



Brief report

Robustness of coherence for multipartite quantum states

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Abstract. In this brief report, we prove that the robustness of coherence (ROC), in contrast with many popular quantitative measures of quantum coherence derived from the resource-theoretic framework of coherence, may be a subadditive for a specific class of multipartite quantum states. We investigate how the subadditivity is affected by admixture with other classes of states for which ROC is superadditive. We show that pairs of quantum states may have different orderings with respect to relative entropy of coherence, l_1 -norm of coherence and ROC and numerically study the difference in ordering for the chosen pairwise coherence measures.

Keywords. Robustness of coherence; subadditivity; l_1 -norm.

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1. Introduction

Quantum mechanics is the theoretical cornerstone underpinning our understanding of the natural world. The abstract laws of quantum mechanics also present us with resources we can harness to perform practical and important information theoretic tasks [1]. Motivated by the importance of quantum entanglement [2] in quantum communication schemes, a general study of the theory of resources within the quantum framework and beyond is being developed at present. One such concrete example of a quantum resource theory is the resource theory of coherence [3–11], which seeks to quantify and study the amount of linear superposition a quantum state possesses with respect to a given basis. As the superposition principle differentiates quantum mechanics from classical particle mechanics, quantum coherence may be viewed as the fundamental signature of non-classicality in physical systems. Coherence may be considered as a resource for certain tasks like better cooling [12,13] or work extraction [14] in nanoscale thermodynamics, quantum algorithms [15–17] or biological processes [18–20]. The relationship of resource theory of quantum coherence with resource theories of entanglement [21–29] and thermodynamics [14,30,31] is also quite close.

However, any resource, including quantum coherence, may decay. One can, thus, quantify quantum resources in terms of how robust they are against mixing with other states. This quantitative measure, introduced in literature [32,33] as the ‘robustness of coherence (ROC)’, follows all the necessary and desirable conditions for a measure of quantum coherence laid down in [4]. In this paper, we point out a surprising property of ROC. Unlike many other measures of quantum coherence, including two most popular measures, viz. ‘ l_1 -norm of coherence’ and ‘relative entropy of coherence’, we show that ROC is not superadditive for multipartite quantum states in general. To this end, we explicitly point out a specific class of quantum states for every member of which, ROC of the multipartite state is less than the ROC of the sum of the reduced states. However, it is worth pointing out that for many classes of multipartite states, e.g. pure states or X-states, ROC is still superadditive. Thus, it is important to study how the superadditivity of quantum coherence gets affected if states from two different classes, one satisfying subadditivity of ROC and the other satisfying superadditivity of ROC, are mixed. Rather interestingly, we numerically observe that when states from a class of quantum states satisfying the superadditivity of ROC are mixed with

states from the class of states satisfying the subadditivity of ROC, provided the mixing weight of the superadditive class of states exceeds a certain value, the ROC of every such resulting mixed state is superadditive.

We also address the issue of non-unanimous ordering of pairs of quantum states with respect to different coherence measures. While the ROC is identical to the l_1 -norm of coherence and quite different from the relative entropy of coherence in two-dimensional systems, we note that as we increase the dimension of the quantum system, a randomly chosen pair of quantum states is more likely to have different ordering with respect to l_1 -norm and ROC rather than with respect to the relative entropy of coherence and ROC. This is in spite of the fact [32,33] that for many multidimensional families of quantum states like pure states or X-states, the ROC is identical to the l_1 -norm of coherence. We observe a similar behaviour for randomly chosen higher rank states with a given dimension.

The paper is organised as follows. In §2, we briefly recall the basic structure of resource theory of coherence and the definition of ROC. In §3, we prove two results for quantum coherence on bipartite systems. In §4, we study the possible subadditivity of ROC. Section 5 deals with the discussion on ordering of quantum states with respect to different coherence measures. We conclude in §6.

2. Robustness of coherence

At first, we shall look into the criteria needed by a functional C to qualify as a measure of coherence. In Baumgratz's framework [4], a functional C , mapping quantum states to a non-negative real numbers, must satisfy the following properties to qualify as a measure of quantum coherence:

- (C1) Firstly, C should vanish on all incoherent states: $C(\rho) = 0, \forall \rho \in I$, where I is the set of all incoherent states in the given basis.
- (C2) Secondly, C should not increase under incoherent operations, which can be of types A and B.
 - (C2a) Under type-A operations, we have monotonicity under incoherent completely positive and trace-preserving maps, i.e. $C(\rho) \geq C(\Phi_{\text{ICPTP}}(\rho)), \forall \Phi_{\text{ICPTP}}$.
 - (C2b) Under type-B operations, we have monotonicity under selective measurements on average, i.e. $C(\rho) \geq \sum_n p_n C(\rho_n), \forall \{K_n\}$ such that $\sum_n K_n^\dagger K_n = \mathbb{I}$ and $K_n I K_n^\dagger \subset I$, where I is the set of all incoherent states in the given basis.

- (C3) Moreover, we would ideally like to ensure that coherence can only decrease under mixing, which leads to our final condition: non-increasing under mixing of quantum states (convexity), i.e. $\sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n)$ for any set of states $\{\rho_n\}$ and any $p_n \geq 0$ with $\sum_n p_n = 1$.

Now, we recall the definition of ROC which satisfies all the aforementioned criteria for a coherence monotone.

ROC: Let $\mathcal{D}(\mathcal{C}^d)$ be the convex set of density operators acting on a d -dimensional Hilbert space. Let $\mathcal{I} \subset \mathcal{D}(\mathcal{C}^d)$ be the subset of incoherent states. Then, the ROC of a state $\rho \in \mathcal{D}(\mathcal{C}^d)$ is defined as

$$C_{\text{ROC}}(\rho) = \min_{\tau \in \mathcal{D}(\mathcal{C}^d)} \left\{ s \geq 0 \left| \frac{\rho + s\tau}{1+s} =: \delta \in \mathcal{I} \right. \right\}. \quad (1)$$

Clearly, $C_{\text{ROC}}(\rho)$ is the minimum weight of another state τ such that its convex mixture with ρ yields an incoherent state δ . It is slightly different from the similarly defined robustness of entanglement [34], in that the mixing is not only ever free, i.e. incoherent states in this case. The reason for such a choice is as follows – incoherent states, unlike separable ones, lie on a zero volume subspace of the original state space. Thus, had ROC been defined in terms of mixing over incoherent state only, that would have led to the ROC blowing up for every non-incoherent state. In this context, we note, however, that a generalisation of robustness of entanglement utilising mixtures with other entangled states as well has been proposed in [35].

The ROC has an operational interpretation as a coherence witness through a semidefinite program. It also means that $C_{\text{ROC}}(\rho)$ can be evaluated via a semidefinite program that finds the optimal coherence witness operator. This semidefinite program has been used to carry out numerical calculations in this paper.

3. Preliminary results

In this section, we derive two results on quantum coherence for joint states. We shall first set the stage by defining what we mean by a superadditive or a subadditive function of a quantum state.

DEFINITION

A function Q defined for a multipartite finite-dimensional quantum state $\rho_{1,2,3,\dots,N}$ as well as for each of the subsystems $1, 2, 3, \dots, N$ is said to be superadditive when $Q(\rho_{1,2,3,\dots,N}) \geq Q(\rho_1) + Q(\rho_2) + \dots + Q(\rho_N)$, and subadditive when $Q(\rho_{1,2,3,\dots,N}) \leq Q(\rho_1) + Q(\rho_2) + \dots + Q(\rho_N)$.

We note that the present definition is not identical to the definition for superadditivity or subadditivity in terms of tensor products of N -copies of identical and uncorrelated states.

It is easy to note that the l_1 -norm of coherence is superadditive. For a bipartite state $\rho_{AB} = \sum_{ijkl} c_{ijkl} |i\rangle_A |j\rangle \otimes |k\rangle_B |l\rangle$, the reduced state ρ_A is given by $\sum_{ijk} c_{ijk} |i\rangle_A |j\rangle$ and the reduced state ρ_B is given by $\sum_{ikl} c_{ikl} |k\rangle_B |l\rangle$. Now the l_1 -norm of coherence of the state ρ_{AB} is given by $C_{l_1}(\rho_{AB}) = \sum_{ijkl; i \neq j || k \neq l} |c_{ijkl}| \geq \sum_{ijkl; i \neq j} |c_{ijkl}| + \sum_{ijkl; k \neq l} |c_{ijkl}|$. Now, $\sum_{ijkl; i \neq j} |c_{ijkl}| + \sum_{ijkl; k \neq l} |c_{ijkl}| \geq \sum_{ijk; i \neq j} |c_{ijk}| + \sum_{ikl; k \neq l} |c_{ikl}|$. The last expression is nothing but $C_{l_1}(\rho_A) + C_{l_1}(\rho_B)$. Thus, the l_1 -norm of coherence is indeed superadditive for all bipartite states. The proof can be similarly extended for multipartite states also.

Result I: For any pure state $|\psi_{AB}\rangle$, ROC is superadditive.

Proof. For any pure state $|\psi_{AB}\rangle$, we have $C_{\text{ROC}}(|\psi_{AB}\rangle) = C_{l_1}(|\psi_{AB}\rangle)$. Now, we use the superadditivity of l_1 -norm of coherence along with the fact that ROC is always upper bounded by the l_1 -norm of coherence to obtain $C_{\text{ROC}}(|\psi_{AB}\rangle) = C_{l_1}(|\psi_{AB}\rangle) \geq C_{l_1}(\rho_A) + C_{l_1}(\rho_B) \geq C_{\text{ROC}}(\rho_A) + C_{\text{ROC}}(\rho_B)$, thus proving the result. \square

We now show that adding an incoherent ancilla does not change the amount of coherence in a system. This intuitively obvious statement is shown below to hold for arbitrary legitimate coherence measures. In order to prove this, we note that inequality (C2b) has been shown as equivalent [36] to the equality condition that if $\rho = p_1\rho_1 \oplus p_2\rho_2$ for $p_1 + p_2 = 1$, then

$$C(p_1\rho_1 \oplus p_2\rho_2) = p_1C(\rho_1) + p_2C(\rho_2). \quad (2)$$

Result II: For any state ρ_A and any incoherent state σ_B , $C(\rho_A \otimes \sigma_B) = C(\rho_A)$ for any legitimate coherence measure C .

Proof. Let us assume that $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = n \geq 2$. Let $X = \rho_A \otimes \sigma_B$. We can always use permutation matrices to transform it to a matrix in block-diagonal form [37] $= d_1\rho_A \oplus d_2\rho_A \oplus \dots \oplus d_n\rho_A$. As permutations correspond merely to relabelling of the basis vectors, the amount of coherence of a system does not depend on such permutations. Now, from (2) [36], we have, for any legitimate coherence measure C , $C(\rho_A \otimes \sigma_B) = C(d_1\rho_A \oplus d_2\rho_A \oplus \dots \oplus d_n\rho_A) = \sum_{i=1}^n d_i C(\rho_A) = C(\rho_A)$, where the last line follows from the unit trace condition for density matrices. \square

4. Subadditivity of ROC

In this section, we explore the possible subadditivity of ROC. To this end, we introduce the following class of n -qubit states $\rho_{A_1 A_2 A_3, \dots, A_n} = (1+k)(\mathbb{I}/2^n) - k|\psi\rangle\langle\psi|$, where \mathbb{I} is the identity matrix, $0 \leq k \leq 1/(2^n - 1)$ and $|\psi\rangle = (1/2^{n/2})(\sum_{i=1}^{2^n} |i\rangle)$ is the maximally coherent n -qubit state.

Theorem 1. For an arbitrary n -qubit system $A_1 A_2 A_3, \dots, A_n$, the ROC for the family Σ of states $\rho_{A_1 A_2 A_3, \dots, A_n} = (1+k)(\mathbb{I}/2^n) - k|\psi\rangle\langle\psi|$, where $0 \leq k \leq 1/(2^n - 1)$ and $|\psi\rangle = (1/2^{n/2})(\sum_{i=1}^{2^n} |i\rangle)$ is the maximally coherent n -qubit state, satisfying the following subadditive relation:

$$C_{\text{ROC}}(\rho_{A_1 A_2 A_3, \dots, A_n}) \leq \sum_{i=1}^n C_{\text{ROC}}(\rho_{A_i}). \quad (3)$$

Proof. Let $\rho_{A_1 A_2 A_3, \dots, A_n} = (1+k)(\mathbb{I}/2^n) - k|\psi\rangle\langle\psi|$, where $0 \leq k \leq 1/(2^n - 1)$ and $|\psi\rangle$ is the maximally coherent n -qubit state. Now, by using definition of ROC (1), we prepare a convex mixture χ of an arbitrary n -qubit state τ and $\rho_{A_1 A_2 A_3, \dots, A_n}$, that is mathematically expressed as

$$\chi = \frac{(1+k)(\mathbb{I}/2^n) - k|\psi\rangle\langle\psi| + s\tau}{1+s}, \quad (4)$$

where s is $C_{\text{ROC}}(\rho_{A_1 A_2 A_3, \dots, A_n})$. Without any loss of generality, when χ in eq. (4) is expanded in n -qubit computational basis, the diagonal elements are of the form

$$\chi_{ii} = \frac{1 + 2^n s \tau_{ii}}{2^n(1+s)}, \quad (5)$$

whereas the off-diagonal elements are of the form

$$\chi_{ij} = \frac{-k + 2^n s \tau_{ij}}{2^n(1+s)}. \quad (6)$$

For χ in eq. (4) to be an incoherent state, we have to ensure that the off-diagonal elements of χ , described by eq. (6), will be zero. So, by equating eq. (6) to zero, we finally arrive at the following condition:

$$s = \frac{k}{2^n \tau_{ij}}. \quad (7)$$

As per definition of ROC (eq. (1)), $s \in \mathfrak{R}$, where \mathfrak{R} is the set of real numbers, has to be minimised. As $s \in \mathfrak{R}$, clearly, $\tau_{ij} \in \mathfrak{R}$. Now, in the trivial case, s is zero when $\rho_{A_1 A_2 A_3, \dots, A_n}$ is already an incoherent state. In the non-trivial case, s is minimum when τ_{ij} takes the maximum value of k , i.e. $\tau_{ij} = 1/(2^n - 1)$. Hence, after substituting $\tau_{ij} = 1/(2^n - 1)$ in eq. (7), we have

$$s = C_{\text{ROC}}(\rho_{A_1 A_2 A_3, \dots, A_n}) = k \left(1 - \frac{1}{2^n} \right). \quad (8)$$

Now, let us consider the single qubit subsystems

$$\begin{aligned} \rho_{A_i} &= \text{Tr}_{A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n} [\rho_{A_1 A_2 A_3, \dots, A_n}] \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{k}{2} \\ \frac{k}{2} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

in computational basis. For single qubit systems, we know that ROC is equal to its l_1 -norm of coherence for a fixed basis. Hence, for single qubit computational basis, $C_{\text{ROC}}(\rho_{A_i}) = C_{l_1}(\rho_{A_i}) = k$.

Finally, we have

$$\begin{aligned} \Lambda &= C_{\text{ROC}}(\rho_{A_1 A_2 A_3, \dots, A_n}) - \sum_{i=1}^n C_{\text{ROC}}(\rho_{A_i}) \\ &= k \left(1 - \frac{1}{2^n} \right) - nk \\ &= k \left[1 - \left(n + \frac{1}{2^n} \right) \right]. \end{aligned} \quad (9)$$

Clearly, for $n \in \mathbb{Z}^+$ and $0 \leq k \leq 1/(2^n - 1)$, we have $\Lambda \leq 0$. \square

Given any pure state, its ROC is identical with its l_1 -norm of coherence, which is always superadditive. We now turn to the scenario when elements of the set of states Σ mentioned in the previous theorem are mixed with a given pure state $|\phi\rangle$ and investigate what happens to the subadditivity property as we increase the mixing.

To this end, we randomly pick a large number of states $\{\sigma\}$ from the family Σ and mix every such state with a chosen pure state $|\phi\rangle$ with mixing parameter p to obtain a large number of states $\Sigma_p = \{(1-p)\sigma + p|\phi\rangle\langle\phi|\}$. We want to know the probability of any randomly chosen element of this set satisfying the subadditivity condition (3). Clearly, if $p = 0$, this set is a random subset of Σ , and thus all the elements will satisfy the subadditivity condition. In the opposite limit, if $p = 1$, this set consists of only $|\phi\rangle$, i.e. always superadditive for ROC. However, it is the intermediate region which is of interest to us. For simplicity, we confine ourselves to the two-qubit scenario. We consider two different pure states $|\phi\rangle$, one being the maximally coherent state $|\phi_1\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ and the other being the maximally entangled state $|\phi_2\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$. For each of them and every value of the mixing weight p , choosing 10,000 random states from Σ_p according to the Haar measure, we calculate the percentage of states in the set Σ_p which satisfy the subadditivity condition. Figure 1 shows the result. Two properties of this figure are quite interesting. First, the plots are almost identical for two very different sets of pure states $|\phi_i\rangle$ ($i = 1, 2$),

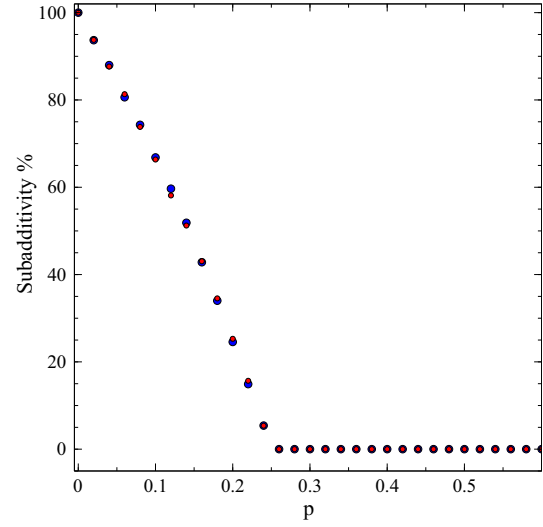


Figure 1. Percentage of randomly chosen two-qubit states from Σ_p which satisfy subadditivity vs. the mixing weight p , where the pure state $|\phi\rangle$ is either the two-qubit maximally coherent state (red dots) or the two-qubit maximally entangled state (blue dots). 1000 randomly generated states are taken for each value of p .

viz. the maximally coherent states (indicated by red points) and the maximally entangled states (indicated by blue points). Secondly, instead of the proportion of states satisfying the subadditivity condition (3) diminishing smoothly as $p \rightarrow 1$, it shows a sudden death at around $p = 0.25$.

5. Ordering of states through different coherence measures

Quantification of any resource through some measure raises the question – what is the operational significance of that particular measure? Indeed, the same resource can be operationally relevant in many different protocols. This naturally leads us to the next question: If the same resource is quantified by different measures motivated by different protocols – then can a state that is ‘bad’ for a particular protocol turn out to be ‘good’ for another protocol utilising the same resource?

For the resource theory of coherence, the central question is: when can one transform a quantum state ρ to σ using incoherent operations? If both input and target states are pure, say $|\psi\rangle$ and $|\chi\rangle$, respectively, a necessary and sufficient condition for such convertibility [38] is given by

$$\vec{c}_\psi \prec \vec{c}_\chi, \quad (10)$$

i.e. the vector \vec{c} corresponding to the input state is majorised by the vector \vec{c} corresponding to the target state, where \vec{c}_ξ for any state $|\xi\rangle$ is the collection of squared moduli of the coefficients of that state when

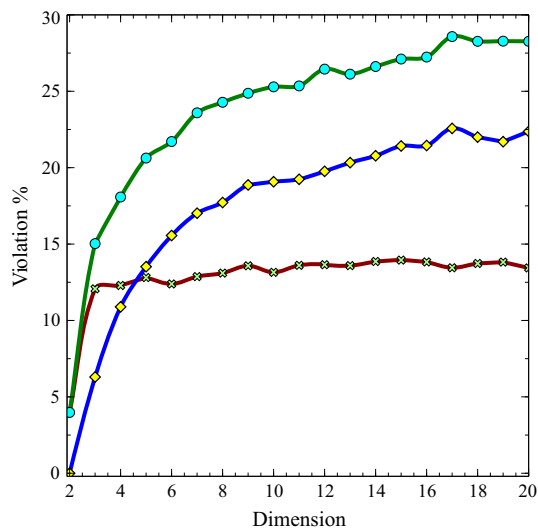


Figure 2. Percentage of pairs of states with different ordering with respect to pairwise chosen coherence measures vs. dimension of states: brown line – l_1 -norm vs. relative entropy of coherence, blue line – l_1 -norm vs. ROC and green line – relative entropy of coherence vs. ROC (taking 10000 randomly chosen pairs of states).

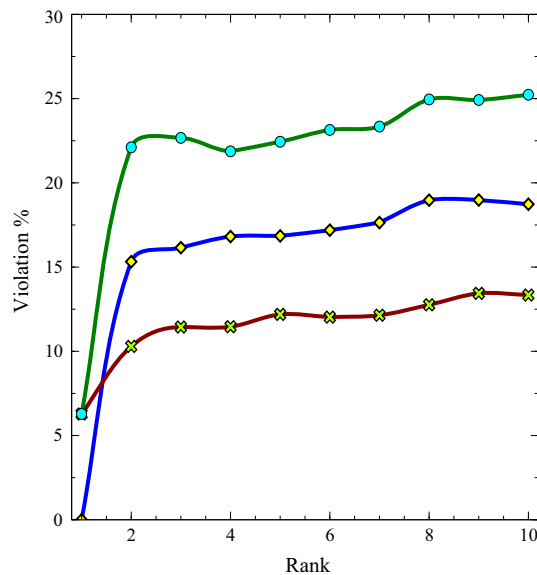


Figure 3. Percentage of pairs of states with different ordering with respect to pairwise chosen coherence measures vs. rank of states: brown line – l_1 -norm vs. relative entropy of coherence, blue line – l_1 -norm vs. ROC and green line – relative entropy of coherence vs. ROC (taking 10000 randomly chosen pairs of states) (dimension $d = 10$).

expanded out on the basis of our choice. Evidently, it is possible to have pairs of pure states for which the collection of coefficients does not majorise each other. This opens the possibility that even for pairs of such pure states, two different coherence measures may give us different ordering. This is indeed confirmed for pure as well as mixed states [39] for C_{rel} and C_{l_1} . In this section, we investigate the statistics of ordering for different coherence measures, viz. C_{rel} , C_{l_1} and C_{ROC} , if random states are chosen from the state space according to the Haar measure. We decided to check the percentage of randomly chosen pairs of states with different ordering with respect to the pairwise chosen coherence measures depending upon the dimension and rank of the chosen states. Why both dimension and rank? The explanation is that for higher-dimensional states, when we generate datasets of random quantum states, we miss out the states of lower ranks which are of measure zero.

From figure 2, it is evident that as the dimension of the quantum state increases, the percentage of ordering violations between ROC and relative entropy measure of coherence (denoted by the green curve) remains greater than that between ROC and l_1 -norm of coherence (denoted by the blue curve) and l_1 -norm and relative entropy of coherence (denoted by the red curve). Moreover, we observe that for dimension $d \leq 5$, the percentage of ordering violation between l_1 -norm and relative entropy of coherence is greater than that between ROC and l_1 -norm of coherence. However, for

dimensions $d > 5$, the percentage of ordering violation between l_1 -norm and ROC is greater than that between relative entropy of coherence and l_1 -norm of coherence.

In figure 3, we observe a similar trend as that in figure 2. Here, as the rank of the quantum state increases, the percentage of ordering violations between ROC and relative entropy measure of coherence is significantly greater than that between ROC and l_1 -norm of coherence and l_1 -norm and relative entropy of coherence. For pure states, i.e. states of rank 1, robustness of coherence is identical to the l_1 norm of coherence. Therefore, there is no ordering violation among them. However, for mixed states, the percentage of ordering violation between l_1 -norm and ROC is greater than that between relative entropy of coherence and l_1 -norm of coherence.

6. Conclusion

We conclude that unlike l_1 -norm or relative entropy of coherence, which are superadditive, ROC can be subadditive for certain classes of states. If we take a mixture of that class of states and pure states, we have found out that beyond a certain range of mixing weight, such mixtures cease to satisfy the subadditive property. We have found that for a pair of randomly generated density matrices, there exists a possibility of ordering

violations corresponding to different legitimate measures of coherence. We welcome further work on implications of subadditivity of ROC for quantum advantage in phase discrimination tasks and quantum information theory in general.

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References

- [1] M A Nielsen and I L Chuang, *Quantum computation and quantum information* (Cambridge University Press, 2010)
- [2] R Horodecki, P Horodecki, M Horodecki and K Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009)
- [3] J Aberg, *Quantifying superposition*, eprint [arXiv:quant-ph/0612146](https://arxiv.org/abs/quant-ph/0612146) (2006)
- [4] T Baumgratz, M Cramer and M B Plenio, *Phys. Rev. Lett.* **113**, 140401 (2014)
- [5] A Winter and D Yang, *Phys. Rev. Lett.* **116**, 120404 (2016)
- [6] A Streltsov, S Rana, P Boes and J Eisert, *Phys. Rev. Lett.* **119**, 140402 (2017)
- [7] A Streltsov, G Adesso and M B Plenio, *Rev. Mod. Phys.* **89**, 041003 (2017)
- [8] E Chitambar and G Gour, *Phys. Rev. Lett.* **117**, 030401 (2016)
- [9] I Chakrabarty, U Kamal Sharma and M K Shukla, *Broadcasting quantum coherence via cloning*, arXiv e-prints (2016)
- [10] T Theurer, N Killoran, D Egloff and M B Plenio, *Phys. Rev. Lett.* **119**, 230401 (2017)
- [11] C Mukhopadhyay, S Das, S Bhattacharya, A S De and U Sen, *Quantum coherence in superposition with non-distinguishable pointers*, arXiv e-prints (2017)
- [12] M T Mitchison, M P Woods, J Prior and M Huber, *New J. Phys.* **17**(11), 115013 (2015)
- [13] J B Brask and N Brunner, *Phys. Rev. E* **92**, 062101 (2015)
- [14] M Lostaglio, D Jennings and T Rudolph, *Nat. Commun.* **6**, 6383 (2015)
- [15] M Hillery, *Phys. Rev. A* **93**, 012111 (2016)
- [16] N Anand and A K Pati, *Coherence and entanglement monogamy in the discrete analogue of analog Grover search*, arXiv e-prints (2016)
- [17] H-L Shi, S-Y Liu, X-H Wang, W-L Yang, Z-Y Yang and H Fan, *Phys. Rev. A* **95**, 032307 (2017)
- [18] A Olaya-Castro, C F Lee, F F Olsen and N F Johnson, *Phys. Rev. B* **78**, 085115 (2008)
- [19] J N Bandyopadhyay, T Paterek and D Kaszlikowski, *Phys. Rev. Lett.* **109**, 110502 (2012)
- [20] M M Wilde, J M McCracken and A Mizel, *Proc. R. Soc. Lond. A* **466**, 1347 (2010)
- [21] A Streltsov, U Singh, H S Dhar, M N Bera and G Adesso, *Phys. Rev. Lett.* **115**, 020403 (2015)
- [22] J K Asbóth, J Calsamiglia and H Ritsch, *Phys. Rev. Lett.* **94**, 173602 (2005)
- [23] Z Xi, Y Li and H Fan, *Sci. Rep.* **5**, 10922 (2015)
- [24] A Streltsov, E Chitambar, S Rana, M N Bera, A Winter and M Lewenstein, *Phys. Rev. Lett.* **116**, 240405 (2016)
- [25] X Qi, T Gao and F Yan, *Coherence concurrence*, arXiv e-prints (2016)
- [26] S Chin, *Conversion of coherence into the generalized entanglement concurrence*, arXiv e-prints (2017)
- [27] H Zhu, Z Ma, Z Cao, S-M Fei and V Vedral, *Phys. Rev. A* **96**, 032316 (2017)
- [28] H Zhu, M Hayashi and L Chen, *Phys. Rev. A* **97**, 022342 (2018)
- [29] D Das, R Sengupta and A Arvind, *Pramana – J. Phys.* **88**: 82 (2017)
- [30] K Korzekwa, M Lostaglio, J Oppenheim and D Jennings, *New J. Phys.* **18**(2), 023045 (2016)
- [31] N A Shah and T Qureshi, *Pramana – J. Phys.* **89**: 80 (2017)
- [32] C Napoli, T R Bromley, M Cianciaruso, M Piani, N Johnston and G Adesso, *Phys. Rev. Lett.* **116**, 150502 (2016)
- [33] M Piani, M Cianciaruso, T R Bromley, C Napoli, N Johnston and G Adesso, *Phys. Rev. A* **93**, 042107 (2016)
- [34] G Vidal and R Tarrach, *Phys. Rev. A* **59**, 141 (1999)
- [35] M Steiner, *Phys. Rev. A* **67**, 054305 (2003)
- [36] X-D Yu, D-J Zhang, G F Xu and D M Tong, *Phys. Rev. A* **94**, 060302 (2016)
- [37] R P Tewarson, *Comput. J.* **10**(3), 300 (1967)
- [38] S Du, Z Bai and Y Guo, *Phys. Rev. A* **91**, 052120 (2015)
- [39] C L Liu, X-D Yu, G F Xu and D M Tong, *Quantum Inf. Process.* **15**(10), 4189 (2016)
- [40] H Zhu, M Hayashi and L Chen, *J. Phys. A* **50**, 475303 (2017)