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Multiple types of exact solutions and conservation laws of new coupled (2 + 1)-dimensional Zakharov–Kuznetsov system with time-dependent coefficients

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Abstract. This paper investigates the new coupled (2 + 1)-dimensional Zakharov–Kuznetsov (ZK) system with time-dependent coefficients for multiple types of exact solutions by using the Lie symmetry transformation method. Similarity transformation reduces the system of equations into ordinary differential equations and further, these are solved for solutions having bright, dark and singular solitons as well as periodic waves. Also, the solutions appeared in terms of time-dependent coefficient $\beta(t)$ and analysed graphically to show the effect of this arbitrary function. It is proved that the given system is nonlinear self-adjoint, and some conservation laws are obtained by applying the new conservation theorem.

Keywords. Lie's infinitesimals criterion; exact solutions; new coupled (2 + 1)-dimensional Zakharov–Kuznetsov system; conservation laws.

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1. Introduction

The nonlinear physical phenomena can be disseminated by solving nonlinear partial differential equations (PDEs). The evolution profile of solutions for such equations plays a key role in mathematical, physical and engineering sciences. Thus, it is required to develop an algorithm to find a variety of solutions. The present work addresses to obtain multiple types of exact solutions for a system of equations by the symmetry transformation method [1–6]. This method reduces the number of independent variables by one at each step. This enables us to reduce the nonlinear PDEs to nonlinear ordinary differential equations (ODEs) which can be solved for exact solutions [7]. Further, the Lie symmetries are used to construct conservation laws [8–15]. The present report describes one such nonlinearly evolved $(2+1)$-dimensional new coupled Zakharov–Kuznetsov (ncZK) system with time-dependent coefficients as follows:

$$\begin{aligned}
\Xi_1 &\equiv u_t + \alpha(t)(uv)_x + \gamma(t)(vw)_x \\
&\quad + \beta(t)(u_{xx} + u_{yy})_x = 0, \\
\Xi_2 &\equiv v_t + \lambda(t)(uw)_x + \beta(t)(v_{xx} + v_{yy})_x = 0, \\
\Xi_3 &\equiv w_t + \lambda(t)(uv)_x + \beta(t)(w_{xx} + w_{yy})_x = 0, \quad (1.1)
\end{aligned}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\lambda(t)$ are arbitrary functions of t.

The ncZK system models dust-acoustic solitary waves evolved in magnetised dusty plasmas, nonlinear ion-acoustic waves and hot isothermal electrons [16–21]. With constant coefficients, system (1.1) is solved successfully by Elboree [22], Wei and Tang [23], Khalique [24] and Fahmy [25] for different types of exact solutions. System (1.1) with time-dependent coefficients has not been investigated yet by the symmetry transformation method for exact solutions as well as for conservation laws. Therefore, the main motive of the present paper is to investigate system (1.1) for exact

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solutions and conservation laws via the symmetry transformation method.

The paper is structured as follows. Section 2 presents the stepwise procedure for determining the Lie infinitesimal symmetries, symmetry groups and optimal systems. Further, in §3, similarity solutions and reduced ODEs are obtained corresponding to vector fields of optimal system, and reduced ODEs are analysed for getting multiple types of exact solutions. Then, some of the solutions are represented graphically. In §4, the nonlinear self-adjointness of the system is proved and the new conservation theorem is employed to determine the non-trivial conservation laws.

2. Lie symmetry transformations

This section presents the Lie symmetry transformation criterion to find exact solutions for system (1.1). For Lie infinitesimal symmetries, consider an infinitesimal generator of the following form [1]:

$$\Gamma = \xi_1\frac{\partial}{\partial t} + \xi_2\frac{\partial}{\partial x} + \xi_3\frac{\partial}{\partial y} + \eta_1\frac{\partial}{\partial u} + \eta_2\frac{\partial}{\partial v} + \eta_3\frac{\partial}{\partial w}, \tag{2.1}$$

where ξ_1, ξ_2, ξ_3, η_1, η_2 and η_3 are functions of (t, x, y, u, v, w). The third-order prolongation $\mathrm{Pr}^3\Gamma$ of Γ for system (1.1) is given as follows:

$$\begin{aligned}\mathrm{Pr}^3\Gamma = \Gamma &+ \eta_1^t\frac{\partial}{\partial u_t} + \eta_1^x\frac{\partial}{\partial u_x} + \eta_1^{xxx}\frac{\partial}{\partial u_{xxx}} \\ &+\eta_1^{xyy}\frac{\partial}{\partial u_{xyy}} + \eta_2^t\frac{\partial}{\partial u_t} + \eta_2^x\frac{\partial}{\partial v_x} \\ &+\eta_2^{xxx}\frac{\partial}{\partial v_{xxx}} + \eta_2^{xyy}\frac{\partial}{\partial v_{xyy}} + \eta_3^t\frac{\partial}{\partial u_t} \\ &+\eta_3^x\frac{\partial}{\partial w_x} + \eta_3^{xxx}\frac{\partial}{\partial w_{xxx}} + \eta_3^{xyy}\frac{\partial}{\partial w_{xyy}},\end{aligned} \tag{2.2}$$

where $\eta_1^t, \eta_2^t, \eta_3^t, \eta_1^x, \eta_2^x, \eta_3^x, \eta_1^{xxx}, \eta_2^{xxx}, \eta_3^{xxx}, \eta_1^{xyy}, \eta_2^{xyy}$ and η_3^{xyy} are the extended infinitesimals [1]. Equation (2.1) represents the Lie point symmetry of the ncZK system if the following conditions hold:

$$\begin{aligned}&\mathrm{Pr}^3\Gamma(\Xi_1)|_{\Xi_1=0,\Xi_2=0,\Xi_3=0} = 0,\\ &\mathrm{Pr}^3\Gamma(\Xi_2)|_{\Xi_1=0,\Xi_2=0,\Xi_3=0} = 0,\\ &\mathrm{Pr}^3\Gamma(\Xi_3)|_{\Xi_1=0,\Xi_2=0,\Xi_3=0} = 0.\end{aligned} \tag{2.3}$$

The following symmetry equations are obtained from eq. (2.3):

$$\begin{aligned}&\eta_1^t + \alpha'(t)\xi_1(uv)_x + \alpha(t)\\ &\quad\times(u\eta_2^x + \eta_1 v_x + \eta_1^x v + u_x\eta_2)\\ &\quad+\gamma'(t)\xi_1(vw)_x + \gamma(t)\\ &\quad\times(v\eta_3^x + \eta_2 w_x + \eta_2^x w + v_x\eta_3)\\ &\quad+\beta'(t)\xi_1 u_{xxx} + \beta(t)\eta_1^{xxx}\\ &\quad+\beta'(t)\xi_1 u_{xyy} + \beta(t)\eta_1^{xyy} = 0,\\ &\eta_2^t + \lambda'(t)\xi_1(uw)_x + \lambda(t)\\ &\quad\times(u\eta_3^x + \eta_1 w_x + \eta_1^x w + u_x\eta_3)\\ &\quad+\beta'(t)\xi_1 v_{xxx} + \beta(t)\eta_2^{xxx}\\ &\quad+\beta'(t)\xi_1 v_{xyy} + \beta(t)\eta_2^{xyy} = 0,\\ &\eta_3^t + \lambda'(t)\xi_1(uv)_x + \lambda(t)\\ &\quad\times(u\eta_2^x + \eta_1 v_x + \eta_1^x v + u_x\eta_2)\\ &\quad+\beta'(t)\xi_1 w_{xxx} + \beta(t)\eta_3^{xxx}\\ &\quad+\beta'(t)\xi_1 w_{xyy} + \beta(t)\eta_3^{xyy} = 0.\end{aligned} \tag{2.4}$$

Using the extended infinitesimals in eq. (2.4) and by equating the coefficients of like derivatives of u, v and w, we have obtained the over-determined system of linear PDEs and the solution of over-determining equations reads as follows:

$$\begin{aligned}&\xi_1 = \frac{1}{\beta(t)}\left(3a_1\int\beta(t)\,\mathrm{d}t + a_4\right),\\ &\xi_2 = a_1x + a_3,\;\; \xi_3 = a_1y + a_2,\\ &\eta_1 = a_5u,\;\; \eta_2 = a_5v,\;\; \eta_3 = a_5w,\end{aligned} \tag{2.5}$$

where a_1, a_2, a_3, a_4, a_5 are arbitrary constants and the coefficients $\alpha(t), \beta(t), \gamma(t), \lambda(t)$ follow the subsequent conditions:

$$\begin{aligned}&\beta(t)\gamma(t)(2a_1 + a_5) + (-\beta_t\gamma(t) + \beta(t)\gamma'(t))\xi_1 = 0,\\ &\beta(t)\alpha(t)(2a_1 + a_5) + (-\beta_t\alpha(t) + \beta(t)\alpha'(t))\xi_1 = 0,\\ &\beta(t)\lambda(t)(2a_1 + a_5) + (-\beta_t\lambda(t) + \beta(t)\lambda'(t))\xi_1 = 0.\end{aligned} \tag{2.6}$$

From eq. (2.5), the one-dimensional Lie algebra [1,11] is generated by the following vector fields:

$$\begin{aligned}&\Gamma_1 = \frac{1}{\beta(t)}\partial t,\quad \Gamma_2 = \partial x,\quad \Gamma_3 = \partial y,\\ &\Gamma_4 = u\partial u + v\partial v + w\partial w,\\ &\Gamma_5 = x\partial x + y\partial y + \frac{3\int\beta(t)\,\mathrm{d}t}{\beta(t)}\partial t.\end{aligned} \tag{2.7}$$

These vector fields generate one-parameter groups $G(\epsilon)$:

$$G_1: (x, t, y, u, v, w) \to \left(x, t + \frac{\epsilon}{\beta(t)}, y, u, v, w\right),$$
$$G_2: (x, t, y, u, v, w) \to (x + \epsilon, t, y, u, v, w),$$
$$G_3: (x, t, y, u, v, w) \to (x, t, y + \epsilon, u, v, w),$$
$$G_4: (x, t, y, u, v, w) \to (x, t, y, u\mathrm{e}^{\epsilon}, v\mathrm{e}^{\epsilon}, w\mathrm{e}^{\epsilon}),$$
$$G_5: (x, t, y, u, v, w) \to \left(x\mathrm{e}^{\epsilon}, t + \epsilon\frac{3\int \beta(t)\,\mathrm{d}t}{\beta(t)}, y\mathrm{e}^{\epsilon}, u, v, w\right), \tag{2.8}$$

where various G_i ($i = 1, 2, \ldots, 5$) represent the symmetry groups and we can state the following theorem that leads to the optimal system [1,11].

Theorem 2.1. *If* $u = P(x, t, y), v = Q(x, t, y), w = R(x, t, y)$ *represents the solution of system* (1.1), *so are the functions*

$$(u^{(1)}, v^{(1)}, w^{(1)}) = \left(P\left(x, t - \frac{\epsilon}{\beta(t)}, y\right), Q\left(x, t - \frac{\epsilon}{\beta(t)}, y\right), \quad R\left(x, t - \frac{\epsilon}{\beta(t)}, y\right)\right),$$
$$(u^{(2)}, v^{(2)}, w^{(2)}) = (P(x - \epsilon, t, y), Q(x - \epsilon, t, y), R(x - \epsilon, t, y)),$$
$$(u^{(3)}, v^{(3)}, w^{(3)}) = (P(x, t, y - \epsilon), Q(x, t, y - \epsilon), R(x, t, y - \epsilon)),$$
$$(u^{(4)}, v^{(4)}, w^{(4)}) = (\mathrm{e}^{\epsilon} P(x, t, y), \mathrm{e}^{\epsilon} Q(x, t, y), \mathrm{e}^{\epsilon} R(x, t, y)),$$
$$(u^{(5)}, v^{(5)}, w^{(5)}) = \left(P\left(x\mathrm{e}^{-\epsilon}, t - \epsilon\frac{3\int \beta(t)\,\mathrm{d}t}{\beta(t)}, y\mathrm{e}^{-\epsilon}\right), Q\left(x\mathrm{e}^{-\epsilon}, t - \epsilon\frac{3\int \beta(t)\,\mathrm{d}t}{\beta(t)}, y\mathrm{e}^{-\epsilon}\right), R\left(x\mathrm{e}^{-\epsilon}, t - \epsilon\frac{3\int \beta(t)\,\mathrm{d}t}{\beta(t)}, y\mathrm{e}^{-\epsilon}\right)\right). \tag{2.9}$$

It describes that for each subgroup of symmetry groups G_i ($i = 1, 2, \ldots, 5$), there will be a family of group-invariant solutions and in the present case there are an infinite number of such subgroups. Hence, it is not feasible to enlist all the possible group-invariant solutions for ncZK system. The group-invariant solution introduces the concept of an optimal system.

The optimal system is obtained using the adjoint representation and is given in terms of the Lie series as follows:

$$\mathrm{Ad}(\exp(\epsilon\Gamma_i))\Gamma_j = \Gamma_j - \epsilon[\Gamma_i, \Gamma_j] + \frac{\epsilon^2}{2}[\Gamma_i, [\Gamma_i, \Gamma_j]] + \cdots, \tag{2.10}$$

Table 1. Adjoint table.

Ad	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	Γ_1	Γ_2	Γ_3	Γ_4	$\Gamma_5 - 3\epsilon\Gamma_1$
Γ_2	Γ_1	Γ_2	Γ_3	Γ_4	$\Gamma_5 - \epsilon\Gamma_2$
Γ_3	Γ_1	Γ_2	Γ_3	Γ_4	$\Gamma_5 - \epsilon\Gamma_3$
Γ_4	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_5	$\Gamma_1\mathrm{e}^{3\epsilon}$	$\Gamma_2\mathrm{e}^{\epsilon}$	$\Gamma_3\mathrm{e}^{\epsilon}$	Γ_4	Γ_5

where ϵ is the real parameter. $[\Gamma_i, \Gamma_j] = \Gamma_i\Gamma_j - \Gamma_j\Gamma_i$ is defined as the Lie bracket. The non-zero Lie brackets from Lie algebra (2.7) are obtained as follows:

$$[\Gamma_1, \Gamma_5] = -[\Gamma_5, \Gamma_1] = 3\Gamma_1,$$
$$[\Gamma_2, \Gamma_5] = -[\Gamma_5, \Gamma_2] = \Gamma_2,$$
$$[\Gamma_3, \Gamma_5] = -[\Gamma_5, \Gamma_3] = \Gamma_3. \tag{2.11}$$

The Lie series (2.10) and Lie commutation relation (2.11) further help to write down the adjoint table (table 1) of the ncZK system.

Then, the adjoint table is used to construct the optimal system generated by the following vector fields:

(i) $\Gamma_5 + \rho\Gamma_4$,
(ii) $\Gamma_4 + \mu\Gamma_3 + \theta\Gamma_2 + \nu\Gamma_1$,
(iii) $\Gamma_3 + r\Gamma_2 + s\Gamma_1$,
(iv) $\Gamma_2 + p\Gamma_1$,
(v) Γ_1,

where $\rho, \mu, \theta, \nu, r, s$ and p are arbitrary constants. These vector fields are used in the next section for similarity reductions.

3. Similarity reductions and exact solutions

For similarity reductions with respect to the vector fields described in the optimal system, the following characteristic equations are used:

$$\frac{\mathrm{d}t}{\xi_1} = \frac{\mathrm{d}x}{\xi_2} = \frac{\mathrm{d}y}{\xi_3} = \frac{\mathrm{d}u}{\eta_1} = \frac{\mathrm{d}v}{\eta_2} = \frac{\mathrm{d}w}{\eta_3}. \tag{3.1}$$

3.1 *Vector field* $\Gamma_5 + \rho\Gamma_4$

The characteristic equation (3.1) for this vector field $\Gamma_5 + \rho\Gamma_4$ gives the following invariants:

$$\zeta_1 = \frac{x}{(\int \beta(t)\,\mathrm{d}t)^{1/3}}, \quad \zeta_2 = \frac{y}{(\int \beta(t)\,\mathrm{d}t)^{1/3}},$$
$$u(x, t, y) = F(\zeta_1, \zeta_2)\left(\int \beta(t)\,\mathrm{d}t\right)^{\rho/3},$$

$$v(x,t,y) = G(\zeta_1,\zeta_2)\left(\int \beta(t)\,\mathrm{d}t\right)^{\rho/3},$$

$$w(x,t,y) = H(\zeta_1,\zeta_2)\left(\int \beta(t)\,\mathrm{d}t\right)^{\rho/3}. \quad (3.2)$$

Using eq. (2.6), the time-dependent coefficients can be written as follows:

$$\alpha(t) = p_1\beta(t)\left(\int \beta(t)\,\mathrm{d}t\right)^{-(2+\rho)/3},$$

$$\gamma(t) = p_2\beta(t)\left(\int \beta(t)\,\mathrm{d}t\right)^{-(2+\rho)/3},$$

$$\lambda(t) = p_3\beta(t)\left(\int \beta(t)\,\mathrm{d}t\right)^{-(2+\rho)/3}, \quad (3.3)$$

where p_1, p_2 and p_3 are arbitrary constants. Using the invariants and coefficient functions given in eqs (3.2) and (3.3), the ncZK system is transformed into the following reduced PDEs:

$$F_{\zeta_1}\zeta_1+F_{\zeta_2}\zeta_2-F\rho-3p_1FG_{\zeta_1}-3p_1F_{\zeta_1}G-3p_2GH_{\zeta_1}$$
$$-3p_2G_{\zeta_1}H-3F_{\zeta_1\zeta_1\zeta_1}-3F_{\zeta_1\zeta_2\zeta_2}=0,$$
$$G_{\zeta_1}\zeta_1+G_{\zeta_2}\zeta_2-G\rho-3p_3HF_{\zeta_1}-3p_3H_{\zeta_1}F$$
$$-3G_{\zeta_1\zeta_1\zeta_1}-3G_{\zeta_1\zeta_2\zeta_2}=0,$$
$$H_{\zeta_1}\zeta_1+H_{\zeta_2}\zeta_2-H\rho-3p_3FG_{\zeta_1}-3p_3F_{\zeta_1}G-3H_{\zeta_1\zeta_1\zeta_1}$$
$$-3H_{\zeta_1\zeta_2\zeta_2}=0. \quad (3.4)$$

We are interested in non-trivial solutions for the given system, but the above reduced PDEs possess only trivial solutions. Hence, this case is not physically important.

3.2 *Vector field* $\Gamma_4+\mu\Gamma_3+\theta\Gamma_2+\nu\Gamma_1$

In this case, the invariants and the corresponding variable coefficients are obtained as follows:

$$\zeta_1 = -\frac{\theta}{\nu}\int\beta(t)\,\mathrm{d}t+x,$$

$$\zeta_2 = -\frac{\mu}{\nu}\int\beta(t)\,\mathrm{d}t+y,$$

$$u(x,t,y) = F(\zeta_1,\zeta_2)\mathrm{e}^{1/\nu\int\beta(t)\,\mathrm{d}t},$$

$$v(x,t,y) = G(\zeta_1,\zeta_2)\mathrm{e}^{1/\nu\int\beta(t)\,\mathrm{d}t},$$

$$w(x,t,y) = H(\zeta_1,\zeta_2)\mathrm{e}^{1/\nu\int\beta(t)\,\mathrm{d}t},$$

$$\alpha(t) = q_1\beta(t)\mathrm{e}^{-1/\nu\int\beta(t)\,\mathrm{d}t},$$

$$\gamma(t) = q_2\beta(t)\mathrm{e}^{-1/\nu\int\beta(t)\,\mathrm{d}t},$$

$$\lambda(t) = q_3\beta(t)\mathrm{e}^{-1/\nu\int\beta(t)\,\mathrm{d}t}, \quad (3.5)$$

where q_1, q_2 and q_3 are arbitrary constants. Invariants (3.5) yield the following reduced PDEs for the ncZK system:

$$F_{\zeta_1}\theta-F_{\zeta_2}\mu+F+q_1\nu FG_{\zeta_1}+q_1\nu F_{\zeta_1}G+q_2\nu GH_{\zeta_1}$$
$$+q_2\nu G_{\zeta_1}H+\nu F_{\zeta_1\zeta_1\zeta_1}+\nu F_{\zeta_1\zeta_2\zeta_2}=0,$$
$$G_{\zeta_1}\theta+G_{\zeta_2}\mu-G-q_3\nu HF_{\zeta_1}-q_3\nu H_{\zeta_1}F$$
$$-\nu G_{\zeta_1\zeta_1\zeta_1}-\nu G_{\zeta_1\zeta_2\zeta_2}=0,$$
$$-H_{\zeta_1}\theta-H_{\zeta_2}\mu+H+q_3\nu FG_{\zeta_1}+q_3\nu F_{\zeta_1}G$$
$$+\nu H_{\zeta_1\zeta_1\zeta_1}+\nu H_{\zeta_1\zeta_2\zeta_2}=0. \quad (3.6)$$

By applying the Lie symmetry transformation algorithm on the reduced PDEs, the following infinitesimals are obtained:

$$\xi^1 = q_5,\quad \xi^2 = q_4,\quad \eta^1=\eta^2=\eta^3=0, \quad (3.7)$$

where q_4 and q_5 are arbitrary constants. The new invariants are obtained using the characteristic equation

$$\frac{\mathrm{d}\zeta_1}{\xi^1}=\frac{\mathrm{d}\zeta_2}{\xi^2}=\frac{\mathrm{d}F}{\eta^1}=\frac{\mathrm{d}G}{\eta^2}=\frac{\mathrm{d}H}{\eta^3}$$

as follows:

$$\zeta = q_4\zeta_1-q_5\zeta_2,\quad F(\zeta_1,\zeta_2)=f(\zeta),$$

$$G(\zeta_1,\zeta_2)=g(\zeta),\quad H(\zeta_1,\zeta_2)=h(\zeta). \quad (3.8)$$

Finally, we obtained the reduced ODEs for the ncZK system as follows:

$$f_\zeta q_4\theta+f_\zeta q_5\mu+f+q_1\nu fg_\zeta q_4+q_1\nu f_\zeta q_4 g+q_2\nu gh_\zeta q_4$$
$$+q_2\nu g_\zeta q_4 h+\nu f_{\zeta\zeta\zeta}q_4^3+\nu f_{\zeta\zeta\zeta}q_5^2q_4=0,$$
$$g_\zeta q_4\theta-g_\zeta q_5\mu-g-q_3\nu hf_\zeta q_4-q_3\nu h_\zeta c_4 f$$
$$-\nu g_{\zeta\zeta\zeta}q_4^3-\nu g_{\zeta\zeta\zeta}q_5^2q_4=0,$$
$$-h_\zeta q_4\theta+h_\zeta q_5\mu+h+q_3\nu fg_\zeta q_4$$
$$+q_3\nu f_\zeta c_4 g+\nu h_{\zeta\zeta\zeta}q_4^3+\nu h_{\zeta\zeta\zeta}q_5^2q_4=0. \quad (3.9)$$

The solution of system (3.9) in the form of power series is considered as follows:

$$f(\zeta)=\sum_{n=0}^{\infty}D_n\zeta^n,\ g(\zeta)=\sum_{n=0}^{\infty}E_n\zeta^n,\ h(\zeta)=\sum_{n=0}^{\infty}K_n\zeta^n, \quad (3.10)$$

where D_n, E_n and K_n are unknown coefficients and are to be determined later. From the substitution of (3.10) into the reduced ODEs (3.9), we obtained the following recurrence relations:

$$D_{n+3} = \frac{-1}{(\nu q_4^3 + \nu q_5^2 q_4)(n+1)(n+2)(n+3)} \times \Bigg[(q_4\theta + q_5\mu)(n+1)D_{n+1} + D_n + \nu q_1 q_4 \left(\sum_{l=0}^{n} (n-l+1)(D_l E_{n-l+1} + E_l D_{n-l+1}) \right) + \nu q_2 q_4 \left(\sum_{l=0}^{n} (n-l+1)(E_l K_{n-l+1} + K_l E_{n-l+1}) \right) \Bigg],$$

$$E_{n+3} = \frac{1}{(\nu q_4^3 + \nu q_5^2 q_4)(n+1)(n+2)(n+3)} \times \Bigg[(q_4\theta - q_5\mu)(n+1)E_{n+1} - E_n - \nu q_3 q_4 \left(\sum_{l=0}^{n} (n-l+1)(K_l D_{n-l+1} + D_l K_{n-l+1}) \right) \Bigg],$$

$$K_{n+3} = \frac{1}{(\nu q_4^3 + \nu q_5^2 q_4)(n+1)(n+2)(n+3)} \times \Bigg[(q_4\theta - q_5\mu)(n+1)K_{n+1} - K_n - \nu q_3 q_4 \left(\sum_{l=0}^{n} (n-l+1)(D_l E_{n-l+1} + E_l D_{n-l+1}) \right) \Bigg], \tag{3.11}$$

where $D_0, D_1, D_2, E_0, E_1, E_2, K_0, K_1, K_2$ are arbitrary constants, and

$$D_3 = \frac{-1}{6(\nu q_4^3 + \nu q_5^2 q_4)}[(q_4\theta + q_5\mu)D_1 + D_0 + \nu q_1 q_4(D_0E_1 + E_0D_1) + \nu q_2 q_4(E_0K_1 + K_0E_1)],$$

$$E_3 = \frac{1}{6(\nu q_4^3 + \nu q_5^2 q_4)}[(q_4\theta - q_5\mu)E_1 - E_0 + \nu q_3 q_4(K_0D_1 + D_0K_1)],$$

$$K_3 = \frac{1}{6(\nu q_4^3 + \nu q_5^2 q_4)}[(q_4\theta - q_5\mu)K_1 - K_0 + \nu q_3 q_4(D_0E_1 + E_0D_1)].$$

In this case, the explicit solutions are obtained in the following form:

$$u(x,t,y) = \mathrm{e}^{1/\nu \int \beta(t)\,\mathrm{d}t} \times \Bigg(D_0 + D_1\zeta + D_2\zeta^2 - \frac{1}{6(\nu q_4^3 + \nu q_5^2 q_4)} \times [(q_4\theta + q_5\mu)D_1 + D_0 + \nu q_1 q_4(D_0E_1 + E_0D_1) + \nu q_2 q_4(E_0K_1 + K_0E_1)]\zeta^3 - \sum_{n=1}^{\infty} \frac{1}{(\nu q_4^3 + \nu q_5^2 q_4)(n+1)(n+2)(n+3)} \times \Bigg[(q_4\theta + q_5\mu)(n+1)D_{n+1} + D_n + \nu q_1 q_4 \left(\sum_{l=0}^{n} (n-l+1)(D_l E_{n-l+1} + E_l D_{n-l+1}) \right) + \nu q_2 q_4 \left(\sum_{l=0}^{n} (n-l+1)(E_l K_{n-l+1} + K_l E_{n-l+1}) \right) \Bigg] \zeta^{n+3} \Bigg),$$

$$v(x,t,y) = \mathrm{e}^{1/\nu \int \beta(t)\,\mathrm{d}t} \times \Bigg(E_0 + E_1\zeta + E_2\zeta^2 + \frac{1}{6(\nu q_4^3 + \nu q_5^2 q_4)} \times [(q_4\theta - q_5\mu)E_1 - E_0 + \nu q_3 q_4(K_0D_1 + D_0K_1)]\zeta^3 + \sum_{n=1}^{\infty} \frac{1}{(\nu q_4^3 + \nu q_5^2 q_4)(n+1)(n+2)(n+3)} \times \Bigg[(q_4\theta - q_5\mu)(n+1)E_{n+1} - E_n + \nu q_3 q_4 \left(\sum_{l=0}^{n} (n-l+1)(K_l D_{n-l+1} + D_l K_{n-l+1}) \right) \Bigg] \zeta^{n+3} \Bigg),$$

$$w(x,t,y) = \mathrm{e}^{1/\nu \int \beta(t)\,\mathrm{d}t} \times \Bigg(K_0 + K_1\zeta + K_2\zeta^2 + \frac{1}{6(\nu q_4^3 + \nu q_5^2 q_4)} \times [(q_4\theta - q_5\mu)K_1 - K_0 + \nu q_3 q_4(D_0E_1 + E_0D_1)]\zeta^3 + \sum_{n=1}^{\infty} \frac{1}{(\nu q_4^3 + \nu q_5^2 q_4)(n+1)(n+2)(n+3)} \times \Bigg[(q_4\theta - q_5\mu)(n+1)K_{n+1} - K_n$$

$$+\nu q_3 q_4\left(\sum_{l=0}^{n}(n-l+1)(D_l E_{n-l+1} + E_l D_{n-l+1})\right)\Bigg]\zeta^{n+3}\Bigg), \tag{3.12}$$

where

$$\zeta = q_4\left(-\frac{\theta}{\nu}\int\beta(t)\,\mathrm{d}t + x\right) - q_5\left(-\frac{\mu}{\nu}\int\beta(t)\,\mathrm{d}t + y\right).$$

3.3 *Vector field* $\Gamma_3 + r\Gamma_2 + s\Gamma_1$

The invariants, coefficient functions and reduced PDEs are found to be as follows:

$$\zeta_1 = -\frac{r}{s}\int\beta(t)\,\mathrm{d}t + x, \quad \zeta_2 = -\frac{1}{s}\int\beta(t)\,\mathrm{d}t + y,$$
$$u(x,t,y) = F(\zeta_1,\zeta_2), \quad v(x,t,y) = G(\zeta_1,\zeta_2),$$
$$w(x,t,y) = H(\zeta_1,\zeta_2), \quad \alpha(t) = c_1\beta(t),$$
$$\gamma(t) = c_2\beta(t),\ \lambda(t) = c_3\beta(t) \tag{3.13}$$

and

$$-F_{\zeta_1}r - F_{\zeta_2} + c_1 sFG_{\zeta_1} + c_1 sF_{\zeta_1}G + c_2 sGH_{\zeta_1} + c_2 sG_{\zeta_1}H + sF_{\zeta_1\zeta_1\zeta_1} + sF_{\zeta_1\zeta_2\zeta_2} = 0,$$
$$G_{\zeta_1}r + G_{\zeta_2} - c_3 sHF_{\zeta_1} - c_3 sH_{\zeta_1}F - sG_{\zeta_1\zeta_1\zeta_1} - sG_{\zeta_1\zeta_2\zeta_2} = 0,$$
$$-H_{\zeta_1}r - H_{\zeta_2} + c_3 sFG_{\zeta_1} + c_3 sF_{\zeta_1}G + sH_{\zeta_1\zeta_1\zeta_1} + sH_{\zeta_1\zeta_2\zeta_2} = 0, \tag{3.14}$$

where c_1, c_2 and c_3 are arbitrary constants. Once again the application of similarity transformation gives the infinitesimals as follows:

$$\xi^1 = c_5, \quad \xi^2 = c_4, \quad \eta^1 = \eta^2 = \eta^3 = 0, \tag{3.15}$$

where c_4 and c_5 are arbitrary constants. The corresponding invariants and ODEs are obtained as follows:

$$\zeta = c_4\zeta_1 - c_5\zeta_2, \quad F(\zeta_1,\zeta_2) = f(\zeta),$$
$$G(\zeta_1,\zeta_2) = g(\zeta), \quad H(\zeta_1,\zeta_2) = h(\zeta) \tag{3.16}$$

and

$$-f_\zeta c_4 r + f_\zeta c_5 + c_1 sfg_\zeta c_4 + c_1 sf_\zeta c_4 g + c_2 sgh_\zeta c_4 + c_2 sg_\zeta c_4 h + sf_{\zeta\zeta\zeta}c_4^3 + sf_{\zeta\zeta\zeta}c_5^2 c_4 = 0,$$
$$g_\zeta c_4 r - g_\zeta c_5 - c_3 shf_\zeta c_4 - c_3 sh_\zeta c_4 f - sg_{\zeta\zeta\zeta}c_4^3 - sg_{\zeta\zeta\zeta}c_5^2 c_4 = 0,$$
$$-h_\zeta c_4 r + h_\zeta c_5 + c_3 sfg_\zeta c_4 + c_3 sf_\zeta c_4 g + sh_{\zeta\zeta\zeta}c_4^3 + sh_{\zeta\zeta\zeta}c_5^2 c_4 = 0. \tag{3.17}$$

These ODEs are solved by the computational software Maple and finally the exact solutions for the ncZK system become

(i)
$$c_2 = \frac{6c_7^2(12c_7^2c_5^2c_4^2 + c_4^2c_1c_8 + 6c_7^2c_4^4 + 6c_7^2c_5^4 + c_5^2c_1c_8)}{c_8^2c_3},$$
$$u(x,t,y) = \frac{8A - c_5 + rc_4}{2c_3c_4s} - \frac{6B}{c_3}\tanh^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right),$$
$$v(x,t,y) = -\frac{c_8(8A - c_5 + rc_4)}{12c_4sB} + c_8\tanh^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right),$$
$$w(x,t,y) = -\frac{c_8(8A - c_5 + rc_4)}{12c_4sB} + c_8\tanh^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right). \tag{3.18}$$

(ii)
$$c_2 = -\frac{3c_1^2}{16c_3},$$
$$u(x,t,y) = \frac{-4A - c_5 + rc_4}{2c_3c_4s} + \frac{6B}{c_3}\mathrm{sech}^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right),$$
$$v(x,t,y) = \frac{2(-4A - c_5 + rc_4)}{3c_1sc_4} + \frac{8B}{c_1}\mathrm{sech}^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right),$$
$$w(x,t,y) = \frac{2(-4A - c_5 + rc_4)}{3c_1sc_4} + \frac{8B}{c_1}\mathrm{sech}^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right). \tag{3.19}$$

(iii)
$$c_2 = \frac{6c_7^2(12c_7^2c_5{}^2c_4^2 + c_4^2c_1c_8 + 6c_7^2c_4^4 + 6c_7^2c_5^4 + c_5^2c_1c_8)}{c_8^2c_3},$$
$$u(x,t,y) = -\frac{-4A - c_5 + rc_4}{2c_3c_4s} + \frac{6B}{c_3}\mathrm{csch}^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right),$$
$$v(x,t,y) = -\frac{c_8(-4A - c_5 + rc_4)}{12c_4sB} + c_8\,\mathrm{csch}^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right),$$
$$w(x,t,y) = \frac{c_8(-4A - c_5 + rc_4)}{12c_4s} - c_8\,\mathrm{csch}^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right). \tag{3.20}$$

(iv)
$$c_2 = \frac{6c_7^2(12c_7^2c_5{}^2c_4^2 + c_4^2c_1c_8 + 6c_7^2c_4^4 + 6c_7^2c_5^4 + c_5^2c_1c_8)}{c_8^2c_3},$$
$$u(x,t,y) = -\frac{-8A - c_5 + rc_4}{2c_4sc_3} + \frac{6B}{c_3}\cot^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right),$$
$$v(x,t,y) = -\frac{c_8(-8A - c_5 + rc_4)}{12c_4sB} + c_8\cot^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right),$$
$$w(x,t,y) = \frac{c_8(-8A - c_5 + rc_4)}{12c_4sB} - c_8\cot^2\left(c_6 + c_7\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right)\right). \tag{3.21}$$

(v)
$$c_2 = -\frac{3c_1^2}{16c_3},$$
$$u(x,t,y) = -\frac{C}{2c_3} + \frac{6Bc_8^2}{c_3c_7^2}\mathrm{JacobiNS}^2\left(c_7 + c_8\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right), c_6\right),$$
$$v(x,t,y) = \frac{2C}{3c_1} - \frac{8Bc_8^2}{c_1c_7^2}\mathrm{JacobiNS}^2\left(c_7 + c_8\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right), c_6\right),$$
$$w(x,t,y) = -\frac{2C}{3c_1} + \frac{8Bc_8^2}{c_1c_7^2}\mathrm{JacobiNS}^2\left(c_7 + c_8\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right), c_6\right). \tag{3.22}$$

(vi)
$$c_2 = -\frac{3c_1^2}{16c_3},$$
$$u(x,t,y) = -\frac{C}{2c_3} + \frac{6c_8^2c_6^2B}{c_3c_7^2} \times \mathrm{JacobiSN}^2\left(c_7 + c_8\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right), c_6\right),$$
$$v(x,t,y) = \frac{2C}{3c_1} - \frac{8c_8^2c_6^2B}{c_1c_7^2} \times \mathrm{JacobiSN}^2\left(c_7 + c_8\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right), c_6\right),$$
$$w(x,t,y) = -\frac{2C}{3c_1} + \frac{8c_8^2c_6^2B}{c_1c_7^2} \times \mathrm{JacobiSN}^2\left(c_7 + c_8\left(-\frac{(rc_4 - c_5)\int\beta(t)\,\mathrm{d}t + s(-c_4x + c_5y)}{s}\right), c_6\right). \tag{3.23}$$

Here

$$A = c_7^2 c_4^3 s + c_7^2 c_4 s c_5^2,$$
$$B = c_7^2\left(c_5^2 + c_4^2\right),$$
$$C = \frac{4c_8^2 c_4^3 s c_6^2 + 4c_8^2 c_4^3 s + 4c_8^2 c_4 s c_5^2 c_6^2 + 4c_8{}^2 c_4 s c_5^2 - c_5 + r c_4}{c_4 s}, \tag{3.24}$$

where c_i are arbitrary constants.

The dark, bright and periodic wave solutions are presented graphically in figures 1–6 for solutions (3.18), (3.19) and (3.23), respectively, by considering suitable parametric values. Figures 1a–1c show the dark soliton solution (3.18) for u, v and w, respectively when $\beta(t) = \sin(t)$ by three-dimensional (3D) plots. Figures 2a–2c describe the effect of coefficient function $\beta(t)$ on the wave profile of the solution by two-dimensional (2D) plots when it is linear in t, exponential function of t and trigonometric function of t. Figures 3a–3c show the bright soliton solution (3.19) at particular $\beta(t) = e^t$ and figures 4a–4c show the effect of various $\beta(t)$ on the solution profile. Figures 5a–5c and 6a–6c show the periodic wave profiles of solution (3.23) by 3D plots for $\beta(t) = t$ and 2D plots for different $\beta(t)$, respectively.

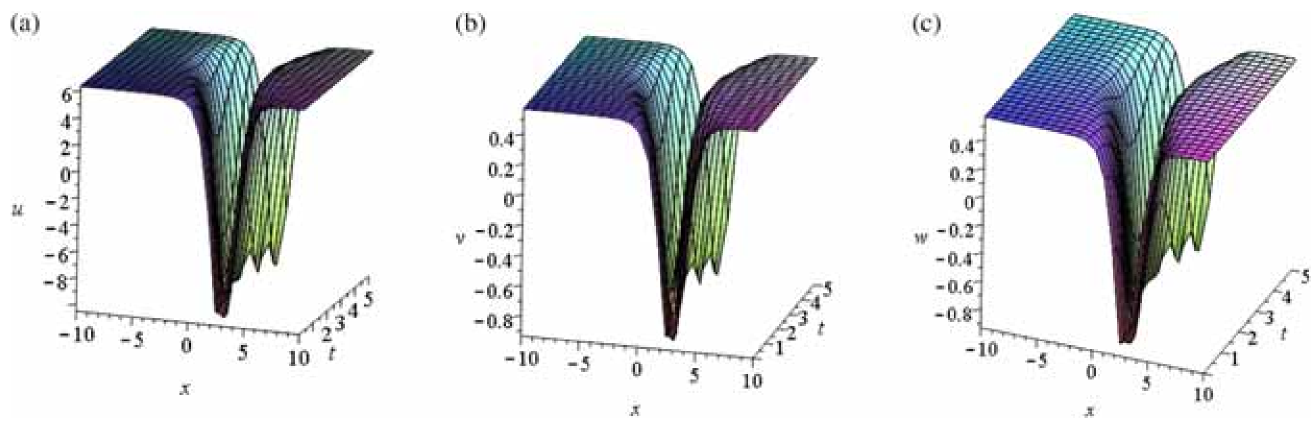

Figure 1. Dark soliton solution (3.18) profile with $c_1 = 0.25$, $c_3 = -1$, $c_4 = 1$, $c_5 = 2$, $c_6 = 0.5$, $c_7 = 0.75$, $c_8 = 1.5$, $r = 0.5$, $s = 1$, $y = 1$, $\beta(t) = \sin(t)$: (**a**) u, (**b**) v, (**c**) w.

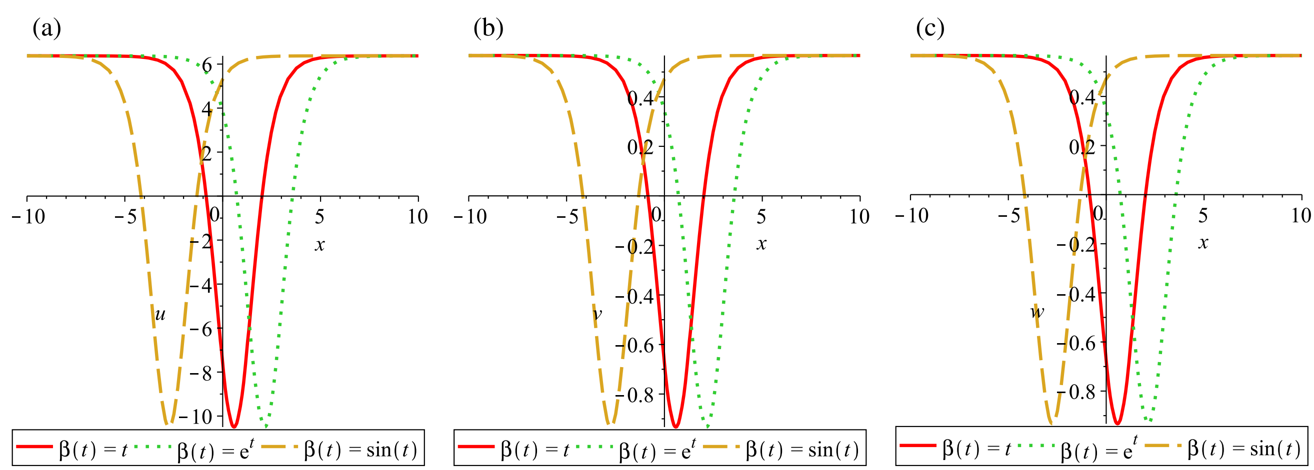

Figure 2. Effect of $\beta(t)$ on the profile of the solution (3.18) with $c_1 = 0.25$, $c_3 = -1$, $c_4 = 1$, $c_5 = 2$, $c_6 = 0.5$, $c_7 = 0.75$, $c_8 = 1.5$, $r = 0.5$, $s = 1$, $y = 1$, $t = 1$: (**a**) u, (**b**) v, (**c**) w.

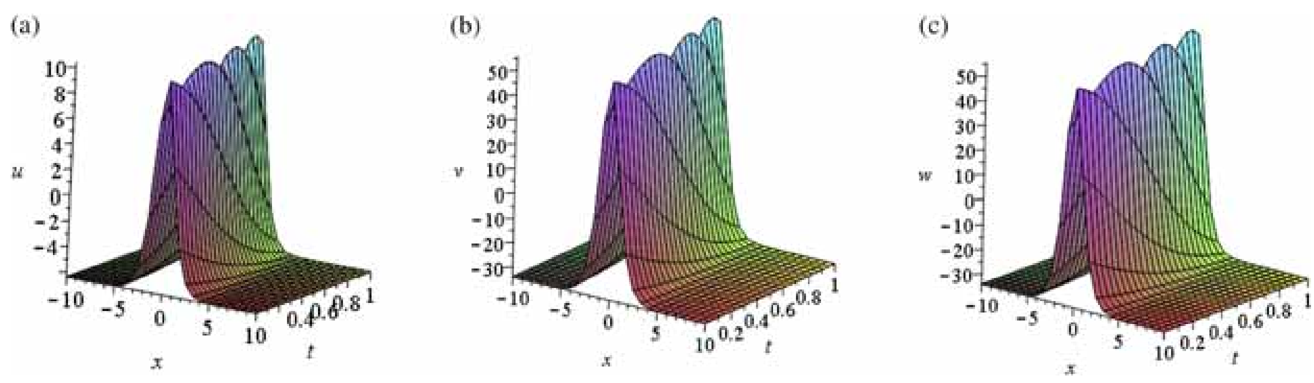

Figure 3. Bright soliton profile of solution (3.19) with $c_1 = 0.25$, $c_3 = 1$, $c_4 = 1$, $c_5 = 2$, $c_6 = 0.5$, $c_7 = 0.75$, $c_8 = 1.5$, $r = 0.5$, $s = 1$, $y = 1$, $\beta(t) = \mathrm{e}^t$: **(a)** u, **(b)** v, **(c)** w.

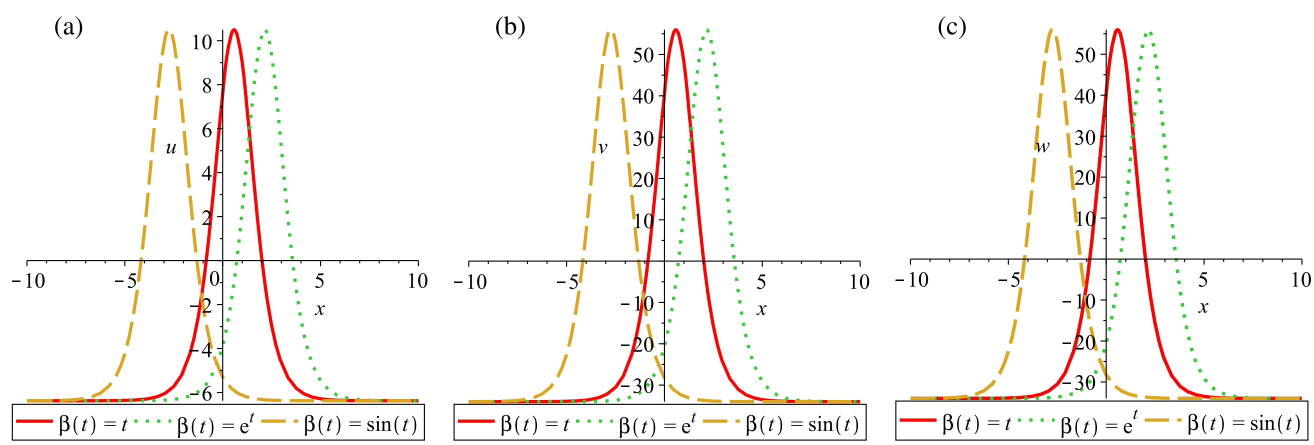

Figure 4. Effect of time-dependent coefficient $\beta(t)$ on solution (3.19) with $c_1 = 0.25$, $c_3 = 1$, $c_4 = 1$, $c_5 = 2$, $c_6 = 0.5$, $c_7 = 0.75$, $c_8 = 1.5$, $r = 0.5$, $s = 1$, $y = 1$, $t = 1$: **(a)** u, **(b)** v, **(c)** w.

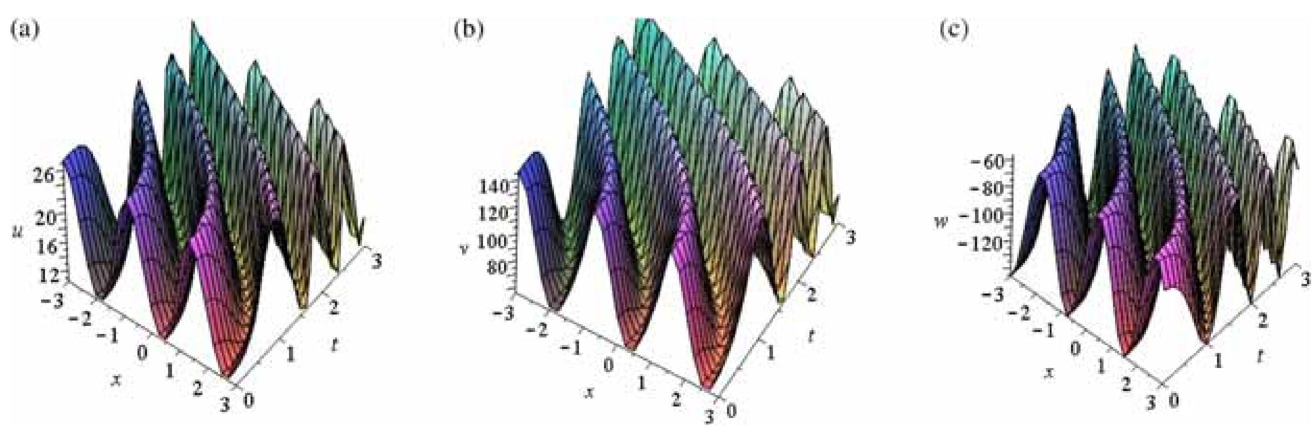

Figure 5. Periodic wave profile of solution (3.23) with $c_1 = 0.25$, $c_3 = -1$, $c_4 = 1$, $c_5 = 2$, $c_6 = 0.5$, $c_7 = 0.75$, $c_8 = 1.5$, $r = 0.5$, $s = 1$, $y = 1$, $\beta(t) = t$: **(a)** u, **(b)** v, **(c)** w.

3.4 *Vector field* $\Gamma_2 + p\Gamma_1$

By using similar procedure as described in the previous section, the reduced ODEs in this case are obtained as follows:

$$-pf_{\zeta\zeta\zeta}r_4^2 + f_\zeta - r_1 pfg_\zeta - r_1 pf_\zeta g - r_2 pgh_\zeta$$
$$-r_2 pg_\zeta h - pf_{\zeta\zeta\zeta}r_5^2 = 0,$$
$$pg_{\zeta\zeta\zeta}r_4^2 - g_\zeta + r_3 phf_\zeta + r_3 ph_\zeta f + pg_{\zeta\zeta\zeta}r_5^2 = 0,$$

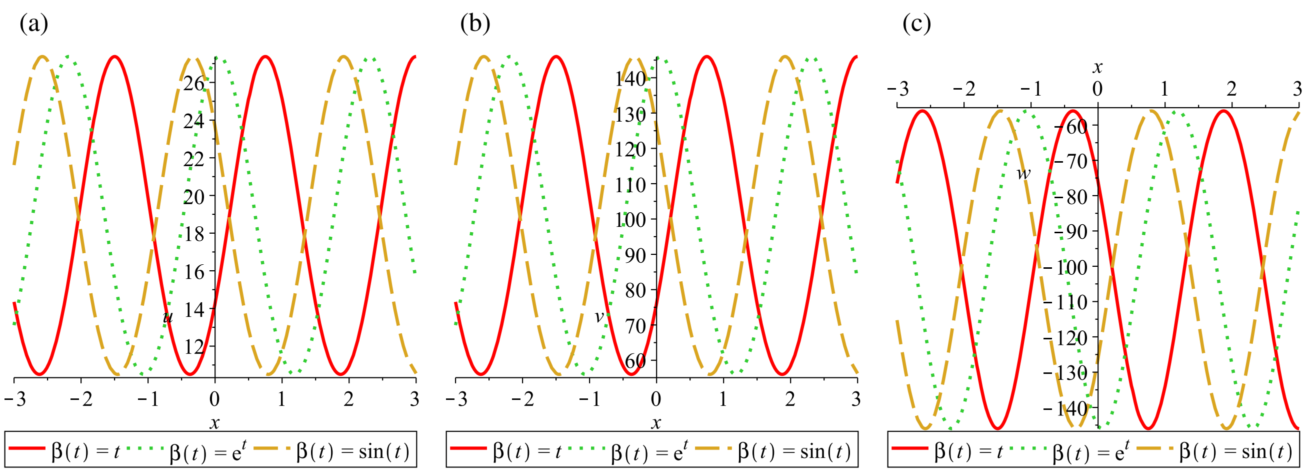

Figure 6. 2D representations of solution (3.23) with $c_1 = 0.25$, $c_3 = -1$, $c_4 = 1$, $c_5 = 2$, $c_6 = 0.5$, $c_7 = 0.75$, $c_8 = 1.5$, $r = 0.5$, $s = 1$, $y = 1$, $t = 1$: (**a**) u, (**b**) v, (**c**) w.

$$-ph_{\zeta\zeta\zeta}r_4^2 + h_\zeta - r_3 p f g_\zeta - r_3 p f_\zeta g - ph_{\zeta\zeta\zeta}r_5^2 = 0, \tag{3.25}$$

where $\zeta = r_5\zeta_1 - r_4\zeta_2$, $\zeta_1 = y$, $\zeta_2 = -(1/p)\int \beta(t)\,\mathrm{d}t + x$, $f(\zeta) = u(x,t,y)$, $g(\zeta) = v(x,t,y)$, $h(\zeta) = w(x,t,y)$, $\alpha(t) = r_1\beta(t)$, $\gamma(t) = r_2\beta(t)$ and $\lambda(t) = r_3\beta(t)$. Here r_i, $i = 1, 2, \ldots, 5$, are arbitrary constants. Using Maple software, the exact solutions for the given system are obtained as follows:

(i) $r_2 = \dfrac{2r_1 r_{10} p + 1}{4r_3 p^2 r_{10}^2}$,

$$u(x,t,y) = -\frac{1}{2pr_3} + \frac{6r_9^2 D}{r_3}\mathrm{WeierstrassP}\left(r_8 + r_9\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right), r_7, r_6\right),$$

$$v(x,t,y) = -r_{10} + E\ \mathrm{WeierstrassP}\left(r_8 + r_9\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right), r_7, r_6\right),$$

$$w(x,t,y) = r_{10} - E\ \mathrm{WeierstrassP}\left(r_8 + r_9\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right), r_7, r_6\right). \tag{3.26}$$

(ii) $r_2 = \dfrac{6r_7^2(r_8 r_5{}^2 r_1 + r_8 r_4^2 r_1 + 6r_7^2 r_4^4 + 12r_7^2 r_4^2 r_5^2 + 6r_7{}^2 r_5{}^4)}{r_8^2 r_3}$,

$$u(x,t,y) = \frac{8J+1}{2pr_3} - \frac{6r_7^2 D}{r_3}\coth^2\left(r_6 + r_7\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right)\right),$$

$$v(x,t,y) = -\frac{r_8(8J+1)}{12r_7^2 pD} + c_8\coth^2\left(r_6 + r_7\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right)\right),$$

$$w(x,t,y) = -\frac{r_8(8J+1)}{12r_7^2 pD} + c_8\coth^2\left(r_6 + r_7\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right)\right). \tag{3.27}$$

(iii) $$r_2 = \frac{6r_7^2(r_8 r_5^2 r_1 + r_8 r_4^2 r_1 + 6r_7^2 r_4^4 + 12 r_7^2 r_4^2 r_5^2 + 6 r_7^2 r_5^4)}{r_8^2 r_3},$$

$$u(x,t,y) = -\frac{4J+1}{2pr_3} + \frac{6r_7^2 D}{r_3}\sec^2\left(r_6 + r_7\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right)\right),$$

$$v(x,t,y) = -\frac{r_8(4J+1)}{12r_7^2 pD} + r_8\sec^2\left(r_6 + r_7\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right)\right),$$

$$w(x,t,y) = \frac{r_8(4J+1)}{12r_7^2 pD} - r_8\sec^2\left(r_6 + r_7\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right)\right). \tag{3.28}$$

(iv) $$r_2 = \frac{6r_7^2(r_8 r_5^2 r_1 + r_8 r_4^2 r_1 + 6r_7^2 r_4^4 + 12 r_7^2 r_4^2 r_5^2 + 6 r_7^2 r_5^4)}{r_8^2 r_3},$$

$$u(x,t,y) = \frac{8J-1}{2pr_3} + \frac{6r_7^2 D}{r_3}\tan^2\left(r_6 + r_7\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right)\right),$$

$$v(x,t,y) = \frac{r_8(8J-1)}{12r_7^2 pD} + r_8\tan^2\left(r_6 + r_7\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right)\right),$$

$$w(x,t,y) = -\frac{r_8(8J-1)}{12r_7^2 pD} - r_8\tan^2\left(r_6 + r_7\left(\frac{r_5 yp + r_4\int\beta(t)\,\mathrm{d}t - r_4 xp}{p}\right)\right). \tag{3.29}$$

Here

$$D = (r_4^2 + r_5^2),$$
$$E = (12r_{10}pr_9^2r_5^2 + 12r_{10}pr_9^2r_4^2),$$
$$J = r_7^2r_4^2p + r_7^2r_5^2p, \tag{3.30}$$

where r_i are arbitrary constants.

Solutions (3.26) and (3.29) are presented graphically for various parametric values. Figures 7a–7c and 8a–8c show the representation of 3D plots for $\beta(t) = t$ and $\beta(t) = \mathrm{e}^t$, respectively, and give singular soliton profile. The effect of function $\beta(t)$ is shown in figures 9a–9c and 10a–10c.

The graphical representation of other solutions given in eqs (3.27) and (3.28) reflects the singular nature which is similar to solutions (3.26) and (3.29). Similarly, other solutions (3.20)–(3.22) also follow singular soliton structure.

In recent years, it has been proved that the nonlinear PDEs possess the soliton-type exact solutions and there is a probability to derive lump solutions for linear and nonlinear PDEs. Lump solutions are recognised as a special type of exact solutions with hidden vital physical information. Recently, the Hirota bilinear method [26–30] is proposed to find lump solutions of nonlinear PDEs. There are reports [31,32] devoted to the interaction of lump solutions with kinks and solitons for nonlinear PDEs.

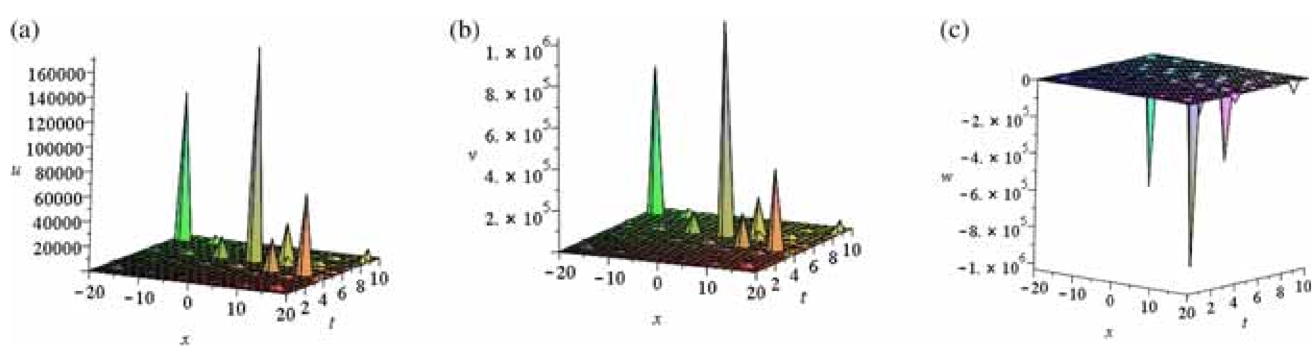

Figure 7. Singular soliton solution (3.26) with $r_1 = 0.25$, $r_3 = 1$, $r_4 = 2$, $r_5 = 1$, $r_6 = 0.5$, $r_7 = 0.25$, $r_8 = 0.5$, $r_9 = 0.75$, $r_{10} = 1.5$, $p = 2$, $y = 1$, $\beta(t) = t$: (**a**) u, (**b**) v, (**c**) w.

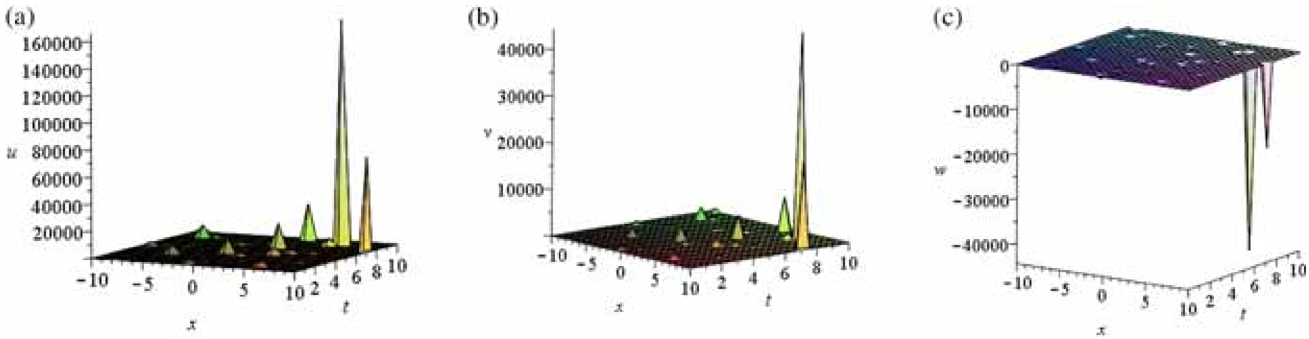

Figure 8. Singular soliton solution (3.29) with $r_1 = 0.25$, $r_3 = 1$, $r_4 = 2$, $r_5 = 1$, $r_6 = 0.5$, $r_7 = 0.25$, $r_8 = 0.5$, $r_9 = 0.75$, $r_{10} = 1.5$, $p = 2$, $y = 1$, $\beta(t) = \mathrm{e}^t$: (**a**) u, (**b**) v, (**c**) w.

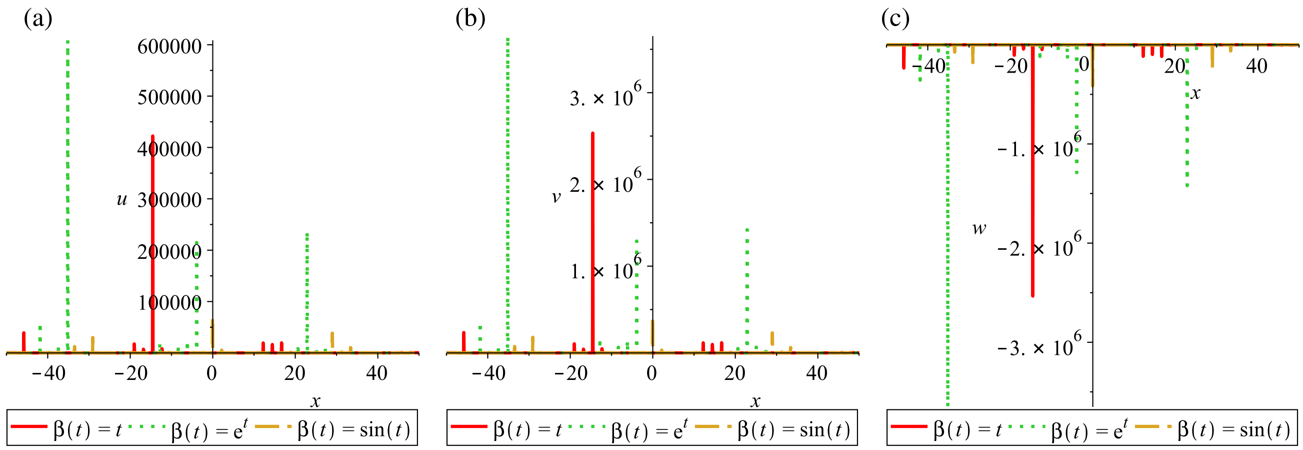

Figure 9. Effect of parameter $\beta(t)$ on the profile of solution (3.26) with $r_1 = 0.25$, $r_3 = 1$, $r_4 = 2$, $r_5 = 1$, $r_6 = 0.5$, $r_7 = 0.25$, $r_8 = 0.5$, $r_9 = 0.75$, $r_{10} = 1.5$, $p = 2$, $y = 1$, $t = 1$: (**a**) u, (**b**) v, (**c**) w.

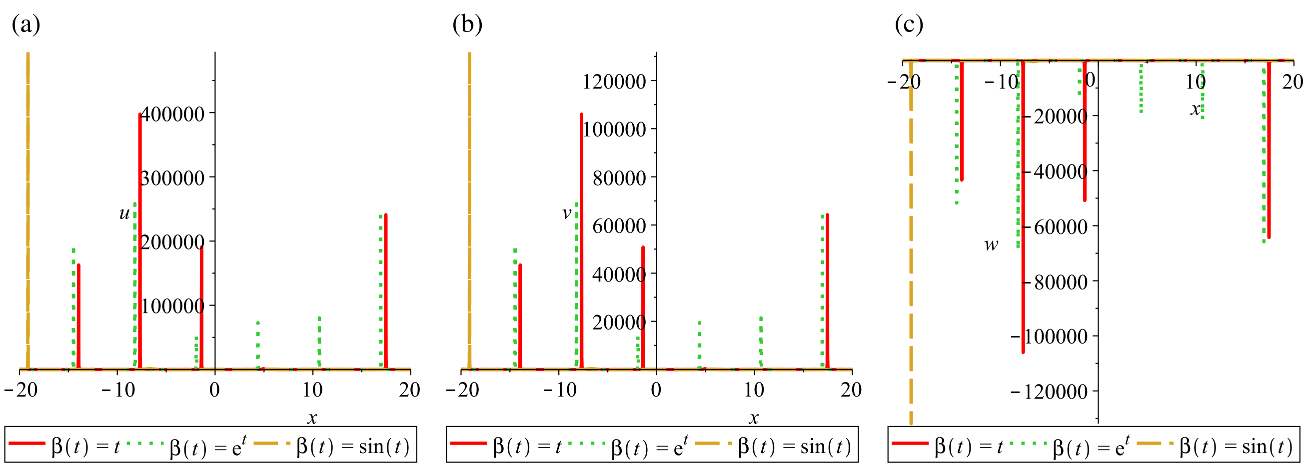

Figure 10. 2D profile of soliton solution (3.29) with $r_1 = 0.25$, $r_3 = 1$, $r_4 = 2$, $r_5 = 1$, $r_6 = 0.5$, $r_7 = 0.25$, $r_8 = 0.5$, $r_9 = 0.75$, $r_{10} = 1.5$, $p = 2$, $y = 1$, $t = 1$: (**a**) u, (**b**) v, (**c**) w.

3.5 *Vector field* Γ_1

In this case, the solutions are constructed by using Maple software as follows:

$$\alpha(t) = s_1\beta(t),\ \gamma(t) = s_2\beta(t),\ \lambda(t) = s_3\beta(t),$$

$$s_2 = \frac{6s_7^2 s_1 s_9 + 6s_7^4 + 12s_7^2 s_8^2 + s_8^2 c_1 s_9 + 6s_8^4}{s_9^2 s_3},$$

$$u(x,t,y) = \frac{6(s_8^2+s_7^2)\,\text{WeierstrassP}(s_6+s_7x+s_8y,s_5,s_4)}{s_3},$$
$$v(x,t,y) = s_9\,\text{WeierstrassP}(s_6+s_7x+s_8y,s_5,s_4),$$
$$w(x,t,y) = -s_9\,\text{WeierstrassP}(s_6+s_7x+s_8y,s_5,s_4), \tag{3.31}$$

where s_i, $i=1,2,\ldots,9$, are arbitrary constants.

Remark. When $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\lambda(t)$ are constant functions, then solutions (3.18)–(3.23) and (3.26)–(3.29) reflect similar type of solutions obtained for constant coefficients ncZK system [22–24]. Also, the given system with constant coefficients is investigated for time-fractional form in [33]. The solutions presented in [33] can be similar to our solutions (3.18)–(3.20) in the present analysis with $\beta(t)=1$ if time-fractional order $\alpha=1$ is considered.

4. Nonlinear self-adjointness and conservation laws

In the theory of nonlinear systems, the integrability of PDEs is judged by the existence of conservation laws [1,34,35]. The local conservation laws for the system possessing Lagrangian have been derived by applying the Noether's theorem [1,35] whereas the adjoint invariance condition provides conservation laws for systems having no Lagrangian formulation [36]. Nonlinear self-adjointness algorithm [12,13] has been proposed to construct the conservation laws for nonlinearly self-adjoint differential equations. Recently, an approach for finding the conservation laws by utilising both symmetries and adjoint symmetries has been reported, and it is applicable to all PDEs in spite of the existence of a Lagrangian [37,38]. Here, we use the concept of nonlinear self-adjointness to prove that system (1.1) is nonlinear self-adjoint and consequently to derive conservation laws.

4.1 *Nonlinear self-adjointness*

This subsection deals with the stepwise procedure to prove nonlinear self-adjointness for system (1.1). For this, the formal Lagrangian [12] for the ncZK system is considered as follows:

$$\begin{aligned}L &= a(x,t,y)(u_t+\alpha(t)(uv_x+u_xv)\\&\quad+\gamma(t)(vw_x+v_xw)+\beta(t)(u_{xxx}+u_{xyy}))\\&\quad+b(x,t,y)(v_t+\lambda(t)(wu_x+w_xu)\\&\quad+\beta(t)(v_{xxx}+v_{xyy}))\\&\quad+c(x,t,y)(w_t+\lambda(t)(uv_x+u_xv)\\&\quad+\beta(t)(w_{xxx}+w_{x,y,y})),\end{aligned} \tag{4.1}$$

where $a(x,t,y)$, $b(x,t,y)$ and $c(x,t,y)$ are the new dependent variables. Adjoint equations for system (1.1) can be determined using the following expressions:

$$\Xi_1^* \equiv \frac{\delta L}{\delta u}=0,\quad \Xi_2^* \equiv \frac{\delta L}{\delta v}=0,\quad \Xi_3^* \equiv \frac{\delta L}{\delta w}=0, \tag{4.2}$$

where $\delta/\delta u$, $\delta/\delta v$ and $\delta/\delta w$ are the variational derivatives defined as follows:

$$\begin{aligned}\frac{\delta}{\delta u} &= \frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}-D_y\frac{\partial}{\partial u_y}\\&\quad-D_{xxx}\frac{\partial}{\partial u_{xxx}}-D_{xyy}\frac{\partial}{\partial u_{xyy}},\\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v}-D_t\frac{\partial}{\partial v_t}-D_x\frac{\partial}{\partial v_x}-D_y\frac{\partial}{\partial v_y}\\&\quad-D_{xxx}\frac{\partial}{\partial v_{xxx}}-D_{xyy}\frac{\partial}{\partial v_{xyy}},\\ \frac{\delta}{\delta w} &= \frac{\partial}{\partial w}-D_t\frac{\partial}{\partial w_t}-D_x\frac{\partial}{\partial w_x}-D_y\frac{\partial}{\partial w_y}\\&\quad-D_{xxx}\frac{\partial}{\partial w_{xxx}}-D_{xyy}\frac{\partial}{\partial w_{xyy}}.\end{aligned} \tag{4.3}$$

From eq. (4.2), adjoint equations for the ncZK system (1.1) read as follows:

$$\begin{aligned}\Xi_1^* &\equiv -a_x\alpha(t)v-b_x\lambda(t)w-c_x\lambda(t)v-a_t\\&\quad-a_{xxx}\beta(t)-a_{xyy}\beta(t)=0,\\ \Xi_2^* &\equiv -a_x\alpha(t)u-a_x\gamma(t)w-c_x\lambda(t)u-b_t\\&\quad-b_{xxx}\beta(t)-b_{xyy}\beta(t)=0,\\ \Xi_3^* &\equiv -a_x\gamma(t)v-b_x\lambda(t)u-c_t\\&\quad-c_{xxx}\beta(t)-c_{xyy}\beta(t)=0.\end{aligned} \tag{4.4}$$

The ncZK system (1.1) is said to be nonlinear self-adjoint if the adjoint equations (4.4) with substitutions of

$$\begin{aligned}a(x,t,y)&=\phi_1(x,t,y,u,v,w),\\ b(x,t,y)&=\phi_2(x,t,y,u,v,w),\\ c(x,t,y)&=\phi_3(x,t,y,u,v,w)\end{aligned} \tag{4.5}$$

satisfy the equations

$$\begin{aligned}\Xi_1^*|_{(4.5)}&=\Lambda_1\Xi_1+\Lambda_2\Xi_2+\Lambda_3\Xi_3,\\ \Xi_2^*|_{(4.5)}&=\Lambda_4\Xi_1+\Lambda_5\Xi_2+\Lambda_6\Xi_3,\\ \Xi_3^*|_{(4.5)}&=\Lambda_7\Xi_1+\Lambda_8\Xi_2+\Lambda_9\Xi_3\end{aligned} \tag{4.6}$$

for all the solutions of u, v and w. Here Λ_i, $i=1,2,\ldots,9$, are the undetermined coefficients and the functions $\phi_i(x,t,y,u,v,w)\neq 0$, simultaneously, $i=1,2,3$.

By equating the coefficients of all possible derivatives of u, v and w from eqs (4.6), the determining equations are obtained. Then, the solution of determining equations with the aid of Maple software is found for ϕ_i as follows:

$$\phi_1(x,t,y,u,v,w) = F_1(y),$$
$$\phi_2(x,t,y,u,v,w) = F_2(y),$$
$$\phi_3(x,t,y,u,v,w) = F_3(y), \tag{4.7}$$

where $F_1(y)$, $F_2(y)$ and $F_3(y)$ are arbitrary functions of y.

4.2 *Conservation laws*

This subsection presents the way to derive conservation laws for the ncZK system (1.1) by applying the new conservation theorem. Conserved quantities corresponding to symmetries in eq. (2.7) can be found by using the following theorem.

Theorem 4.1 [12]. *Any symmetry given in eq. (2.7) for the ncZK system (1.1) as well as adjoint equations (4.4) gives conservation laws using* $D_t\Psi^1+D_x\Psi^2+D_y\Psi^3=0$. *Here* Ψ^i, *for* $i=1,2,3$, *represent the conserved vector components as follows*:

$$\Psi^1 = \xi_1 L + W^1\frac{\partial L}{\partial u_t} + W^2\frac{\partial L}{\partial v_t} + W^3\frac{\partial L}{\partial w_t},$$

$$\begin{aligned}\Psi^2 = {} & \xi_2 L + W^1\left(\frac{\partial L}{\partial u_x} + D_{yy}\left(\frac{\partial L}{\partial u_{xyy}}\right)\right. \\ & \left.+D_{xx}\left(\frac{\partial L}{\partial u_{xxx}}\right)\right) - D_x(W^1)D_x\left(\frac{\partial L}{\partial u_{xxx}}\right) \\ & - D_y(W^1)D_y\left(\frac{\partial L}{\partial u_{xyy}}\right) + D_{xx}(W^1)\left(\frac{\partial L}{\partial u_{xxx}}\right) \\ & + D_{yy}(W^1)\left(\frac{\partial L}{\partial u_{xyy}}\right)+ W^2\left(\frac{\partial L}{\partial v_x} + D_{yy}\left(\frac{\partial L}{\partial v_{xyy}}\right)\right. \\ & \left.+D_{xx}\left(\frac{\partial L}{\partial v_{xxx}}\right)\right) - D_x(W^2)D_x\left(\frac{\partial L}{\partial v_{xxx}}\right) \\ & + D_y(W^2)\left(-D_y\left(\frac{\partial L}{\partial v_{xyy}}\right)\right) \\ & + D_{xx}(W^2)\left(\frac{\partial L}{\partial v_{xxx}}\right) + D_{yy}(W^2)\left(\frac{\partial L}{\partial v_{xyy}}\right) \\ & + W^3\left(\frac{\partial L}{\partial w_x}+D_{yy}\left(\frac{\partial L}{\partial w_{xyy}}\right)+D_{xx}\left(\frac{\partial L}{\partial w_{xxx}}\right)\right) \\ & - D_x(W^3)D_x\left(\frac{\partial L}{\partial w_{xxx}}\right)-D_y(W^3)D_y\left(\frac{\partial L}{\partial w_{xyy}}\right) \\ & + D_{xx}(W^3)\left(\frac{\partial L}{\partial w_{xxx}}\right) + D_{yy}(W^3)\left(\frac{\partial L}{\partial w_{xyy}}\right),\end{aligned}$$

$$\begin{aligned}\Psi^3 = {} & \xi_3 L + W^1\left(D_{yx}\left(\frac{\partial L}{\partial u_{yyx}}\right) + D_{xy}\left(\frac{\partial L}{\partial u_{yxy}}\right)\right) \\ & -D_y(W^1)D_x\left(\frac{\partial L}{\partial u_{yyx}}\right)-D_x(W^1)D_y\left(\frac{\partial L}{\partial u_{yxy}}\right) \\ & +D_{xy}(W^1)\left(\frac{\partial L}{\partial u_{yxy}}\right) + D_{yx}(W^1)\left(\frac{\partial L}{\partial u_{yyx}}\right) \\ & +W^2\left(D_{yx}\left(\frac{\partial L}{\partial v_{yyx}}\right) + D_{xy}\left(\frac{\partial L}{\partial v_{yxy}}\right)\right) \\ & -D_y(W^2)D_x\left(\frac{\partial L}{\partial v_{yyx}}\right) - D_x(W^2)D_y\left(\frac{\partial L}{\partial v_{yxy}}\right) \\ & +D_{xy}(W^2)\left(\frac{\partial L}{\partial v_{yxy}}\right) + D_{yx}(W^2)\left(\frac{\partial L}{\partial v_{yyx}}\right) \\ & +W^3\left(D_{yx}\left(\frac{\partial L}{\partial w_{yyx}}\right) + D_{xy}\left(\frac{\partial L}{\partial w_{yxy}}\right)\right) \\ & -D_y(W^3)D_x\left(\frac{\partial L}{\partial w_{yyx}}\right) - D_x(W^3)D_y\left(\frac{\partial L}{\partial w_{yxy}}\right) \\ & +D_{xy}(W^3)\left(\frac{\partial L}{\partial w_{yxy}}\right)+D_{yx}(W^3)\left(\frac{\partial L}{\partial w_{yyx}}\right),\end{aligned} \tag{4.8}$$

where $W^1=\eta_1-\xi_1u_t-\xi_2u_x-\xi_3u_y$, $W^2=\eta_1-\xi_1v_t-\xi_2v_x-\xi_3v_y$ *and* $W^3=\eta_1-\xi_1w_t-\xi_2w_x-\xi_3w_y$ *are the Lie characteristic functions and the Lagrangian* L *is considered in the symmetric form for the mixed derivative terms, i.e.* u_{xyy}, v_{xyy} *and* w_{xyy} *as follows*:

$$\begin{aligned}L = {} & a(x,t,y)\Big(u_t + \alpha(t)(uv_x + u_xv) \\ & +\gamma(t)(vw_x + v_xw) \\ & +\beta(t)\Big(u_{xxx} + \frac{1}{3}(u_{xyy} + u_{yxy} + u_{yyx})\Big)\Big) \\ & +b(x,t,y)\Big(v_t + \lambda(t)(wu_x + w_xu) \\ & +\beta(t)\Big(v_{xxx} + \frac{1}{3}(v_{xyy} + v_{yxy} + v_{yyx})\Big)\Big) \\ & +c(x,t,y)\Big(w_t + \lambda(t)(uv_x + u_xv) \\ & +\beta(t)\Big(w_{xxx} + \frac{1}{3}(w_{xyy} + w_{yxy} + w_{yyx})\Big)\Big).\end{aligned}$$

The nonlinear self-adjointness substitutions (4.7) and the above theorem provide the conservation laws for the ncZK system in the following cases.

Case 1. For symmetry $\Gamma_1 = \partial/\partial x$, the Lie characteristic functions are $W^1=-u_x$, $W^2=-v_x$ and $W^3=-w_x$. Thus, the conserved vector components in this case are obtained as follows:

$$\begin{aligned}
\Psi^1 &= -u_x F_1 - v_x F_2 - w_x F_3,\\
\Psi^2 &= \frac{1}{3}u_t F_1 + \frac{1}{3}w_t F_3 + \frac{1}{3}v_t F_2 - \frac{1}{3}u_x F_{1yy}\beta(t)\\
&\quad + \frac{1}{3}u_{xy}F_{1y}\beta(t) - \frac{2}{3}F_1\alpha(t)uv_x - \frac{2}{3}F_1\alpha(t)u_x v\\
&\quad - \frac{2}{3}F_1\gamma(t)vw_x - \frac{2}{3}F_1\gamma(t)v_x w - \frac{2}{3}F_1\beta(t)u_{xxx}\\
&\quad - \frac{1}{3}v_x F_{2yy}\beta(t) + \frac{1}{3}v_{xy}F_{2y}\beta(t)\\
&\quad - \frac{2}{3}F_2\lambda(t)wu_x - \frac{2}{3}F_2\lambda(t)w_x u - \frac{2}{3}F_2\beta(t)v_{xxx}\\
&\quad - \frac{1}{3}w_x F_{3yy}\beta(t) + \frac{1}{3}w_{xy}F_{3y}\beta(t)\\
&\quad - \frac{2}{3}F_3\lambda(t)uv_x - \frac{2}{3}F_3\lambda(t)u_x v - \frac{2}{3}F_3\beta(t)w_{xxx},\\
\Psi^3 &= -\frac{1}{3}\beta(t)(-u_{xx}F_{1y} + 2u_{xxy}F_1 - v_{xx}F_{2y}\\
&\quad + 2v_{xxy}F_2 - w_{xx}F_{3y} + 2w_{xxy}F_3). \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{3}u_{ty}F_{1y} + \frac{1}{3}w_{ty}F_{3y} - \frac{1}{3}v_t F_{2yy}\\
&- \frac{1}{3}u_t F_{1yy} - \frac{1}{3}w_t F_{3yy} + \frac{1}{3}v_{ty}F_{2,y}\\
&+ \frac{1}{\beta(t)}((-u_t\lambda(t)v - v_t\lambda(t)u)F_3\\
&+(-u_t\lambda(t)w - w_t\lambda(t)u)F_2)\\
&+ \frac{1}{\beta(t)}(-u_t\alpha(t)v - v_t\gamma(t)w - w_t\gamma(t)v\\
&-v_t\alpha(t)u)F_1,\\
\Psi^3 &= \frac{1}{3}u_{tx}F_{1y} - \frac{2}{3}u_{txy}F_1 + \frac{1}{3}v_{tx}F_{2y} - \frac{2}{3}v_{txy}F_2\\
&+ \frac{1}{3}w_{tx}F_{3y}\frac{1}{\beta(t)} - \frac{2}{3}w_{txy}F_3. \qquad (4.10)
\end{aligned}$$

In this case, the divergence condition becomes

$$D_t\Psi^1 + D_x\Psi^2 + D_y\Psi^3 = D_x\left(\frac{(-F_3\lambda(t)uv - F_2\lambda(t)wu + (-\alpha(t)uv - \gamma(t)vw)F_1)\beta'(t)}{\beta(t)^2} + \frac{\lambda'(t)uvF_3 + \lambda'(t)wuF_2 + (\alpha'(t)uv + \gamma'(t)vw)F_1}{\beta(t)}\right). \qquad (4.11)$$

By taking the terms of D_x in the left-hand side, we get

$$D_t\Psi^1 + D_x\left(\Psi^2 - \frac{(-F_3\lambda(t)uv - F_2\lambda(t)wu + (-\alpha(t)uv - \gamma(t)vw)F_1)\beta'(t)}{\beta(t)^2} - \frac{\lambda'(t)uvF_3 + \lambda'(t)wuF_2 + (\alpha'(t)uv + \gamma'(t)vw)F_1}{\beta(t)}\right) + D_y\Psi^3 = 0. \qquad (4.12)$$

Here, the divergence condition $D_t\Psi^1 + D_x\Psi^2 + D_y\Psi^3 = 0$ is satisfied using Maple software for all the solutions of the said system (1.1).

Case 2. The symmetry $\Gamma_2 = (1/\beta(t))(\partial/\partial t)$ with the Lie characteristic functions $W^1 = -((1/\beta(t))u_t)$, $W^2 = -((1/\beta(t))v_t)$ and $W^3 = -((1/\beta(t))w_t)$ gives the following non-trivial conservation laws:

$$\begin{aligned}
\Psi^1 &= -\frac{u_t F_1 + v_t F_2 + w_t F_3}{\beta(t)},\\
\Psi^2 &= \left(-\frac{1}{3}w_{tyy} - w_{txx}\right)F_3 + \left(-v_{txx} - \frac{1}{3}v_{tyy}\right)F_2\\
&\quad + \left(-u_{txx} - \frac{1}{3}u_{tyy}\right)F_1
\end{aligned}$$

Therefore, the components of conserved vectors are modified as follows:

$$\begin{aligned}
\tilde{\Psi}^1 &= \Psi^1,\\
\tilde{\Psi}^2 &= \frac{1}{3}u_{ty}F_{1y} - \frac{1}{3}w_{tyy}F_3 - \frac{1}{3}w_t F_{3yy}\\
&\quad - \frac{1}{3}v_{tyy}F_2 - \frac{1}{3}u_{tyy}F_1 + \frac{1}{3}w_{ty}F_{3y}\\
&\quad - u_{txx}F_1 - v_{txx}F_2 - w_{txx}F_3 + \frac{1}{3}v_{ty}F_{2y}\\
&\quad - \frac{1}{3}v_t F_{2yy} + \frac{1}{\beta(t)}\Big((-F_1\gamma'(t)v - F_2\lambda'(t)u\\
&\quad - v_t F_1\gamma(t) - u_t F_2\lambda(t))w\\
&\quad + \Big(-\frac{1}{3}(3F_3\lambda'(t) + 3F_1\alpha'(t))u - u_t F_1\alpha(t)
\end{aligned}$$

$$-u_tF_3\lambda(t) - w_tF_1\gamma(t)\Big)v$$
$$-\frac{1}{3}(3v_tF_1\alpha(t) + 3v_tF_3\lambda(t) + 3w_tF_2\lambda(t))u\Big)$$
$$-\frac{1}{3}u_tF_{1yy} + \frac{1}{\beta(t)^2}\Big((F_1\beta'(t)\gamma(t)v$$
$$+F_2\beta'(t)\lambda(t)u)w - \frac{1}{3}(-3F_3\beta'(t)\lambda(t)$$
$$-3F_1\beta'(t)\alpha(t))uv\Big),$$
$$\tilde{\Psi}^3 = \Psi^3. \tag{4.13}$$

Case 3. The symmetry

$$\Gamma_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{3\int\beta(t)\mathrm{d}t}{\beta(t)}\frac{\partial}{\partial t}$$

with

$$W^1 = -xu_x - yu_y - \frac{3\int\beta(t)\,\mathrm{d}t}{\beta(t)}u_t,$$
$$W^2 = -xv_x - yv_y - \frac{3\int\beta(t)\,\mathrm{d}t}{\beta(t)}v_t$$

and

$$W^3 = -xw_x - yw_y - \frac{3\int\beta(t)\,\mathrm{d}t}{\beta(t)}w_t$$

determines the following conservation laws:

$$\Psi^1 = -\frac{3\int\beta(t)\,\mathrm{d}t(u_tF_1 + v_tF_2 + w_tF_3)}{\beta(t)}$$
$$-y(F_2v_y + F_3w_y + F_1u_y)$$
$$-x(w_xF_3 + v_xF_2 + u_xF_1),$$
$$\Psi^2 = \Big(-\frac{1}{3}(9w_{txx} + 3w_{tyy})F_3$$
$$-\frac{1}{3}(9v_{txx} + 3v_{tyy})F_2 + v_{ty}F_{2y}$$
$$-u_tF_{1yy} - v_tF_{2yy} - w_tF_{3yy} + u_{ty}F_{1y}$$
$$+w_{ty}F_{3y} - \frac{1}{3}(9u_{txx} + 3u_{tyy})F_1$$
$$+\frac{1}{\beta(t)}\Big(-\frac{1}{3}(9u_t\lambda(t)v + 9v_t\lambda(t)u)F_3$$
$$-\frac{1}{3}(9v_t\alpha(t)u + 9u_t\alpha(t)v + 9w_t\gamma(t)v$$
$$+9v_t\gamma(t)w)F_1 - \frac{1}{3}(9w_t\lambda(t)u + 9u_t\lambda(t)w)F_2\Big)\Big)$$
$$\times\Big(\int\beta(t)\,\mathrm{d}t\Big) + \Big(-\frac{1}{3}(2xw_{xxx} + 2w_{yy}$$
$$+3yw_{xxy} + 6w_{xx} + yw_{yyy})F_3$$
$$-\frac{1}{3}(yu_{yyy} + 2xu_{xxx}$$
$$+3yu_{xxy} + 2u_{yy} + 6u_{xx})F_1$$
$$-\frac{1}{3}(3yv_{xxy} + yv_{yyy} + 6v_{xx}$$
$$+2v_{yy} + 2xv_{xxx})F_2 + \frac{1}{3}F_{1y}yu_{yy}$$
$$+\frac{1}{3}F_{2y}xv_{xy} + \frac{1}{3}F_{2y}yv_{yy} - \frac{1}{3}xv_xF_{2yy}$$
$$-\frac{1}{3}yu_yF_{1yy} - \frac{1}{3}xw_xF_{3yy} + \frac{1}{3}F_{1y}xu_{xy}$$
$$+\frac{1}{3}F_{3y}yw_{yy} + \frac{1}{3}F_{2y}v_y - \frac{1}{3}xu_xF_{1yy}$$
$$+\frac{1}{3}F_{3y}xw_{xy} + \frac{1}{3}F_{1y}u_y$$
$$-\frac{1}{3}yv_yF_{2yy} + \frac{1}{3}F_{3y}w_y - \frac{1}{3}yw_yF_{3yy}\Big)\beta(t)$$
$$-\frac{1}{3}(2x\lambda(t)uv_x + 3yu_y\lambda(t)v$$
$$+2x\lambda(t)u_xv + 3yv_y\lambda(t)u - xw_t)F_3$$
$$-\frac{1}{3}(3yu_y\lambda(t)w + 3yw_y\lambda(t)u$$
$$+2x\lambda(t)w_xu - xv_t + 2x\lambda(t)wu_x)F_2$$
$$-\frac{1}{3}(3yv_y\alpha(t)u + 2x\alpha(t)u_xv + 3yv_y\gamma(t)w$$
$$+2x\alpha(t)uv_x + 3yu_y\alpha(t)v + 3yw_y\gamma(t)v$$
$$+2x\gamma(t)v_xw + 2x\gamma(t)vw_x - xu_t)F_1,$$
$$\Psi^3 = \Big(\frac{2}{3}yF_1vw_x + \frac{2}{3}yF_1v_xw\Big)\gamma(t)$$
$$+\Big(\frac{2}{3}yF_1uv_x + \frac{2}{3}yF_1u_xv\Big)\alpha(t)$$
$$+\Big(\frac{2}{3}yF_2w_xu + \frac{2}{3}yF_3u_xv + \frac{2}{3}yF_2wu_x$$
$$+\frac{2}{3}yF_3uv_x\Big)\lambda(t)$$
$$+\Big(\frac{1}{3}F_{3y}w_x + \frac{1}{3}F_{1y}yu_{xy} + \frac{1}{3}F_{3y}yw_{xy}$$
$$-\frac{2}{3}F_1xu_{xxy} + \frac{1}{3}F_{3y}xw_{xx} - \frac{2}{3}F_3xw_{xxy}$$
$$-\frac{4}{3}F_3w_{xy} + \frac{1}{3}F_{1y}xu_{xx} + \frac{2}{3}yF_1u_{xxx}$$
$$+\frac{1}{3}F_{1y}u_x + \frac{2}{3}yF_2v_{xxx} - \frac{4}{3}F_1u_{xy}$$

$$+\frac{2}{3}yF_3w_{xxx}-\frac{4}{3}F_2v_{xy}+\frac{1}{3}F_{2y}xv_{xx}-\frac{2}{3}F_2xv_{xxy}+\frac{1}{3}F_{2y}yv_{xy}+\frac{1}{3}F_{2y}v_x\Big)\beta(t)+\frac{2}{3}yF_2v_t+\frac{2}{3}yF_3w_t+\left(\int\beta(t)\mathrm{d}t\right)\times(v_{tx}F_{2y}-2w_{txy}F_3+u_{tx}F_{1y}-2u_{txy}F_1-2v_{txy}F_2+w_{tx}F_{3y})+\frac{2}{3}yF_1u_t. \tag{4.14}$$

The total divergence condition gives the following result:

$$+\frac{1}{3}F_{1y}yu_{yy}-\frac{1}{3}xw_xF_{3yy}-\frac{1}{3}yF_1u_{yyy}+\frac{1}{3}F_{2y}xv_{xy}-\frac{1}{3}yw_yF_{3yy}+\frac{2}{3}F_1xu_{xyy}+\frac{1}{3}F_{3y}yw_{yy}-yF_2v_{xxy}-\frac{1}{3}xv_xF_{2yy}-\frac{1}{3}xu_xF_{1yy}+\frac{1}{3}F_{1y}xu_{xy}+\frac{2}{3}F_2xv_{xyy}-yF_3w_{xxy}-\frac{1}{3}yu_yF_{1yy}-\frac{1}{3}yF_2v_{yyy}-\frac{1}{3}yv_yF_{2yy}-\frac{1}{3}yF_3w_{yyy}-yF_1u_{xxy}$$

$$D_t\Psi^1+D_x\Psi^2+D_y\Psi^3 = D_x\left(\frac{((-3\alpha(t)uv-3\gamma(t)vw)F_1-3F_2\lambda(t)wu)\left(\int\beta(t)\,\mathrm{d}t\right)\beta'(t)}{\beta(t)^2}+\frac{(3\lambda'(t)uvF_3+3\lambda'(t)wuF_2+(3\alpha'(t)uv+3\gamma'(t)vw)F_1)\left(\int\beta(t)\,\mathrm{d}t\right)}{\beta(t)}+2F_2\lambda(t)wu+2F_3\lambda(t)uv+(2\gamma(t)vw+2\alpha(t)uv)F_1\frac{-3F_3\lambda(t)uv}{\beta(t)^2}\right). \tag{4.15}$$

Then the modified components of the conserved vectors are obtained as follows:

$$\tilde{\Psi}^1=\Psi^1,$$

$$\tilde{\Psi}^2=xF_3w_t+xF_1u_t+xF_2v_t+\big(-w_{tyy}F_3-3w_{txx}F_3-w_tF_{3yy}-u_tF_{1yy}+w_{ty}F_{3y}+v_{ty}F_{2y}+u_{ty}F_{1y}-u_{tyy}F_1-3u_{txx}F_1-v_tF_{2yy}-3v_{txx}F_2-v_{tyy}F_2\big)\left(\int\beta(t)\,\mathrm{d}t\right)-yv_yF_1\alpha(t)u-yv_yF_1\gamma(t)w-yv_yF_3\lambda(t)u-yw_yF_1\gamma(t)v-yw_yF_2\lambda(t)u-yu_yF_1\alpha(t)v-yu_yF_3\lambda(t)v-yu_yF_2\lambda(t)w-2F_2\lambda(t)wu-2F_3\lambda(t)uv-2F_1\gamma(t)vw-2F_1\alpha(t)uv+\Big(\frac{1}{3}F_{1y}u_y-\frac{2}{3}F_2v_{yy}-\frac{2}{3}F_3w_{yy}-2F_1u_{xx}-2F_3w_{xx}+\frac{1}{3}F_{2y}v_y-\frac{2}{3}F_1u_{yy}-2F_2v_{xx}+\frac{1}{3}F_{3y}w_y+\frac{1}{3}F_{2y}yv_{yy}+\frac{2}{3}F_3xw_{xyy}+\frac{1}{3}F_{3y}xw_{xy}\Big)\beta(t)+\left(\frac{\int\beta(t)\,\mathrm{d}t}{\beta(t)}\right)\times\big(-3F_1\alpha'(t)uv-3F_3\lambda'(t)uv-3F_2\lambda'(t)wu-3F_1\gamma'(t)vw-3v_tF_1\alpha(t)u-3u_tF_3\lambda(t)v-3v_tF_1\gamma(t)w-3u_tF_2\lambda(t)w-3w_tF_1\gamma(t)v-3w_tF_2\lambda(t)u-3u_tF_1\alpha(t)v-3v_tF_3\lambda(t)u\big)+\frac{\left(\int\beta(t)\,\mathrm{d}t\right)\left(\beta'(t)\right)}{\beta(t)^2}(3F_3\lambda(t)uv+3F_1\gamma(t)vw+3F_1\alpha(t)uv+3F_2\lambda(t)wu),$$

$$\tilde{\Psi}^3=\Psi^3. \tag{4.16}$$

Case 4. The symmetry $\Gamma_4=\partial/\partial y$ yields the following conserved vectors:

$$\Psi^1=-u_yF_1-v_yF_2-w_yF_3,$$

$$\Psi^2=\left(\left(-w_{xxy}-\frac{1}{3}w_{yyy}\right)F_3+\left(-v_{xxy}-\frac{1}{3}v_{yyy}\right)F_2+\left(-u_{xxy}-\frac{1}{3}u_{yyy}\right)F_1+\frac{1}{3}w_{yy}F_{3y}-\frac{1}{3}v_yF_{2yy}-\frac{1}{3}u_yF_{1yy}\right.$$

$$+\frac{1}{3}u_{yy}F_{1y}+\frac{1}{3}v_{yy}F_{2y}-\frac{1}{3}w_yF_{3yy}\Big)\beta(t)$$
$$+(-u_y\lambda(t)v-v_y\lambda(t)u)F_3$$
$$+(-u_y\lambda(t)w-w_y\lambda(t)u)F_2$$
$$+(-w_y\gamma(t)v-v_y\alpha(t)u$$
$$-v_y\gamma(t)w-u_y\alpha(t)v)F_1,$$

$$\Psi^3=\left(\frac{2}{3}F_1u_xv+\frac{2}{3}F_1uv_x\right)\alpha(t)$$
$$+\left(\frac{2}{3}F_1vw_x+\frac{2}{3}F_1v_xw\right)\gamma(t)$$
$$+\Big(\frac{2}{3}F_2wu_x+\frac{2}{3}F_2w_xu+\frac{2}{3}F_3u_xv$$
$$+\frac{2}{3}F_3uv_x\Big)\lambda(t)+\Big(\frac{1}{3}u_{xy}F_{1y}+\frac{1}{3}v_{xy}F_{2y}$$
$$+\frac{2}{3}F_3w_{xxx}+\frac{2}{3}F_2v_{xxx}+\frac{2}{3}F_1u_{xxx}$$
$$+\frac{1}{3}w_{xy}F_{3y}\Big)\beta(t)+\frac{2}{3}F_1u_t+\frac{2}{3}F_2v_t+\frac{2}{3}F_3w_t. \tag{4.17}$$

The divergence condition yields the following expression:

$$D_t\Psi^1+D_x\Psi^2+D_y\Psi^3$$
$$=D_t(F_{1y}u+F_{2y}v+F_{3y}w)$$
$$+D_x((F_{1y}\gamma(t)v+F_{2y}\lambda(t)u)w$$
$$+(F_{3y}\lambda(t)+F_{1y}\alpha(t))uv$$
$$+((w_{xx}+w_{yy})F_{3y}+(v_{xx}+v_{yy})F_{2y}$$
$$+(u_{yy}+u_{xx})F_{1y})\beta(t)). \tag{4.18}$$

Then the components of conserved vectors are changed as follows:

$$\tilde{\Psi}^1=-F_1u_y-F_2v_y-F_3w_y-F_{1y}u-F_{2y}v-F_{3y}w,$$
$$\tilde{\Psi}^2=(-v_yF_3u-F_{2y}wu-F_{3y}uv-u_yF_3v$$
$$-u_yF_2w-w_yF_2u)\lambda(t)$$
$$+(-v_yF_1u-u_yF_1v-F_{1y}uv)\alpha(t)$$
$$+(-F_{1y}vw-w_yF_1v-v_yF_1w)\gamma(t)$$
$$+\Big(-\frac{2}{3}v_{yy}F_{2y}-\frac{1}{3}v_yF_{2yy}-F_{2y}v_{xx}$$
$$-\frac{2}{3}w_{yy}F_{3y}-\frac{2}{3}u_{yy}F_{1y}-\frac{1}{3}w_yF_{3yy}-u_{xxy}F_1$$
$$-\frac{1}{3}u_{yyy}F_1-\frac{1}{3}w_{yyy}F_3-F_{3y}w_{xx}-v_{xxy}F_2$$
$$-\frac{1}{3}u_yF_{1yy}-w_{xxy}F_3-F_{1y}u_{xx}-\frac{1}{3}v_{yyy}F_2\Big)\beta(t),$$
$$\tilde{\Psi}^3=\Psi^3. \tag{4.19}$$

Case 5: For symmetry

$$\Gamma_5=u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+w\frac{\partial}{\partial w},$$

the conservation laws are determined as follows:

$$\Psi^1=F_1u+vF_2+wF_3,$$
$$\Psi^2=(2F_3uv+2F_2wu)\lambda(t)$$
$$+\Big(\left(w_{xx}+\frac{1}{3}w_{yy}\right)F_3+\left(v_{xx}+\frac{1}{3}v_{yy}\right)F_2$$
$$+\left(u_{xx}+\frac{1}{3}u_{yy}\right)F_1-\frac{1}{3}F_{3y}w_y$$
$$-\frac{1}{3}F_{2y}v_y+\frac{1}{3}uF_{1yy}-\frac{1}{3}F_{1y}u_y$$
$$+\frac{1}{3}wF_{3yy}+\frac{1}{3}vF_{2yy}\Big)\beta(t)+(2\gamma(t)vw$$
$$+2\alpha(t)uv)F_1,$$
$$\Psi^3=-\frac{1}{3}\beta(t)\big(u_xF_{1y}-2u_{xy}F_1+v_xF_{2y}$$
$$-2v_{xy}F_2+w_xF_{3y}-2w_{xy}F_3\big). \tag{4.20}$$

The total divergence condition has the following form:

$$D_t\Psi^1+D_x\Psi^2+D_y\Psi^3=D_x(((\gamma(t)w$$
$$+\alpha(t)u)F_1+uF_3\lambda(t))v+F_2\lambda(t)wu). \tag{4.21}$$

Then the conserved vectors are modified as follows:

$$\tilde{\Psi}^1=\Psi^1,$$
$$\tilde{\Psi}^2=(uF_2w+uF_3v)\lambda(t)$$
$$+\Big(-\frac{1}{3}u_yF_{1y}+u_{xx}F_1+\frac{1}{3}u_{yy}F_1+\frac{1}{3}uF_{1yy}$$
$$+\frac{1}{3}vF_{2yy}-\frac{1}{3}v_yF_{2y}+v_{xx}F_2+\frac{1}{3}v_{yy}F_2$$
$$+\frac{1}{3}wF_{3yy}-\frac{1}{3}w_yF_{3y}+w_{xx}F_3+\frac{1}{3}w_{yy}F_3\Big)\beta(t)$$
$$+uF_1\alpha(t)v+vF_1\gamma(t)w,$$
$$\tilde{\Psi}^3=\Psi^3. \tag{4.22}$$

5. Concluding remarks

In this paper, we have presented an algorithm to find Lie symmetries, similarity solutions and conservation laws for the (2 + 1)-dimensional ncZK system with time-dependent coefficients $\alpha(t)$, $\beta(t)$, $\lambda(t)$, $\gamma(t)$. The solutions are obtained from the reduced ODEs in the form of power series, trigonometric, hyperbolic, Jacobi and Weierstrass functions. The graphical analysis of the solutions reveals bright, dark, singular solitons and periodic profiles. The effect of arbitrary coefficient function $\beta(t)$ on the wave profile of the solutions is successfully represented by 2D and 3D plots. The system is found to possess non-trivial infinitely many conservation laws because of arbitrary functions $F_1(y)$, $F_2(y)$, $F_3(y)$. The authenticity of the solutions and conservation laws is verified by Maple software.

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