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Solitary waves in strongly non-local media with a harmonic potential

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Abstract. An exact analytical solution in strongly non-local media with a harmonic potential has been studied. Two-dimensional Bessel solitary wave clusters have been obtained by a self-similar method. The intensity distributions of optical beam with different parameters have been discussed in detail. It is found that the solitary waves have a symmetric necklace distribution and the number of facular points is double the value of the quantum number m. The modulation of the external potential field to the width of light beam is also discussed.

Keywords. Solitons; self-similar method; non-local Schrödinger equation.

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1. Introduction

In recent years, the study of solitary waves in nonlinear media has been the subject of much interest in theory and experiment [1–9]. Non-local nonlinearities, which exist in some optical nonlinearity media and ultracold Rydberg atom gases, have attracted significant interest recently [10–15]. Non-local nonlinear response can suppress modulation instability, prevent the collapse of self-focussing beams [16], support fundamental and vortex solitons [17], and can stabilise dipole or multipole optical solitary waves [18,19]. Strong non-locality of nonlinear media means that the spatial range of the beam is much larger than its size. The phenomena of strong non-locality have been found in some experiments in nonlinear media [20,21]. It has been demonstrated that the non-locality of nonlinear media can be described by a general non-local nonlinear Schrödinger equation (NNLSE). In order to solve nonlinear problems in a simpler way, many methods have been given [22–24]. In 1997, Snyder and Mitchell [5] proposed a non-local linear model to describe the propagation of optical beam in a non-local nonlinear medium in the case of strong non-locality. Subsequently, some generalised nonlinear models have been proposed and a series of solutions, such as Hermit–Gauss, Laguerre–Gauss, necklace solitons, etc., have been obtained in different dimensional coordinates [3,25–27].

Self-similar solutions have been explored not only in some areas such as hydrodynamics and quantum field theory [28], but also in nonlinear optics [29]. As examples, self-similar behaviour in stimulated Raman scattering [24], exact self-similar solutions in non-local Schrödinger equation (NLSE) with distributed coefficients or some solutions in strongly non-local nonlinear media [3,30] and self-similar evolutions in various nonlinear media [31,32] were extensively investigated. Such self-similar solutions have many features similar to the ideal solitons, and so it is also called self-similar solitary wave.

In this paper, by a self-similar transformation, we give an exact analytical Bessel-type solution for a generalised NLSE with a harmonic potential in a strongly non-local limit.

2. Methods and discussions

The NLSE of two-dimensional optical beams in the case of strong non-locality can be written in a dimensionless form as [3,5,25]

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$$\mathrm{i}\frac{\partial \psi}{\partial z} + \frac{1}{2}\nabla_{\perp}^2 \psi - sr^2 \psi = 0, \tag{1}$$

where $\psi = \psi(r, z, \varphi)$ is a paraxial beam and s is the normalised unit corresponding to the beam in the transverse plane. $r = \sqrt{x^2 + y^2}$ and x, y are the axes of Cartesian coordinates. $\nabla_{\perp}^2 = (\partial^2/\partial r^2) + (1/r)(\partial/\partial r) + (1/r^2)(\partial^2/\partial \varphi^2)$ is the Laplacian operator in two-dimensional polar coordinates and z is the axis of the light propagation. In this paper, we consider the evolution of a scalar wave field $\psi = \psi(r, z, \varphi)$ governed by the general NLSE with an external potential in the scaled form [33]

$$\mathrm{i}\frac{\partial \psi}{\partial z} + \frac{1}{2}\nabla_{\perp}^2 \psi - sr^2 \psi - V(r)\psi = 0. \tag{2}$$

Let us now consider the external potential $V(r)$ to be a harmonic external potential, i.e., $V(r) = (1/2)kr^2$ where k is the potential parameter. Now, we consider that the beam has the form $\psi = u(r, z)\phi(\varphi)$ by a separation of variables. Equation (2) can be separated into two functions as follows:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\varphi^2} + m^2\phi = 0, \tag{3}$$

$$\mathrm{i}\frac{\partial u}{\partial z} + \frac{1}{2}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{m^2}{r^2}u\right) - s'r^2 u = 0. \tag{4}$$

The parameter m is an integer and stands for the physical quantum number of ϕ. The parameter $s' = s + (1/2)k$. The solution of eq. (3) is $\varphi = \cos(m\phi) + \mathrm{i}q \sin(m\phi)$. The parameter q determines the depth of the azimuthal modulation. The next step is to find self-similar solutions of eq. (4). As pointed out in [3,27,28], the complex field can be defined as $u(r, z) = A(r, z)\mathrm{e}^{\mathrm{i}B(z,r)}$, where $A(r, z)$ and $B(r, z)$ are real functions. If we substitute the form of $u(r, z)$ into eq. (4) and separate the real part and imaginary part, we get the following equations:

$$-\frac{\partial B}{\partial z} + \frac{1}{2}\left[\frac{1}{A}\frac{\partial^2 A}{\partial r^2} - \left(\frac{\partial B}{\partial r}\right)^2 + \frac{1}{rA}\frac{\partial A}{\partial r} - \frac{m^2}{r^2}\right] - s'r^2 = 0, \tag{5}$$

$$\frac{1}{A}\frac{\partial A}{\partial z} + \frac{1}{2}\left[\frac{2}{A}\frac{\partial B}{\partial r}\frac{\partial A}{\partial r} + \frac{\partial^2 B}{\partial r^2} + \frac{1}{r}\frac{\partial B}{\partial r}\right] = 0. \tag{6}$$

To search for a self-similar solution, we introduce a set of self-similar transformations [34]

$$A(r, z) = \frac{F(\theta)}{w(z)}, \tag{7}$$

$$B(z, r) = a(z) + b(z)r + c(z)r^2, \tag{8}$$

where $w(z)$ is the width of the light beam, $F(\theta)$ is a self-similar function and $\theta(r, z) = r^2/w^2$ is a self-similar variable. The parameters $a(z)$, $b(z)$ and $c(z)$, respectively, stand for the phase offset, the frequency shift and the wave front curvature. After substituting eqs (7) and (8) into eq. (6), we obtain a function with the variable r:

$$\left(\frac{1}{F}\frac{\partial F}{\partial z} - \frac{1}{w}\frac{\partial w}{\partial z} + \frac{b}{F}\frac{\partial F}{\partial r} + 2c\right)r + \frac{2c}{F}\frac{\partial F}{\partial r}r^2 + \frac{b}{2} = 0. \tag{9}$$

From this function, we can get $b(z) = 0$ and $c(z) = (1/2w)(\partial w/\partial z)$. Substituting these results into eq. (5), we have

$$-\frac{\mathrm{d}a}{\mathrm{d}z} - \frac{r^2}{2w}\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - sr^2 - \frac{2r^2}{w^4} + \frac{2}{Fr^2}\left[\theta^2\frac{\mathrm{d}^2 F}{\mathrm{d}\theta^2} + \theta\frac{\mathrm{d}F}{\mathrm{d}\theta} + \left(\theta^2 - \left(\frac{m}{\sqrt{2}}\right)^2 F\right)\right] = 0. \tag{10}$$

This equation can be rewritten as follows:

$$\theta^2\frac{\mathrm{d}^2 F}{\mathrm{d}\theta^2} + \theta\frac{\mathrm{d}F}{\mathrm{d}\theta} + (\theta^2 - m'^2 F) = 0, \tag{11}$$

$$a = 0, \tag{12}$$

$$-\frac{1}{2w}\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - s' - \frac{2}{w^4} = 0. \tag{13}$$

Here the parameter $m' = m/\sqrt{2}$. The solution of eq. (11) is a type of Bessel function and it can be written as

$$F(\theta) = J_{m'}(\theta). \tag{14}$$

In order to find the solution of eq. (13), we give a transformation $\mathrm{d}w/\mathrm{d}z = W$. According to eq. (13), we obtain $(\mathrm{d}W/\mathrm{d}z) = -2s'w - (4/w^3)$ and $(\mathrm{d}w/\mathrm{d}W) = W/(-2s'w - (4/w^3))$. Based on the initial physical conditions:

$$\begin{cases} w|_{z=0} = w_0, \\ W = (\mathrm{d}w/\mathrm{d}z)|_{z=0} = 0, \end{cases}$$

we get the following equation:

$$\frac{1}{2}(\mathrm{d}w/\mathrm{d}z)^2 = -s'w^2 + \frac{2}{w^2}. \tag{15}$$

The solution of eq. (15) can be easily obtained as

$$w^2 = w_0^2[\cos^2(\sqrt{2s'}z) + \lambda \sin^2(\sqrt{2s'}z)], \tag{16}$$

where $\lambda = (1/w_0^2)\sqrt{2/s'}$. Then the expression of $c(z)$ can be obtained as

$$c(z) = \frac{\sqrt{2s'}w_0^2(\lambda - 1)\sin(2\sqrt{2s'}z)}{2w^{3/2}}. \tag{17}$$

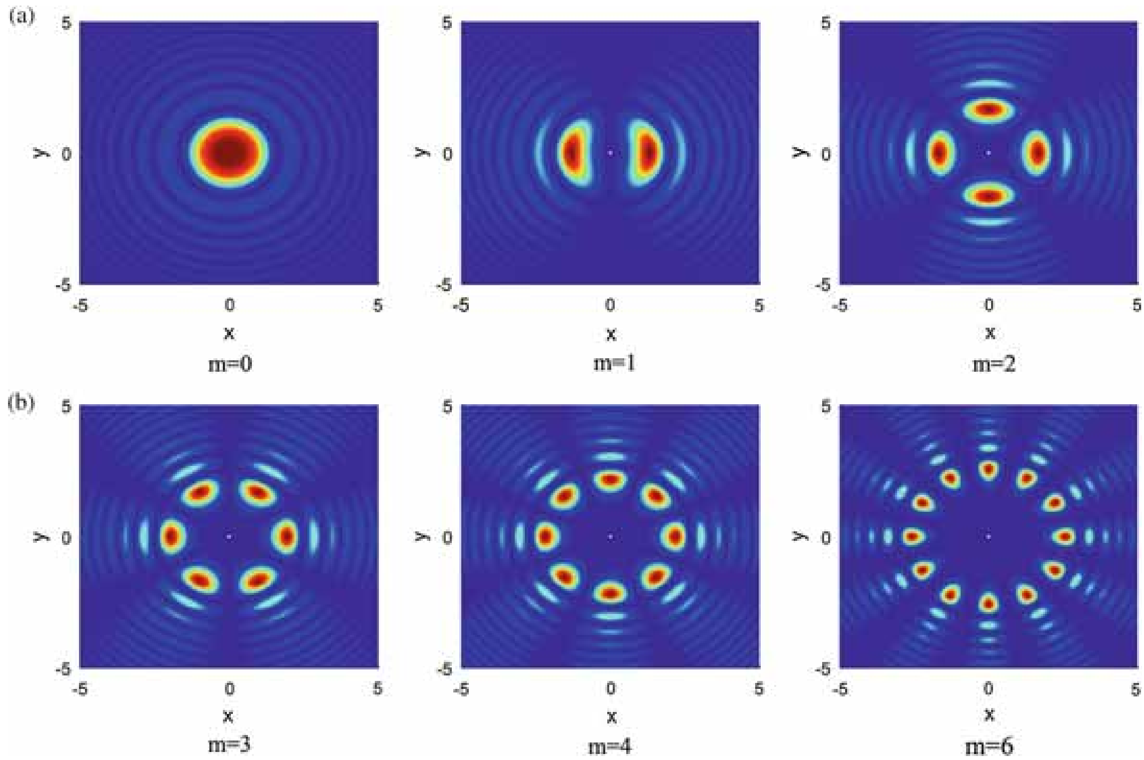

Figure 1. Contour plot of symmetric intensity distributions of self-similar solitary waves for different parameters when $z = \pi/6$. The parameters $w_0 = 1$, $s' = 1$, $m = 0{-}4$ and 6 are as shown in the figure.

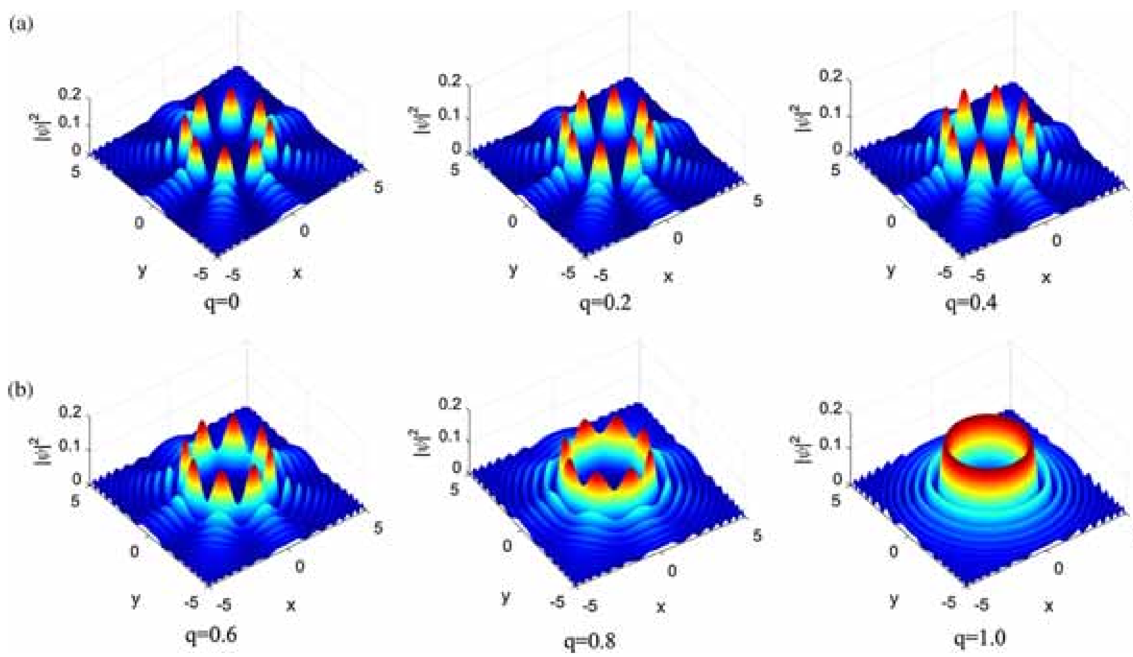

Figure 2. Intensity distributions of solitons with different values of q when $z = \pi/6$. The parameters $w_0 = 1$, $s' = 1$ and $m = 4$.

Finally, we get the exact self-similar solution of eq. (2):

$$\psi(r, z, \varphi) = \frac{J_{m'}(\theta)}{w}[\cos(m\varphi) + \mathrm{i}q\sin(m\varphi)]\mathrm{e}^{\mathrm{i}(-nz+c(z))}. \quad (18)$$

In the following paragraphs, we shall explore the intensity distribution of self-similar solitary waves by our analytical solutions. Figure 1 shows the intensity distributions of self-similar soliton clusters with different m. Here the initial parameters are chosen as

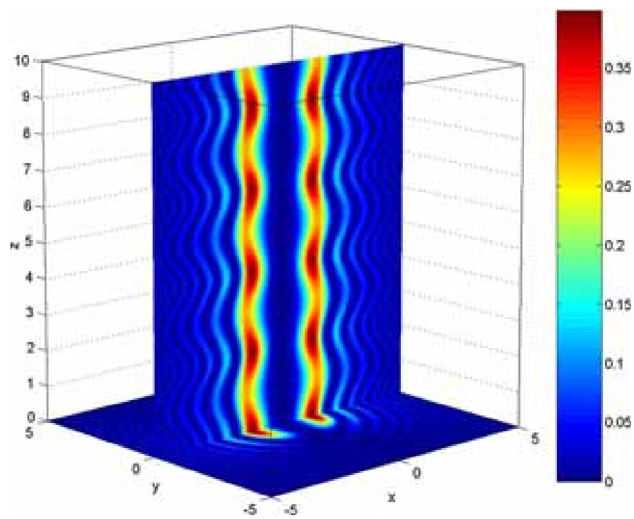

Figure 3. The variation of intensity distribution of the soliton with the three-dimensional coordinate. The parameters $w_0 = 1$, $s' = 1$, $q = 0$ and $m = 1$.

$w_0 = 1$, $s' = 1$ and $q = 0$. It is clearly seen that the intensities of soliton clusters have a symmetric distribution. This phenomenon comes from the additional strongly non-local condition [16]. It can be noted that the solution becomes the usual Bessel type when $m = 0$. With the increase of m, there is a necklace-type distribution around the central point for the facula points and the number of facular points is double the value of m.

It can be seen from eq. (18) that the intensity distribution of self-similar solitary waves is greatly affected by the azimuthal modulation parameter q unless $q = 1$. As shown in figure 2, the light intensity of the ring wave is located closest to the transmission shaft and the transmission in the centre axis of the wave intensity is zero. With the increase in modulation parameter q, the effect of angle modulation is more and more obvious. The necklace will become more and more obscure with the increase of q until it becomes a Bessel ring at $q = 1$.

In figure 3, we show the three-dimensional distribution of the solitary wave's intensity as a function of the coordinate axis. Obviously, when the wave travels in space, its intensity varies periodically with the change of transmission distance z. Because of the non-local nonlinearity, the polarisation of the medium is related not only to the electric field in the region but also to the electric field in the surrounding region of the medium. It is the influence of the surrounding electric field that leads to the attenuation of the light field [3]. These phenomena can be seen from figures 1–3. Therefore, the light field formed by self-similar solitary waves is an attenuation field.

In general, the modulation of the soliton motion by the applied potential field is very obvious. In figure 4, as an example, we take the width of light beam at $z = 10$ and 20 to discuss the modulation process of the optical harmonic oscillator potential field. It can be seen from the figure that the influence of the harmonic oscillator potential field on the width of light beam is obvious. With the gradual increase of the harmonic oscillator potential field, the peak value of the light beam width appears similar to periodic fluctuation, and the overall trend is that the period increases gradually with the potential parameter k.

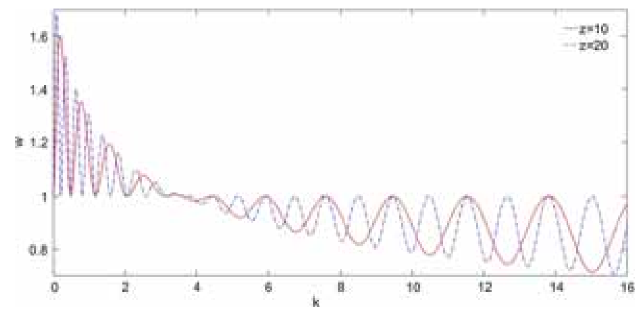

Figure 4. The variation of the width of the light beam with the trap frequency k when $z = 10$ and 20. The parameters $w_0 = 1$ and $s' = 0.1$.

3. Conclusion

In summary, using a self-similar method, the exact solutions of a generalised NLSE with a harmonic potential in the strongly non-local limit have been studied. A two-dimensional solution of Bessel solitary waves has been obtained. The intensity distributions of the optical beam have been discussed in detail. It is found that the soliton clusters have a necklace-type symmetric distribution around the central point and the number of facular points is double the value of m. The three-dimensional distribution of wave propagation in space is also discussed. In addition, we also discussed the modulation of the external potential field to the width of the light beam.

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