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q-Deformed oscillator algebra in fermionic and bosonic limits

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Abstract. In this paper, the structure function corresponding to the q-deformed harmonic oscillator algebra is considered, where we construct the Hamiltonian by using creation and annihilation operators. Finally, the problem is investigated by evaluating the partition function of the system in finite- and infinite-dimensional Fock space for both fermionic and bosonic limits. Other thermodynamic properties such as the internal energy and the specific heat of the system are also calculated.

Keywords. Fock space; fermionic limit; bosonic limit; deformed oscillator algebra; thermodynamic properties.

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1. Introduction

Over the last couple of years, quantum groups and q-deformed algebras have been the subject of intense investigation [1–8]. Biedenharn [9], Macfarlane [10] and Arik and Coon [11] have studied the generalised coherent states associated with q-deformed harmonic oscillator commutation relation. Recently, the q-deformed oscillator systems [12–16] have been applied to many areas of research in physics as well as mathematics. Chaichian *et al* [16] used the deformed formalism to describe the generalised Jaynes–Cummings model. Chung [17,18] introduced the Wigner algebras in deformed formalism. Then he has investigated the effects of deformed Wigner algebras on thermodynamics and he derived in another paper the q-deformation of fermion algebra with q being the root of unity. Chaturvedi *et al* [19] suggested the Tamm–Dancoff deformation of the boson oscillator algebra and some of the Hopf algebraic aspects of the Tamm–Dancoff deformation were also discussed in ref. [19]. The difference between fermions and bosons is the mean number of particles occupying, which for fermions and bosons is 1 and infinity, respectively. The intermediate statistics contains two types of statistics called Bose–Einstein statistics and Fermi–Dirac statistics [20–22]. The statistical distribution function interpolates continuously between the Fermi–Dirac and the Bose–Einstein limits. In this paper, we consider the structure function corresponding to the q-deformed harmonic oscillator. Then, we construct the Hamiltonian by using the creation and annihilation operators to calculate the partition function in the finite- and infinite-dimensional Fock space for the fermionic and the bosonic limits. Finally, we obtain the internal energy and the specific heat in q-deformed formalism. The structure of this paper is as follows: Section 2 is a brief introduction of the standard q-harmonic oscillator algebra. Then, we studied the q-boson algebra with q being the root of unity. We investigate the statistical properties and Hamiltonian of the q-deformed harmonic oscillator algebra in §3. Finally, §4 gives conclusion.

2. q-deformed oscillator algebra

The ordinary algebra can be written as [23]

$$aa^{\dagger} \mp a^{\dagger}a = 1, \quad [N, a^{\dagger}] = a^{\dagger}, \quad [N, a] = -a, \tag{1}$$

where the $-$ $(+)$ sign corresponds to the boson (fermion) algebra.

Arik and Coon utilised the following form [11]:

$$aa^{\dagger} - qa^{\dagger}a = 1, \quad [N, a^{\dagger}] = a^{\dagger}, \quad [N, a] = -a, \tag{2}$$

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where q-number is defined as

$$[N] = \frac{q^N - q^{-N}}{q - q^{-1}}. \tag{3}$$

This algebra introduces the interpolation of bosons and fermions. If we choose $q = 1, -1$ the algebra reduces to boson and fermion algebras. The algebra is for q real in infinite-dimensional Fock space. Then, many researchers used the formalism and they introduced this algebra in different forms [9–26]. The new q-deformed oscillator algebra should obey the following conditions:

1. This algebra should have finite- and infinite-dimensional Fock spaces.
2. This algebra should have both fermionic and bosonic limits.
3. This algebra should be well defined when q is real or a complex root of unity.

Now, we construct the q-deformed algebra in finite- and infinite-dimensional Fock space and two bosonic and fermionic limits, and we consider the structure function as [27]

$$f(N) = \frac{1}{2}([N] + [N]^*). \tag{4}$$

We consider $q = \mathrm{e}^{\mathrm{i}\theta}$ and the structure function is written as

$$f(N) = \frac{\sin(N\theta)}{\sin(\theta)}. \tag{5}$$

We suppose $\theta = 0, \pi$ for bosonic limit and fermionic limit, respectively. We have

$$q = \mathrm{e}^{\mathrm{i}\pi/(p-1)}, \quad q^{p-1} = -1. \tag{6}$$

In this case the structure function is given by

$$f(N) = \frac{\sin(N\pi/(p-1))}{\sin(\pi/(p-1))}. \tag{7}$$

If $p = \infty$ and 2 the algebra reduces to the bosonic and fermionic limits, respectively. The matrix representation of this algebra for the fermionic and the bosonic limits is as follows:

$$N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \ldots, p-1,$$

$$a|n\rangle = \sqrt{f(n)}|n-1\rangle = \sqrt{\frac{\sin(n\pi/(p-1))}{\sin(\pi/(p-1))}}|n-1\rangle,$$

$$a^\dagger|n\rangle = \sqrt{f(n+1)}|n+1\rangle = \sqrt{\frac{\sin((n+1)\pi/(p-1))}{\sin(\pi/(p-1))}}|n+1\rangle. \tag{8}$$

3. Thermal properties of q-deformed harmonic oscillator algebra

In this section we are going to obtain the partition function by the Hamiltonian of the problem. The ordinary boson algebra Hamiltonian is

$$H_\mathrm{B} = H_{p=\infty} = \frac{w}{p}(a^\dagger a + a a^\dagger) \tag{9}$$

and the ordinary fermion algebra Hamiltonian is

$$H_\mathrm{F} = H_{p=2} = \frac{w}{2}(a^\dagger a - a a^\dagger). \tag{10}$$

Thus, the Hamiltonian for finite p is written as

$$H = \frac{w}{p}\left[\csc\left(\frac{\pi}{(p-1)}\right)\left[\sin\left(\frac{n\pi}{(p-1)}\right) - \sin\left(\frac{(n+1)\pi}{(p-1)}\right)\right]\right]. \tag{11}$$

The partition function is given by

$$Z_p = \sum_{n=0}^{p-1} \mathrm{e}^{-\frac{\beta w}{p}\left[\csc\left(\frac{\pi}{(p-1)}\right)\left[\sin\left(\frac{n\pi}{(p-1)}\right) - \sin\left(\frac{(n+1)\pi}{(p-1)}\right)\right]\right]}. \tag{12}$$

When $p = 2M$ or $p = 2M+1$, $M = 1, 2, \ldots$, we have

$$Z_{2M} = \sum_{n=0}^{2M-1} \mathrm{e}^{-\frac{\beta w}{2M}\left[\csc\left(\frac{\pi}{(2M-1)}\right)\left[\sin\left(\frac{n\pi}{(2M-1)}\right) - \sin\left(\frac{(n+1)\pi}{(2M-1)}\right)\right]\right]},$$

$$Z_{2M+1} = \sum_{n=0}^{2M} \mathrm{e}^{-\frac{\beta w}{2M+1}\left[\csc\left(\frac{\pi}{(2M)}\right)\left[\sin\left(\frac{n\pi}{(2M)}\right) - \sin\left(\frac{(n+1)\pi}{(2M)}\right)\right]\right]}. \tag{13}$$

For the first few p, we have

$$Z_3 = 2\cosh\left[\frac{\beta w}{3}\right] + \mathrm{e}^{-\beta w/3},$$

$$Z_4 = 2\cosh\left[\frac{\beta w}{4}\right] + 1 + \mathrm{e}^{-\beta w/4},$$

$$Z_5 = 2\cosh\left[\frac{\beta w}{5}\right] + \mathrm{e}^{-\beta w/5} + 2\cosh\left[\frac{\beta w}{5}\left[1 - \sqrt{2}\right]\right],$$

$$Z_6 = 1 + 3\cosh\left[\frac{\beta w}{6}\right] + 2\cosh\left[\frac{\beta w}{12}\left(-1 + \sqrt{5}\right)\right] - \sinh\left[\frac{\beta w}{6}\right],$$

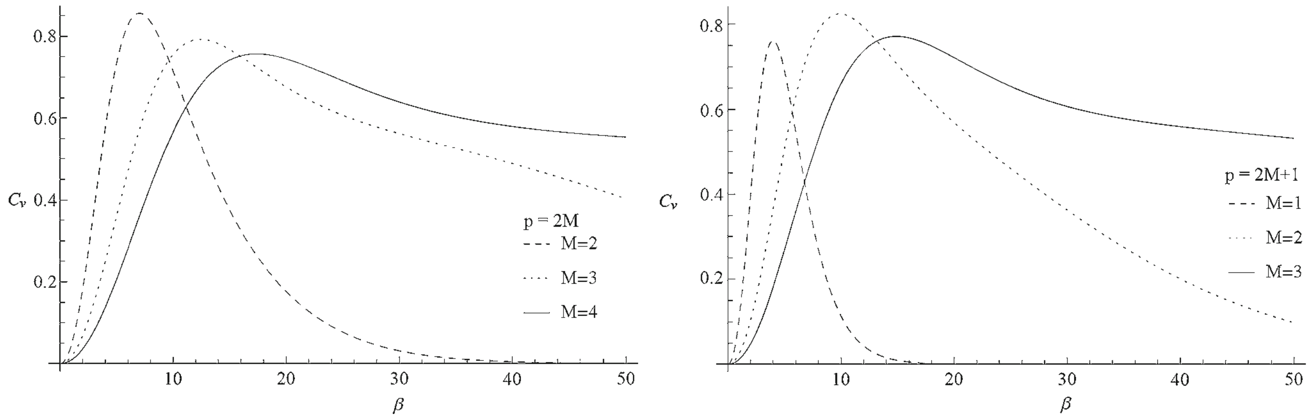

Figure 1. The specific heat vs. β in both cases of different values of the parameter $p = 2M, 2M + 1$.

$$Z_7 = 3\cosh\left[\frac{\beta w}{7}\right] + 2\cosh\left[\frac{\beta w}{7}\left[-2+\sqrt{3}\right]\right] + \cosh\left[\frac{\beta w}{7}\left[-1+\sqrt{3}\right]\right] - \sinh\left[\frac{\beta w}{7}\right]. \quad (14)$$

The internal energy is [28–33]

$$U_{2M} = \sum_{n=0}^{2M-1}\left(\frac{w e^{-\frac{\beta w}{2M}\left[\csc\left(\frac{\pi}{(2M-1)}\right)\left[\sin\left(\frac{n\pi}{(2M-1)}\right)-\sin\left(\frac{(n+1)\pi}{(2M-1)}\right)\right]\right]}}{2M\left(e^{-\frac{\beta w}{2M}\left[\csc\left(\frac{\pi}{(2M-1)}\right)\left[\sin\left(\frac{n\pi}{(2M-1)}\right)-\sin\left(\frac{(n+1)\pi}{(2M-1)}\right)\right]\right]}\right)}\right) \times \left[\csc\left(\frac{\pi}{(2M-1)}\right)\left[\sin\left(\frac{n\pi}{(2M-1)}\right) - \sin\left(\frac{(n+1)\pi}{(2M-1)}\right)\right]\right],$$

$$U_{2M+1} = \sum_{n=0}^{2M}\left(\frac{\left(\frac{w e^{-\frac{\beta w}{2M+1}\left[\csc\left(\frac{\pi}{(2M)}\right)\left[\sin\left(\frac{n\pi}{(2M)}\right)-\sin\left(\frac{(n+1)\pi}{(2M)}\right)\right]\right]}}{2M+1}\left[\csc\left(\frac{\pi}{2M}\right)\left[\sin\left(\frac{n\pi}{2M}\right) - \sin\left(\frac{(n+1)\pi}{2M}\right)\right]\right]\right)}{e^{-\frac{\beta w}{2M+1}\left[\csc\left(\frac{\pi}{2M}\right)\left[\sin\left(\frac{n\pi}{2M}\right)-\sin\left(\frac{(n+1)\pi}{2M}\right)\right]\right]}}\right). \quad (15)$$

The specific heat in the two limits for the set ($p = 2M$, $p = 2M + 1$) is shown in figure 1.

4. Conclusion

In this paper, we considered the structure function corresponding to a q-deformed harmonic oscillator. We first discussed the representation of new deformed algebra. Using this, we found the deformed boson algebra and fermion algebra that are called intermediate statistics. We constructed the Hamiltonian interpolating bosonic and fermionic limits, and then we obtained partition function, internal energy and specific heat.

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