rho(x, y, t)$ and (d) $p(x, y, t)$ with respect to t when $\hbar = -1$, $n = 1$, $x = 0.1$ and $y = 0.1$ with diverse μ .

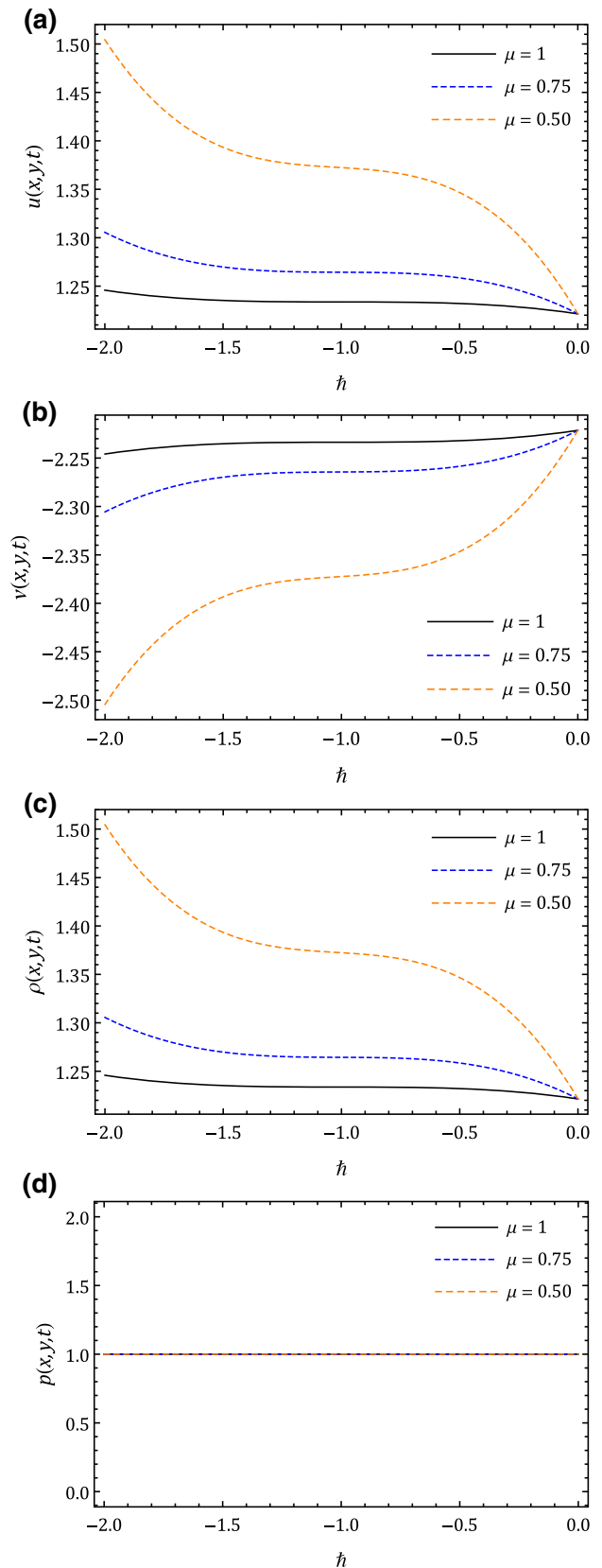


Figure 7. The \hbar -curves drawn for (a) $u(x, y, t)$, (b) $v(x, y, t)$, (c) $\rho(x, y, t)$ and (d) $p(x, y, t)$ with diverse μ when $n = 1$, $t = 0.01$, $x = 0.1$ and $y = 0.1$.

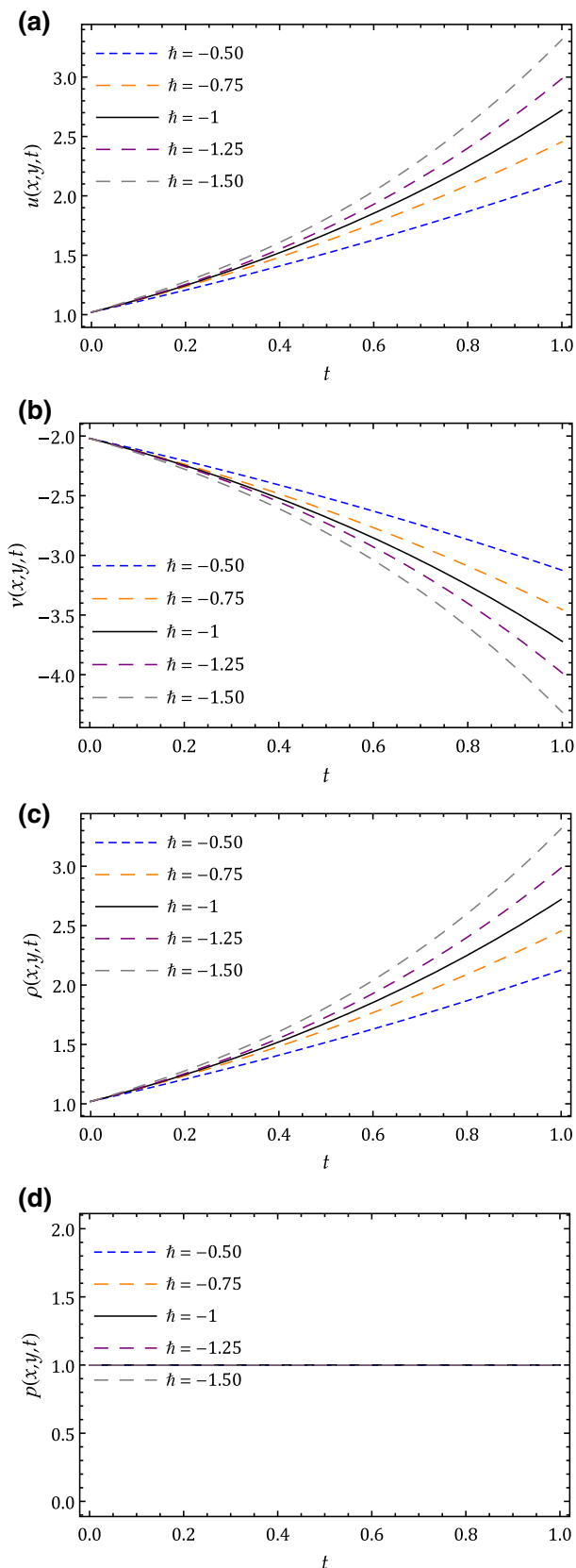


Figure 8. Behaviour of the obtained solution for (a) $u(x, y, t)$, (b) $v(x, y, t)$, (c) $\rho(x, y, t)$ and (d) $p(x, y, t)$ with diverse \hbar when $n = 1$, $\mu = 1$, $x = 0.1$ and $y = 0.1$.

$$\begin{aligned}
 p_1(x, y, t) &= 0, \\
 u_2(x, y, t) &= \frac{-(n + \hbar)\hbar e^{x+y}t^\mu}{\Gamma[\mu + 1]} + \frac{\hbar^2 e^{x+y}t^{2\mu}}{\Gamma[2\mu + 1]}, \\
 v_2(x, y, t) &= \frac{(n + \hbar)\hbar e^{x+y}t^\mu}{\Gamma[\mu + 1]} - \frac{\hbar^2 e^{x+y}t^{2\mu}}{\Gamma[2\mu + 1]}, \\
 \rho_2(x, y, t) &= \frac{-(n + \hbar)\hbar e^{x+y}t^\mu}{\Gamma[\mu + 1]} + \frac{\hbar^2 e^{x+y}t^{2\mu}}{\Gamma[2\mu + 1]}, \\
 p_2(x, y, t) &= 0, \\
 u_3(x, y, t) &= \frac{-(n + \hbar)^2 \hbar e^{x+y}t^\mu}{\Gamma[\mu + 1]} \\
 &\quad + \frac{(n + \hbar)\hbar^2 e^{x+y}t^{2\mu}}{\Gamma[2\mu + 1]} - \frac{\hbar^3 e^{x+y}t^{3\mu}}{\Gamma[3\mu + 1]}, \\
 v_3(x, y, t) &= \frac{(n + \hbar)^2 \hbar e^{x+y}t^\mu}{\Gamma[\mu + 1]} - \frac{(n + \hbar)\hbar^2 e^{x+y}t^{2\mu}}{\Gamma[2\mu + 1]} \\
 &\quad + \frac{\hbar^3 e^{x+y}t^{3\mu}}{\Gamma[3\mu + 1]}, \\
 \rho_3(x, y, t) &= \frac{-(n + \hbar)^2 \hbar e^{x+y}t^\mu}{\Gamma[\mu + 1]} \\
 &\quad + \frac{(n + \hbar)\hbar^2 e^{x+y}t^{2\mu}}{\Gamma[2\mu + 1]} - \frac{\hbar^3 e^{x+y}t^{3\mu}}{\Gamma[3\mu + 1]}, \\
 p_3(x, y, t) &= 0, \\
 u_4(x, y, t) &= \frac{-(n + \hbar)^3 \hbar e^{x+y}t^\mu}{\Gamma[\mu + 1]} \\
 &\quad + \frac{(n + \hbar)^2 \hbar^2 e^{x+y}t^{2\mu}}{\Gamma[2\mu + 1]} \\
 &\quad - \frac{(n + \hbar)\hbar^3 e^{x+y}t^{3\mu}}{\Gamma[3\mu + 1]} + \frac{\hbar^4 e^{x+y}t^{4\mu}}{\Gamma[4\mu + 1]}, \\
 v_4(x, y, t) &= \frac{(n + \hbar)^3 \hbar e^{x+y}t^\mu}{\Gamma[\mu + 1]} - \frac{(n + \hbar)^2 \hbar^2 e^{x+y}t^{2\mu}}{\Gamma[2\mu + 1]} \\
 &\quad + \frac{(n + \hbar)\hbar^3 e^{x+y}t^{3\mu}}{\Gamma[3\mu + 1]} - \frac{\hbar^4 e^{x+y}t^{4\mu}}{\Gamma[4\mu + 1]}, \\
 \rho_4(x, y, t) &= \frac{-(n + \hbar)^3 \hbar e^{x+y}t^\mu}{\Gamma[\mu + 1]} \\
 &\quad + \frac{(n + \hbar)^2 \hbar^2 e^{x+y}t^{2\mu}}{\Gamma[2\mu + 1]} \\
 &\quad - \frac{(n + \hbar)\hbar^3 e^{x+y}t^{3\mu}}{\Gamma[3\mu + 1]} + \frac{\hbar^4 e^{x+y}t^{4\mu}}{\Gamma[4\mu + 1]},
 \end{aligned}$$

Table 1. Error analysis of the obtained solution for the proposed problem with distinct x and t when $\hbar = -1, n = 1, y = 1$ and $\mu = 1$.

x	t	$ u_{\text{Exac.}} - u_{q\text{-HATM}} $	$ v_{\text{Exac.}} - v_{q\text{-HATM}} $	$ \rho_{\text{Exac.}} - \rho_{q\text{-HATM}} $	$ p_{\text{Exac.}} - p_{q\text{-HATM}} $
00.2	0.2	8.64762×10^{-9}	8.64762×10^{-9}	8.64762×10^{-9}	0
	0.4	1.05622×10^{-8}	1.05622×10^{-8}	1.05622×10^{-8}	
	0.6	1.29007×10^{-8}	1.29007×10^{-8}	1.29007×10^{-8}	
	0.8	1.57570×10^{-8}	1.57570×10^{-8}	1.57570×10^{-8}	
	1	1.92456×10^{-8}	1.92456×10^{-8}	1.92456×10^{-8}	
0.4	0.2	1.13576×10^{-6}	1.13576×10^{-6}	1.13576×10^{-6}	
	0.4	1.38723×10^{-6}	1.38723×10^{-6}	1.38723×10^{-6}	
	0.6	1.69436×10^{-6}	1.69436×10^{-6}	1.69436×10^{-6}	
	0.8	2.06950×10^{-6}	2.06950×10^{-6}	2.06950×10^{-6}	
	1	2.52769×10^{-6}	2.52769×10^{-6}	2.52769×10^{-6}	
0.6	0.2	1.99220×10^{-5}	1.99220×10^{-5}	1.99220×10^{-5}	
	0.4	2.43328×10^{-5}	2.43328×10^{-5}	2.43328×10^{-5}	
	0.6	2.97201×10^{-5}	2.97201×10^{-5}	2.97201×10^{-5}	
	0.8	3.63002×10^{-5}	3.63002×10^{-5}	3.63002×10^{-5}	
	1	4.43372×10^{-5}	4.43372×10^{-5}	4.43372×10^{-5}	
0.8	0.2	1.53301×10^{-4}	1.53301×10^{-4}	1.53301×10^{-4}	
	0.4	1.87240×10^{-4}	1.87240×10^{-4}	1.87240×10^{-4}	
	0.6	2.28696×10^{-4}	2.28696×10^{-4}	2.28696×10^{-4}	
	0.8	2.79330×10^{-4}	2.79330×10^{-4}	2.79330×10^{-4}	
	1	3.41174×10^{-4}	3.41174×10^{-4}	3.41174×10^{-4}	
1	0.2	7.51252×10^{-4}	7.51252×10^{-4}	7.51252×10^{-4}	
	0.4	9.17582×10^{-4}	9.17582×10^{-4}	9.17582×10^{-4}	
	0.6	1.12074×10^{-3}	1.12074×10^{-3}	1.12074×10^{-3}	
	0.8	1.36887×10^{-3}	1.36887×10^{-3}	1.36887×10^{-3}	
	1	1.67194×10^{-3}	1.67194×10^{-3}	1.67194×10^{-3}	

$$\begin{aligned}
 p_4(x, y, t) &= 0, \\
 u_5(x, y, t) &= \frac{-(n + \hbar)^4 \hbar e^{x+y} t^\mu}{\Gamma[\mu + 1]} \\
 &\quad + \frac{(n + \hbar)^3 \hbar^2 e^{x+y} t^{2\mu}}{\Gamma[2\mu + 1]} \\
 &\quad - \frac{(n + \hbar)^2 \hbar^3 e^{x+y} t^{3\mu}}{\Gamma[3\mu + 1]} \\
 &\quad + \frac{(n + \hbar) \hbar^4 e^{x+y} t^{4\mu}}{\Gamma[4\mu + 1]} - \frac{\hbar^5 e^{x+y} t^{5\mu}}{\Gamma[5\mu + 1]}, \\
 v_5(x, y, t) &= \frac{(n + \hbar)^4 \hbar e^{x+y} t^\mu}{\Gamma[\mu + 1]} \\
 &\quad - \frac{(n + \hbar)^3 \hbar^2 e^{x+y} t^{2\mu}}{\Gamma[2\mu + 1]} \\
 &\quad + \frac{(n + \hbar)^2 \hbar^3 e^{x+y} t^{3\mu}}{\Gamma[3\mu + 1]} \\
 &\quad - \frac{(n + \hbar) \hbar^4 e^{x+y} t^{4\mu}}{\Gamma[4\mu + 1]} + \frac{\hbar^5 e^{x+y} t^{5\mu}}{\Gamma[5\mu + 1]},
 \end{aligned}$$

$$\begin{aligned}
 \rho_5(x, y, t) &= \frac{-(n + \hbar)^4 \hbar e^{x+y} t^\mu}{\Gamma[\mu + 1]} \\
 &\quad + \frac{(n + \hbar)^3 \hbar^2 e^{x+y} t^{2\mu}}{\Gamma[2\mu + 1]} \\
 &\quad - \frac{(n + \hbar)^2 \hbar^3 e^{x+y} t^{3\mu}}{\Gamma[3\mu + 1]} \\
 &\quad + \frac{(n + \hbar) \hbar^4 e^{x+y} t^{4\mu}}{\Gamma[4\mu + 1]} - \frac{\hbar^5 e^{x+y} t^{5\mu}}{\Gamma[5\mu + 1]}, \\
 p_5(x, y, t) &= 0, \\
 &\quad \vdots
 \end{aligned}$$

Similarly, we can get the rest of the term. Then, the q -HATM series solution of system (23) is given by

$$\begin{aligned}
 u(x, y, t) &= u_0(x, y, t) \\
 &\quad + \sum_{m=1}^{\infty} u_m(x, y, t) \left(\frac{1}{n}\right)^m, \\
 v(x, y, t) &= v_0(x, y, t)
 \end{aligned}$$

Table 2. Numerical simulation between the consecutive iterations of q -HATM solutions with distinct x and t when $\hbar = -1, n = 1, y = 1$ and $\mu = 1$.

x	t	$ u_{\text{Exac.}} - u_{q\text{-HATM}}^{(3)} $	$ u_{\text{Exac.}} - u_{q\text{-HATM}}^{(4)} $	$ u_{\text{Exac.}} - u_{q\text{-HATM}}^{(5)} $	$ u_{\text{Exac.}} - u_{q\text{-HATM}}^{(6)} $
0.25	0.25	5.97721×10^{-4}	2.96316×10^{-5}	1.22715×10^{-6}	4.36271×10^{-8}
	0.50	7.67489×10^{-4}	3.80478×10^{-5}	1.57569×10^{-6}	5.60183×10^{-8}
	0.75	9.85476×10^{-4}	4.88543×10^{-5}	2.02322×10^{-6}	7.19289×10^{-8}
	1	1.26538×10^{-3}	6.27302×10^{-5}	2.59787×10^{-6}	9.23586×10^{-8}
0.50	0.25	1.00799×10^{-2}	9.90457×10^{-4}	8.15136×10^{-5}	5.76830×10^{-6}
	0.50	1.29428×10^{-2}	1.27177×10^{-3}	1.04666×10^{-4}	7.40664×10^{-6}
	0.75	1.66189×10^{-2}	1.63299×10^{-3}	1.34393×10^{-4}	9.51031×10^{-6}
	1	2.13391×10^{-2}	2.09680×10^{-3}	1.72564×10^{-4}	1.22115×10^{-5}
0.75	0.25	5.38822×10^{-2}	7.86696×10^{-3}	9.64674×10^{-4}	1.01888×10^{-4}
	0.50	6.91861×10^{-2}	1.01014×10^{-2}	1.23867×10^{-3}	1.30827×10^{-4}
	0.75	8.88368×10^{-2}	1.29704×10^{-2}	1.59048×10^{-3}	1.67985×10^{-4}
	1	1.14069×10^{-1}	1.66544×10^{-2}	2.04222×10^{-3}	2.15697×10^{-4}
1	0.25	1.80155×10^{-1}	3.47237×10^{-2}	5.63747×10^{-3}	7.89770×10^{-4}
	0.50	2.31323×10^{-1}	4.45861×10^{-2}	7.23865×10^{-3}	1.01408×10^{-3}
	0.75	2.97025×10^{-1}	5.72496×10^{-2}	9.29461×10^{-3}	1.30211×10^{-3}
	1	3.81387×10^{-1}	7.35101×10^{-2}	1.19345×10^{-2}	1.67194×10^{-3}

$$+ \sum_{m=1}^{\infty} v_m(x, y, t) \left(\frac{1}{n}\right)^m,$$

$$\rho(x, y, t) = \rho_0(x, y, t)$$

$$+ \sum_{m=1}^{\infty} \rho_m(x, y, t) \left(\frac{1}{n}\right)^m,$$

$$p(x, y, t) = p_0(x, y, t)$$

$$+ \sum_{m=1}^{\infty} p_m(x, y, t) \left(\frac{1}{n}\right)^m. \tag{30}$$

If we set $\mu = 1, \hbar = -1, n = 1$ and $N \rightarrow \infty$ then the obtained solution

$$\sum_{m=0}^N u_m(x, y, t) \left(\frac{1}{n}\right)^m,$$

$$\sum_{m=0}^N v_m(x, y, t) \left(\frac{1}{n}\right)^m,$$

$$\sum_{m=1}^N \rho_m(x, y, t) \left(\frac{1}{n}\right)^m$$

and

$$\sum_{m=1}^N p_m(x, y, t) \left(\frac{1}{n}\right)^m,$$

respectively, converges to the exact solution

$$u(x, y, t) = e^{x+y+t},$$

$$v(x, y, t) = -1 - e^{x+y+t},$$

$$\rho(x, y, t) = e^{x+y+t},$$

$$p(x, y, t) = \eta.$$

5. Numerical results and discussion

Here, we present the numerical simulation of the obtained solution for four systems of differential equations describing the unsteady flow of a polytropic gas having an arbitrary order. In the present investigation, we find the sixth order q -HATM solution. The behaviour of the q -HATM solution, exact solution and absolute error for each case are presented in figures 1–4. The surface of the obtained solution for four systems of equation is captured in figure 5. In figure 6, the response of the q -HATM solution for the diverse fractional Brownian motion and the standard motion is presented for four different cases. The \hbar -curves are plotted in figure 7 for velocity (u, v), density (ρ) and pressure (p) components which help us to control the convergence region of the obtained series solution. The behaviour of the q -HATM solution for diverse values of \hbar are cited in figure 8 which helps us to understand the effect of the homotopy parameter considered in the future technique.

Moreover, numerical evaluation has been conducted to ensure that the proposed scheme is efficient and accurate. In table 1, we show the numerical study in terms

of the absolute error for components of velocity, density and pressure. The comparison of the consecutive iterations with diverse values of x and t is presented in table 2. It is clear from table 2 that as the number of iterations increases, the obtained solution gets closer to the exact solution.

6. Conclusion

In the present framework, we find the approximated analytical solution for a system of equations describing the flow of a polytropic gas with the aid of q -HATM. The numerical simulation has been presented in terms of plots and tables. The main advantage of the method is that it solves nonlinear differential equations directly without linearisation, discretisation and perturbation. From the obtained results we can see that the future technique gives a straightforward solution procedure and it is simple to find out the accurate region of \hbar to obtain a series solution which converges by means of the so-called \hbar -curves. The outcomes expose that the results achieved with the aid of q -HATM are more general and contain the results of many traditional techniques as a particular case ($\hbar = 1$ and $n = 1$). Finally, we can conclude that the proposed technique is highly methodical and more accurate, and it can be used to study nonlinear problems arising in complex phenomena.

References

- [1] A Singh, S Das, S H Ong and H Jafari, *J. Comput. Nonlinear Dyn.* **14**(4), 041003 (2019)
- [2] M Badr, A Yazdani and H Jafari, *Numer. Methods Partial Differ. Equ.* **34**, 1459 (2018)
- [3] D G Prakasha, P Veerasha and H M Baskonus, *Fractal Fract.* **3**(1), (2019), <https://doi.org/10.3390/fractalfract3010009>
- [4] S S Roshan, H Jafari and D Baleanu, *Math. Methods Appl. Sci.* **41**, 9134 (2018)
- [5] R W Ibrahim, H Jafari, H A Jalab and S B Hadid, *Adv. Differ. Equ.* (2019), <https://doi.org/10.1186/s13662-019-2033-4>
- [6] P Veerasha, D G Prakasha and D Baleanu, *Mathematics* **7**(3), (2019), <https://doi.org/10.3390/math7030265>
- [7] M A Firoozjaee, H Jafari, A Lia and D Baleanu, *J. Comput. Appl. Math.* **339**, 367 (2018)
- [8] H Jafari and S Seifi, *Commun. Nonlinear Sci.* **14**, 2006 (2009)
- [9] A Prakash, P Veerasha, D G Prakasha and M Goyal, *Pramana – J. Phys.* **93**: 6 (2019), <https://doi.org/10.1007/s12043-019-1763-x>
- [10] T A Sulaiman, H M Baskonus and H Bulut, *Pramana – J. Phys.* **91**: 58 (2018), <https://doi.org/10.1007/s12043-018-1635-9>
- [11] M Caputo, *Elasticita Dissipazione* (Zanichelli, Bologna, 1969)
- [12] K S Miller and B Ross, *An introduction to fractional calculus and fractional differential equations* (A Wiley, New York, 1993)
- [13] I Podlubny, *Fractional differential equations* (Academic Press, New York, 1999)
- [14] A A Kilbas, H M Srivastava and J J Trujillo, *Theory and applications of fractional differential equations* (Elsevier, Amsterdam, 2006)
- [15] C S Drapaca and S Sivaloganathan, *J. Elast.* **107**, 105 (2012)
- [16] H Nasrolahpour, *Commun. Nonlinear Sci.* **18**, 2589 (2013)
- [17] A Deshpande and V D Gejji, *Pramana – J. Phys.* **87**: 49 (2016), <https://doi.org/10.1007/s12043-016-1231-9>
- [18] H Bulut, T A Sulaiman and H M Baskonus, *Optik* **163**, 49 (2018)
- [19] V K Shchigolev, *Commun. Theor. Phys.* **56**(2), 389 (2011)
- [20] D G Prakasha, P Veerasha and H M Baskonus, *Comp. Math. Methods* **2**(1), (2019), <https://doi.org/10.1002/cmm4.1021>
- [21] P Veerasha, D G Prakasha and H M Baskonus, *Chaos* **29**, 013119 (2019), <https://doi.org/10.1063/1.5074099>
- [22] A Atangana, *Chaos Solitons Fractals* **114**, 347 (2018)
- [23] D G Prakasha, P Veerasha and M S Rawashdeh, *Math. Methods Appl. Sci.* (2019), <https://doi.org/10.1002/mma.5533>
- [24] J C Dalsgard, *Lecture notes on stellar structure and evolution* (Aarhus University Press, Aarhus, 2004)
- [25] I Klebanov, A Panov, S Ivanov and O Maslova, *Commun. Nonlinear Sci.* **59**, 437 (2018)
- [26] H Moradpour and A Abri, *Int. J. Mod. Phys. D* **12**(1), (2016), <https://doi.org/10.1142/S0218271816500140>
- [27] M Matinfar and S J Nodeh, *J. Math. Ext.* **3**(2), 61 (2009)
- [28] M Matinfar and M Saeidy, *World Appl. Sci. J.* **9**(9), 980 (2010)
- [29] S J Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D. thesis (Shanghai Jiao Tong University, 1992)
- [30] S J Liao, *Appl. Math. Mech.* **19**, 957 (1998)
- [31] J Singh, D Kumar and R Swroop, *Alexandria Eng. J.* **55**(2), 1753 (2016)
- [32] A Prakash, P Veerasha, D G Prakasha and M Goyal, *Eur. Phys. J. Plus* **134**(19), (2019), <https://doi.org/10.1140/epjp/i2019-12411-y>
- [33] H M Srivastava, D Kumar and J Singh, *Appl. Math. Model* **45**, 192 (2017)
- [34] J Singh, D Kumar, D Baleanu and S Rathore, *Appl. Math. Comput.* **335**, 12 (2018)

- [35] H Bulut, D Kumar, J Singh, R Swroop and H M Baskonus, *Math. Nat. Sci.* **2**(1), 33 (2018)
- [36] D Kumar, R P Agarwal and J Singh, *J. Comput. Appl. Math.* **399**, 405 (2018)
- [37] P Veerasha, D G Prakasha, N Magesh, M M Nandeppanavar and A J Christopher, [arXiv:1810.06311v2](https://arxiv.org/abs/1810.06311v2) [math.NA] (2019)
- [38] A Prakash, D G Prakasha and P Veerasha, *Nonlinear Eng.* <https://doi.org/10.1515/nleng-2018-0080> (2019)
- [39] P Veerasha, D G Prakasha and H M Baskonus, *Math. Sci.* **13**, 33 (2019), <https://doi.org/10.1007/s40096-019-0276-6>
- [40] M A Mohamed, *Appl. Appl. Math.* **4**, 52 (2009)
- [41] H M Cherif, D Ziane and K Belghaba, *Nonlinear Stud.* **25**(4), 53 (2018)
- [42] A J Al-Saif and F A Al-Saadawi, *J. Phys. Sci. Appl.* **5**, 38 (2015)