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Determination of classical behaviour of the Earth for large quantum numbers using quantum guiding equation

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Abstract. For quantum systems, we expect to see the classical behaviour at the limit of large quantum numbers. Hence, we apply Bohmian approach for describing the evolution of Earth around the Sun. We obtain possible trajectories of the Earth system with different initial conditions which converge to a certain stable orbit, known as the Kepler orbit, after a given time. The trajectories are resulted from the guiding equation $p = \nabla S$ in the Bohmian mechanics, which relates the momentum of the system to the phase part of the wave function. Except at some special situations, Bohmian trajectories are not Newtonian in character. We show that the classic behaviour of the Earth can be interpreted as the consequence of the guiding equation at the limit of large quantum numbers.

Keywords. Bohmian mechanics; quantum trajectories; correspondence principle.

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1. Introduction

Quantum mechanics works exceedingly well in all practical applications. No examples of conflict between its predictions and experiments are known. The main problem arises when we emphasise on macroscopic systems which behave classically. The transition of quantum to classical mechanics is an unsolved fundamental problem for years. Most physicists believe that macroscopic systems are quantum-mechanical in nature. Therefore, the way they behave classically is not clear yet. In the most general sense, the correspondence principle determines the guidelines of scientific theories. However, not only the latest achievements require to make predictions that the earlier studies were inadequate to address, but also they need to confirm the correct predictions of the previous theories [1]. In this issue, the Bohmian mechanics, which comes up with trajectories for quantum systems, can be an appropriate option for the development of other studies describing the microworld.

The Bohmian mechanics is a deterministic and distinctly non-Newtonian reformulation of quantum mechanics, which the wave function itself is responsible for guiding the motion of the particles [2]. Bohmian mechanics is in agreement with quantum mechanics in results, but it proposes a different explanation [3]. Through self-experiences and the classical description of nature, we comprehend macrosystems with trajectories. Therefore, there is no way to understand the quantum–classical transition, until we reach an explanation for classical trajectories based on the quantum description. As Bohmian mechanics keeps the quantum mechanics results based on its casual implementation, it can potentially provide a proper connection between the quantum–classical domain. In this regard, many efforts have been made to extend the Bohmian description to a variety of physical problems such as deriving the relativistic quantum potential [4], following the Bohmian approach in nonlinear Klein–Gordon equation [5], extension of a classical motion of a constrained particle to the quantum domain considering Bohm's view of quantum mechanics [6] and space–time transformation for the propagator in the Bohm theory [7]. It should be noted that the trajectories that the Bohmian mechanics imply are entirely different from the Newtonian trajectories, so that the theory has its particular description [8]. Interestingly, in this regard, periodic orbits have been, since Kepler, considered as the key concepts for describing and understanding the classical dynamics

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which, since Bohr, could be gainful in apprehending the quantum–classical transition [9].

Describing the Sun–Earth dynamics was always major problem in physics [10]. The first illustration of Earth's orbit was computed by Lagrange (1781, 1782), and improved by Pontecoulant (1834), Agassiz (1840) and others. Since then, many works have been carried out to modify the theory and ameliorate the results [11]. In recent times, many quantum research studies have been done to shed light on the problem. For instance, Flöthmann and Welge [9] showed the classical dynamics of electron motion of hydrogen atom under the cross magnetic–electric field, in connection with the gravitational field of the Sun–Earth problem. Also, Battista and Esposito [12] studied the effect of quantum corrections on the Newtonian potential for the evaluation of equilibrium points.

The real issue still has been remained unsolved. However, the dynamics of the famous classical Earth–Sun problem has unanswered questions. Meanwhile, the quantum aspects of the problem have been considered unsolved in general [13–16]. Nevertheless, both quantum and classical features of the model appear in all subfields of physics [17–26]. Regarding the classical Sun–Earth dynamics, many efforts have been made to describe the system in a quantum fashion. Keeports [1] considered the Earth as a quantum object and discussed its quantum properties compared to classical ones. Also, studies on quantum potential correction terms are recently reported [27–29].

In this paper, we are going to present the quantum aspects which are responsible for the Earth to behave classically. In §2, we present the non-relativistic hydrogen-like Hamiltonian for the Sun–Earth system to show that the classical energy of the Earth is dependent upon the principal and the angular momentum quantum numbers n and l, respectively. In §3, we discuss the role of the Bohmian mechanics and its guiding equation in the appearance of the Earth dynamics. Then, in the next section, we show as one of our main results how the high values of magnetic quantum number m affect the possible trajectories of the Earth via the Bohmian guiding equation. Later, we discussed the wave function. Finally, in §6, we addressed the results of our work.

2. Definition of the Earth Hamiltonian

We know the Earth as a classical object. Nonetheless there is a belief that the behaviour of macroscopic systems, even the Earth in the solar gravitational field, is based on their quantum mechanical nature. Therefore, different forms of Hamiltonians for describing the Earth dynamics have been proposed in [1,10,16]. Here, we introduce a Hamiltonian which contains two terms: (1) the hydrogen-like Hamiltonian H_0 containing the kinetic and gravitational potential energy terms and (2) an additional energy term denoted by K:

$$H = H_0 + K, \tag{1}$$

$$H_0 = -\frac{\hbar^2}{2m_\mathrm{e}}\nabla^2 - \frac{Gm_\mathrm{s}m_\mathrm{e}}{r}, \tag{2}$$

where G is the universal gravitational constant, and m_s and m_e are the masses of the Sun and the Earth, respectively.

We included the additional energy term K regarding the derived Hamiltonian from two-body problem post-Newtonian (PN) approximation. Accordingly, r^{-1}, r^{-2} and the lower-order energy terms of r are included in the expressions of 1PN, 2PN and 3PN contributions to the Hamiltonian [30]. Also, as the dynamics here is based on the hydrogen atom two-body problem, relativistic perturbations result in additional r^0, r^{-1} and r^{-2} correction terms to the kinetic energy [31]. Moreover, an ensemble of trajectories, which are solutions to the equations of motions, correspond to a well-known theorem of mechanics [32]. Thus, concerning the recent discussion, we defined the additional r^0, r^{-1} and r^{-2} energy terms, in which the obtained equations of motion are consistent with the classical description of the problem. Evidently, including the exact terms of 1PN, 2PN and 3PN, correction terms in the Hamiltonian lead to a more accurate and also a more complicated solution. It should be noted that here K is not supposed to have a small effect compared to H_0. It should be considered as a real part of the Earth Hamiltonian H, without which a proper description of the Earth's dynamics is not possible, even in a classical-like (Bohmian) approach. We introduce the additional term K in the next section.

Following the Bohmian approach in describing the problem [33], we consider $\psi(x,t) = R\mathrm{e}^{-\mathrm{i}S/\hbar}$ as an eigenfunction of the Hamiltonian. Thus, by dividing the real and the imaginary parts of the Schrödinger equation, one obtains the following equations:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m_\mathrm{e}} - \frac{\hbar^2}{2m_\mathrm{e}}\frac{\nabla^2 R}{R} - \frac{Gm_\mathrm{s}m_\mathrm{e}}{r} + K = 0, \tag{3}$$

$$\frac{\partial R^2}{\partial t} + \nabla\cdot\left(\frac{R^2\nabla S}{m_\mathrm{e}}\right) = 0. \tag{4}$$

Relation (3) is the well-known quantum Hamilton–Jacobi equation. Keeports [1] showed how the hydrogen-like energy of the Earth is related to its classical one. He showed that if the Earth–Sun Hamiltonian is assumed to be similar to the hydrogen atom, regardless of the additional term K, the energy levels could be calculated as $E_n = -G^2m_\mathrm{e}^3m_\mathrm{s}^2/2\hbar^2n^2$. On the other hand, by

Table 1. Important quantities and quantum numbers of the Earth in quantum dynamics.

Symbol	Quantity	Value
r_{eq}	The equilibrium distance between the Earth and the centre of the coordinates	2.116×10^{11} m
a	The Earth's semimajor axis	1.496×10^{11} m
n	Principal quantum number	2.524×10^{74}
l	Azimuthal quantum number	2.524×10^{74}
m	Magnetic quantum number	10^{73}
A	$m^2\hbar^2/m_e^2$	10^{31} m^{4}s^{-2}
b	$\hbar^2/Gm_e^2m_s$	2.348×10^{-138} m

considering the continuous limit of the Earth's classical energy, he obtained

$$E_n = -\frac{G^2 m_e^3 m_s^2}{2\hbar^2 n^2} = -\frac{G m_e^3 m_s^2}{2\hbar^2 l(l+1)}, \tag{5}$$

where n and l are the principal and the azimuthal quantum numbers. As it is clear, he suggested that we should have $n = l \to \infty$. Also, Keeports [1] showed that the wave function of the Earth is the same as that of the hydrogen atom when the quantum numbers n and l tend to large values.

The hydrogen-like Hamiltonian H_0 explains the classical energy of the Earth very well, as we mentioned above. So, it is reasonable to assume that the energy remains unchanged in this new framework. This point is crucial in analysing the role of different terms in the Hamilton–Jacobi equation (3). Moreover, the term $-(\hbar^2/2m_e)(\nabla^2 R/R)$ is negligible due to the large mass of the Earth. Considering the energy relation (5), for Hamiltonian (1) the term $(\nabla S)^2/2m_e + K$ should be equivalent to $(\nabla S)^2/2m_e$ for hydrogen-like Hamiltonian (H_0). Accordingly, we can write the following expression in spherical coordinates:

$$\frac{(\nabla S)^2}{2m_e} + K = \frac{m^2\hbar^2}{2m_e r^2 \sin^2\theta}, \tag{6}$$

where m denotes the so-called magnetic quantum number.

The difference here from the system with hydrogen-like Hamiltonian is that both r and θ are supposed to be time dependent. Thereupon our first assumption here is that the energy of the Earth's wave function is in a form similar to the same system with hydrogen-like Hamiltonian (for hydrogen-like Earth–Sun system, see [1]). The first assumption is an appropriate statement because our explanation should be in complete agreement with the classical results. Hence, we seek for an answer to the Schrödinger equation which achieves energy (5). Furthermore, we discuss the trajectories that resulted from the wave function.

Hence, the study of the Earth in the Bohmian approach necessitates the accordance with quantum mechanical results, reaching the energy relation (5) in the same way. For the hydrogen atom, the gradient of the phase function is $(\nabla S)^2/2m_e = m^2\hbar^2/2m_e r^2 \sin^2\theta$ [3]. Hence, eq. (3) should have the same form as in the hydrogen atom. This condition, however, leads to a new phase function and also provides a different dynamics afterwards.

3. Bohmian dynamics of the Earth

Keeping the energy unchanged as discussed in the previous section and according to eq. (6), K plays an essential role in the determination of the Earth's trajectories. For the reasons discussed earlier, we define

$$K = \frac{A m_e}{2(r^2 - Z_h^2)} - \mu m_e\left(\frac{1}{r} - \frac{1}{2a}\right), \tag{7}$$

where a is the semimajor axis of the Earth (see table 1 for its value) and μ is defined as $\mu = G(m_s + m_e) \simeq G m_s$, A is a constant which will be defined later and Z_h represents the z-axis of the coordinate system as shown in figure 1. Then in (6) we have

$$\frac{(\nabla S)^2}{2m_e} = \frac{m^2\hbar^2}{2m_e r^2 \sin^2\theta} - \frac{A m_e}{2(r^2 - Z_h^2)} + \frac{m_e\mu}{r} - \frac{m_e\mu}{2a}. \tag{8}$$

Let us consider the xy-plane parallel to the plane of the Earth's orbit. The z-axis is then perpendicular to this plane. So, the z-coordinate of the Earth's position must be constant, i.e.

$$r\cos\theta = Z_h, \tag{9}$$

where one can write

$$r^2 \sin^2\theta = r^2 - Z_h^2. \tag{10}$$

According to the negligible changes in radius, r, and polar angle, θ, during the Earth's evolution, Z_h is almost

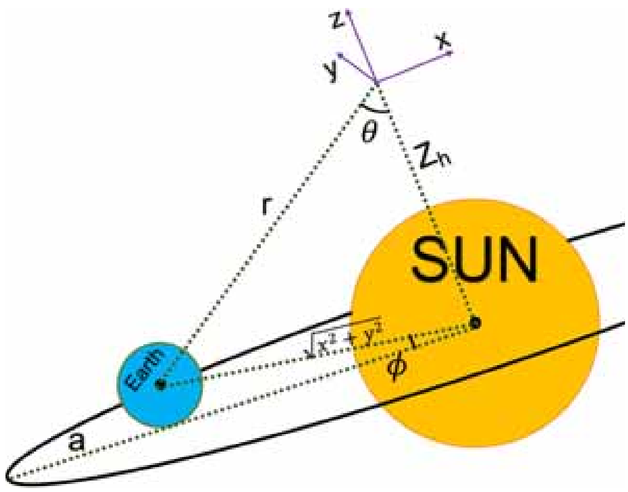

Figure 1. Coordination axes, variables and parameters in the Earth–Sun system.

considered as a constant, as is obvious in figure 1. The value of Z_h is pretty arbitrary as its value does not affect the dynamics. However, theoretically, it can be considered to be zero (the centre of the coordination system could be chosen as the centre of the Sun). Nevertheless, it is still not easy to solve the equations. A reasonable choice for Z_h is a constant in the same order of magnitude of the Earth–Sun distance. In this case (according to figure 1), the radius r would also be in the same order of magnitude ($r = \sqrt{2}(x^2+y^2)^{1/2}$). Thus, Hamiltonian (1) does not change due to the coordinate transformation. We would discuss this choice later by investigating the trajectories. As a suitable choice, we also define the constant A in a way that the first two terms on RHS of (8) cancel out each other. So, we adopt

$$A = \frac{m^2\hbar^2}{m_{\mathrm{e}}^2}. \tag{11}$$

Consequently, in (8) one can show that the Bohmian guiding equation is equal to

$$(\nabla S)^2 = m_{\mathrm{e}}^2\left(\frac{2\mu}{r} - \frac{\mu}{a}\right). \tag{12}$$

As the guiding equation $\nabla S = p$, where p is the linear momentum, we obtain

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}, \tag{13}$$

which is the well-known vis-viva relation in Newtonian mechanics. Regarding eq. (13), velocity only depends on the distance between the Sun and the Earth.

In the Bohmian mechanics, the dynamics of the wave function is defined by the Schrödinger equation and the dynamics of the system is determined by the guiding equation. There is a wrong general belief that the Bohmian mechanics becomes Newtonian for macroscopic systems or for systems with large masses. This misleading concept happens when we ignore the guiding equation $\nabla S = p$. Dürr and Teufel [8] show that for many systems the Bohmian trajectories obtained from the guiding equation are different from the Newtonian ones. The nature of Bohmian trajectories depends on the circumstances that the guiding equation is at work. We shall show that, for the Sun–Earth system, the guiding equation leads to the trajectories which are classical in large values of quantum numbers.

We must emphasise that as long as the wave function satisfies the Schrödinger equation, that is $v = \nabla S/m$, and we have the ensemble with $|\psi|^2$ probability density, then the Bohmian results are quite identical with the one that quantum mechanics proposes [32], as given in this paper. Indeed, we use the results obtained from the standard quantum dynamics [1], to extend them into the Bohmian mechanics as the latter provides trajectories and leads to a better understanding of the problem in the macroscopic regime.

To show that our results here are entirely compatible with the standard quantum mechanics, we need to discuss the three main aspects of the problem: Energy, wave function and the probability density. As we argued before, the dynamics here are based on the energy relation (5), calculated by Keeports [1], that in high values of n are completely compatible with classical results. Moreover, we previously computed the phase of the wave function S using the quantum Hamilton–Jacobi equation (3) regarding the energy relation (5) and the fact that the term $-(\hbar^2/2m_{\mathrm{e}})(\nabla^2 R/R)$ (quantum potential) is negligible because of the mass of the Earth. Further in §5, we shall calculate the amplitude of the wave function R. Therefore, $\psi = R\mathrm{e}^{-\mathrm{i}S/\hbar}$ is the wave function of Hamiltonian (1) with the energy eigenvalues equal to (5) regardless of the Copenhagen or Bohmian interpretation. Furthermore, we discuss in §5 that the spatial probability density of the Earth, $|\psi|^2$, is entirely compatible with our calculated trajectories. In the following, we are going to study the Bohmian solution of the problem and to calculate the system trajectories.

4. Earth's trajectories

The electron in the hydrogen atom has a circular trajectory in the Bohmian mechanics with a constant radius in the xy-plane. Azimuthal angle $\phi(t)$ is the only parameter that changes with time, which has a linear time dependency [3, pp. 148–153]. Similarity of the hydrogen system with the Earth is a good reason to assume a similar relation for $\phi(t)$ in the Earth–Sun case. So, our second assumption here, is to consider the time dependency of the azimuthal angle $\phi(t)$ as the same as the solution of the hydrogen atom in Bohmian mechanics, albeit in its form. However, in this case, the radius r and the polar angle θ are supposed to be time-dependent, contrary to the case of the hydrogen atom. So we assume that

$$\phi(t) = \frac{m\hbar t}{m_e r^2(t)\sin^2\theta(t)} + \phi_0. \tag{14}$$

This is an appropriate assumption according to the classical dynamics of the Earth's motion around the Sun. Accordingly, the time dependence of the azimuthal angle is in the order of t. Relation (14), which is inspired from the dynamics of the hydrogen atom, signifies the role of the magnetic quantum number m. The suggested relation needs to be in accordance with the Schrödinger equation. Furthermore, the important consequence of this choice is the direct effect of the quantum number m on the dynamics of the system. The aforementioned is the main accomplishment of the Bohmian approach which appears in this problem. Our second assumption is not a surprise but is completely understandable. It is clear that the idea comes from the circular movement of the Earth around the Sun, and eq. (14) is the simplest way to describe this movement.

Now, for the angular velocity v_ϕ, we have

$$v_\phi = r\sin\theta\dot{\phi} = \frac{m\hbar}{m_e r^2\sin^2\theta} - \frac{2m\hbar t}{m_e r^3\sin^2\theta}\dot{r} - \frac{2m\hbar t\cos\theta}{m_e r^2\sin^3\theta}\dot{\theta}, \tag{15}$$

where $v^2 = v_r^2 + v_\theta^2 + v_\phi^2$. Also, using eq. (9), one gets

$$v_\theta = r\dot{\theta} = \frac{Z_h\dot{r}}{r\sin\theta}. \tag{16}$$

Using (15), (16) and (13), the time-dependent equation for r is obtained as

$$\left(1 + \frac{4At^2}{r^4\sin^2\theta} + \frac{Z_h^2}{r^2\sin^2\theta} + \frac{4AZ_h^4t^2}{r^8\sin^6\theta} + \frac{8AZ_h^2t^2}{r^6\sin^4\theta}\right)\dot{r}^2 - \left(\frac{4At}{r^3\sin^2\theta} + \frac{4AZ_h^2t}{r^5\sin^4\theta}\right)\dot{r} + \frac{A}{r^2\sin^2\theta} - \frac{2\mu}{r} + \frac{\mu}{a} = 0. \tag{17}$$

Here, we need a solution for the radial velocity $\dot{r}$ to show the trajectories. To do this, we should know the magnitude of each term to make some approximations.

The order of magnitude of the constant A depends on the Earth's mass, Planck constant and the magnetic quantum number m. Therefore, its order must be determined owing to the Earth's dynamics. As we know, it takes one year for the Earth to revolve around the Sun. So, for the azimuthal angle $\phi(t)$, we have $\phi(\tau + 1\ \text{year}) = \phi(\tau) + 2\pi$. The radial time dependency of the Earth's motion is negligible, and its value remains nearly unchanged. Then, according to the definition of $\phi(t)$ in (14), we can estimate the magnitude of the magnetic quantum number $m \approx 10^{73}$. As we expect, the Earth's dynamics appears in the large values of quantum numbers l and m.

Thus, from (11) we have $A \approx 10^{31}\ \text{m}^4\,\text{s}^{-2}$ and due to eq. (9), the constant Z_h should be in the order of the Earth–Sun distance. Now, let us assume that

$$\dot{r} = \xi\sin\theta\sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}, \tag{18}$$

where ξ is a dimensionless constant which will be determined later. Then, for eq. (17), we have

$$\overbrace{\left(1 + \frac{4AZ_h^2t^2}{r^6\sin^4\theta}(1+\sin^2\theta) + \frac{4At^2}{r^4}\right)}^{\alpha}\left(\frac{2\mu}{r} - \frac{\mu}{a}\right)\xi^2 - \frac{A\xi}{r^2\sin^2\theta}\underbrace{\frac{4t}{r\sin\theta}\sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}}_{\beta} + \frac{A}{r^2\sin^2\theta} - \left(\frac{2\mu}{r} - \frac{\mu}{a}\right) = 0. \tag{19}$$

With respect to the Earth–Sun distance, which is approximately constant, and the fact that we are seeking the trajectories for the time domain of 10^7–10^8 s magnitude, the coefficients α and β are almost constant (about 10^1) during the revolution of the Earth. Moreover, ξ is assumed to be constant due to the limited time domain that we supposed here. For long time variations, ξ is time-dependent. But then eq. (19) cannot be solved rigorously.

Regarding eq. (18), the radial velocity $\dot{r}$ changes with time due to the time dependency of $r(t)$ and $\theta(t)$. There is no analytical solution for this kind of nonlinear differential equation. Yet, by applying some physical and mathematical assumptions, eq. (18) can give us proper trajectories, which are appropriate at 1 year time domain. The method is used here to obtain the following solution:

$$r(t) = \xi\frac{\sqrt{F^2(t) - \sqrt{F^2(t)(F^2(t) - 4Z_h^2)}}}{\sqrt{2}}, \tag{20}$$

$$F(t) = \frac{-B_2 + C\tan[\frac{1}{2}C(t+\tau)]}{B_3}. \tag{21}$$

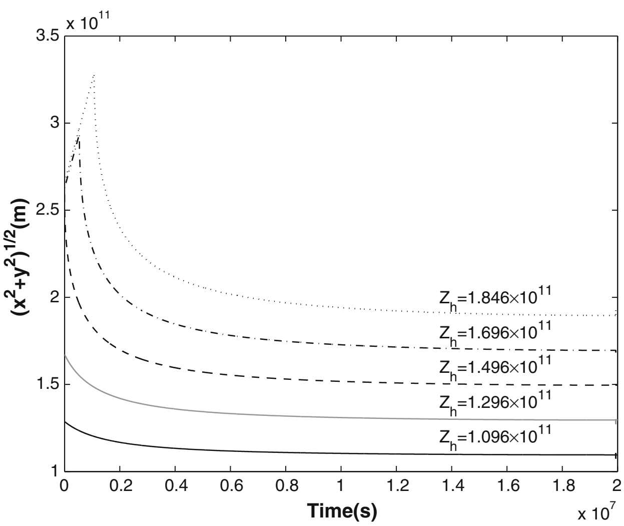

Figure 2. The Earth–Sun distance vs. time for different values of the constant Z_h and the typical value of $\xi = 1.414$.

In (21) we define

$$B_1 := B - \frac{\mu^2}{8B^3 r_{\mathrm{eq}}^2} + \frac{\mu}{2Br_{\mathrm{eq}}}, \quad B_3 := \frac{\mu^2}{8B^3 r_{\mathrm{eq}}^4},$$

$$B_2 := \frac{\mu^2}{4B^3 r_{\mathrm{eq}}^3} - \frac{\mu}{2Br_{\mathrm{eq}}^2}, \quad C := \sqrt{-B_2^2 + 4B_1 B_3},$$

$$B := \sqrt{\frac{2\mu}{r_{\mathrm{eq}}} - \frac{\mu}{a}}, \tag{22}$$

where τ is a constant with time dimension. The details of calculating (20) from (18) are given in the Appendix.

According to eqs (9), (14) and (20) we know how $\phi(t)$, $\theta(t)$ and $r(t)$ vary with time. Hence, it is now possible to draw the trajectories. However, some points should be made clear first. If we take a closer look at (21), we shall find a periodic 'tangent' function with singularity points which means that our trajectories become discrete in these points. So, there is no continuous path prediction for all definite times $0 \leq t < \infty$ in this model. We can only follow and draw the trajectories in limited time domains, e.g. for 1 year (10^7–10^8 s). In each time interval (which can be extended even to several years), we can obtain closed cycles, nearly demonstrating the Earth orbit around the Sun.

Equation (20) shows us that r decreases while time goes forward, then it reaches an equilibrium value (r_{eq}). Such a behaviour is shown in figure 2. The centre of coordinates here is not located on the Sun, but is displaced by Z_h. Hence, the radius r is equal to $\sqrt{x^2 + y^2 + Z_h^2}$. Nevertheless, when we speak about the Earth–Sun distance, we mean $\sqrt{x^2 + y^2}$, and as discussed earlier, Z_h, r and Earth–Sun distance are in the same order of magnitude. As is shown in figure 1, Z_h is just a displacement in the centre of coordinates, and the choice of its value is arbitrary due to the method of solution. According to figure 2, irrespective of the initial conditions, the Earth–Sun distance decreases rapidly and tends to an equilibrium value. In other words, different trajectories generated from different initial conditions converge to a particular equilibrium situation when sufficient time is passed.

As we mentioned before, the trajectory equation (20) is not the exact answer of the differential equation (19), though by an accurate choice for the values of Z_h and ξ, which could be made by the classical data, eq. (20) would be practically an appropriate solution for the Earth trajectory in the Bohmian framework. For $Z_h = 1.496 \times 10^{11}$ m, $\phi_0 = 0$ and $\tau = 0$, as the main initial conditions, figure 3 shows how the Earth trajectory finally approaches a stable closed cycle around the Sun with an equilibrium distance 1.496×10^{11} m ($r_{\mathrm{eq}} = \sqrt{2} \times 1.496 \times 10^{11}$). We can assume that $\phi(t) = 2\pi t$. Then, for the x and y components of the velocity, we have

$$\dot{x} = \frac{\sqrt{x^2+y^2}}{x^2+y^2+Z_h^2} \times \left(1 + \frac{Z_h^2}{x^2+y^2}x\right)\sqrt{\frac{2\mu}{\sqrt{x^2+y^2+Z_h^2}} - \frac{\mu}{a}} - 2\pi y, \tag{23}$$

$$\dot{y} = \frac{\sqrt{x^2+y^2}}{x^2+y^2+Z_h^2} \times \left(1 + \frac{Z_h^2}{x^2+y^2}y\right)\sqrt{\frac{2\mu}{\sqrt{x^2+y^2+Z_h^2}} - \frac{\mu}{a}} + 2\pi x. \tag{24}$$

Figure 4 shows the stream of the vector field $(\dot{x}, \dot{y})$ which is described by eqs (23) and (24). As figure 4 indicates, the final equilibrium dynamics of the system is independent of initial conditions.

5. The wave function

So far we have discussed the Earth's dynamics and its trajectory. Here, we are going to investigate the Earth wave function. Let us note the time-independent amplitude of the wave function, $R(r, \theta)$. Considering eq. (4) we have

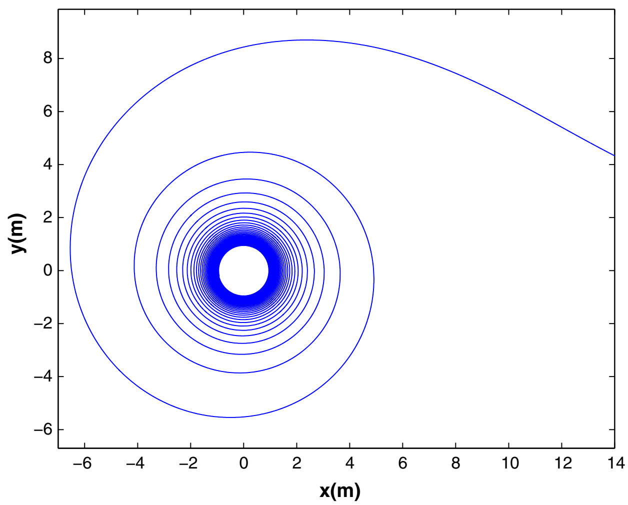

Figure 3. The Earth's trajectory around the Sun. After sufficient time (at the order of 10^7 s), the Earth–Sun distance tends to its classical equilibrium value of 1.496×10^{11} m. The axes are 10^{11} times smaller.

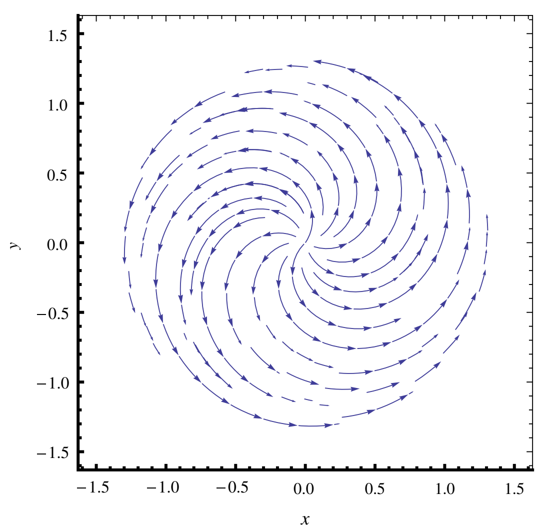

Figure 4. The stream of Earth's vector field $(\dot{x}, \dot{y})$ as a function of x and y coordinates in its final form.

$$R^2\nabla \cdot \nabla S + \nabla S \cdot \nabla R^2 = 0. \tag{25}$$

In spherical coordinates, one can write ∇S as

$$\nabla S = m_e \dot{r} \vec{\rho}_r + m_e \cot\theta \dot{r} \vec{\rho}_\theta + \frac{\sqrt{A}}{r\sin\theta}\left(1 - \frac{2\dot{r}t}{r} - \frac{2\dot{r}\cos^2\theta t}{r\sin^3\theta}\right)\vec{\rho}_\phi, \tag{26}$$

where $\dot{r}$ is defined in (18). If we consider R as $R(r,\theta) = \Re(r)\Theta(\theta)$, similar to the hydrogen-like wave function, with large values of l and m quantum numbers, one can write $\Theta(\theta) \propto \sin^m\theta$. According to $\Theta(\theta)$ relation, the function Θ^2 as a probability distribution is approximately either one for $\theta = \pi/2$ or zero for the other values of θ due to the large value of m. On the one hand, for $\Theta^2 = 0$ relation (25) is simply valid, but on the other hand, for $\Theta^2 = 1$ we have $R^2(r,\theta) \simeq \Re^2(r)$ and the validity of (25) should be studied. By substituting ∇S from (26) in (25) and after some algebraic calculations, one can show that

$$\frac{\partial \Re^2}{\partial r} = \left(\frac{a}{2ar - r^2} - \frac{1}{r}\right)\Re^2. \tag{27}$$

Equation (27) imposes constraints on the amplitude of the wave function $R(r,\theta)$. Inspired by the hydrogen atom wave function we assume that $\Re^2(r)$ is in the following form:

$$\Re^2(r) = f(r)\mathrm{e}^{-2r/bn}, \tag{28}$$

where $b = \hbar^2/Gm_e^2 m_s$ and $f(r) = \sum_i^{n-1} C_i r^i$. Accordingly, regarding the large value of n and the fact that a is in the same order of magnitude as r, by substituting (28) into (27) one can conclude that

$$R^2(r,\theta) = \Re^2(r)\Theta^2(\theta) = C\frac{2^{n+2}}{n^{n+2}\Gamma(n+2)b^{n-1}} r^{n-1}\mathrm{e}^{-2r/bn}\sin^m\theta, \tag{29}$$

where $C = b^{-3}$ is the normalisation factor and for positive integers $\Gamma(m) = (m-1)!$. The radial probability density $r^2\Re(r)^2$ is a Gaussian-like distribution. We see that for higher values of the principal quantum number n, the function is more sharp around the most probable area with an exponential growth as shown in figure 5. By maximising the probability density, one gets $r_{mp} = n(n+1)b/2$ as the most probable radius. Applying b and equivalating the most probable radius with r_{eq} from table 1, we obtain $n = 2.524 \times 10^{74}$. As the quantum number n is quite large, the normalised probability distribution $r^2\Re^2(r)$ is perfectly sharp in r_{mp}. Accordingly, we can write $r^2\Re^2(r) = \delta(r - r_{mp})$. The provided trajectories are in complete agreement with the probability density as discussed earlier. It has to be noted that the dynamics are settled regarding the conservation of total probabilities, if all trajectories exist for all times.

Here the wave function is an energy eigenstate as discussed before. However, with large values of n, the differences between the energy levels are negligible, and the energy becomes continuous. Therefore, although the wave function is an energy eigenstate, the energy values are continual, which is in agreement with classical results.

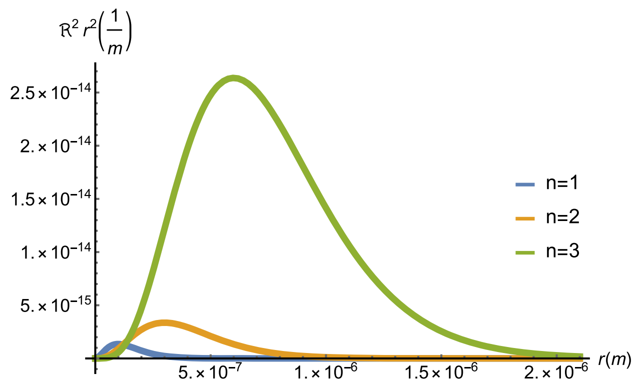

Figure 5. Radial probability distribution $r^2\Re^2$ vs. radius r for $n = 1$–3 for a typical value of $b = 1 \times 10^{-7}$ m. For higher principal quantum number values, the probability density is more sharp around the most probable range. For values of n and b mentioned in table 1, the radial probability distribution tends to the Dirac delta function, i.e. $\delta(r - r_{\rm mp})$.

6. Conclusion

Here, we have considered the Earth as a quantum object to show that the quantum guiding equation makes it behave classically in the limit of large quantum numbers. Through the quantum formalism, the Earth's dynamics is determined by the Schrödinger equation. By describing the Sun–Earth system as a model of the hydrogen atom, we introduced a new Hamiltonian with an additional kinetic energy term K. To obtain the predicted quantum trajectory of the Earth, we used the guiding equation $p = \nabla S$ (see (12)) in the Bohmian regime.

Astonishingly, we see that the large quantum numbers directly ascertain the Earth's dynamics. As Keeports [1] has already shown, at large quantum numbers n and l, the classic energy of the Earth is obtained. Also, we showed here that the Earth's dynamics depends on the high values of the quantum number m which affect the Earth's trajectory via the guiding equation (12). The investigations led us to the well-known Newton's vis-viva equation (13) as the Earth's velocity, which leads to acceptable Bohmian trajectories, after some reasonable approximations.

Interestingly, the main result is independent of the initial conditions. The rotation distance of the Earth–Sun system decreases rapidly and tends to an equilibrium one. All trajectories with different initial conditions approach a stable closed cycle, illustrating how the Earth orbits around the Sun with negligible deviation. As a macrosystem, this is a magnificent achievement to approximately obtain the Earth trajectory in the Bohmian framework, which enables us to see quantum footprints in a classical domain.

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Appendix A

The differential equation (18) can be solved in any time domain in which ξ is considered as a constant. One can write this equation as

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \xi \sin\theta \sqrt{\frac{2\mu}{r(t)} - \frac{\mu}{a}}, \tag{A.1}$$

where both θ and r are time-dependent, but have no dependency on each other. For θ we have from (9)

$$\sin\theta = \sqrt{1 - \frac{Z_h^2}{r^2(t)}}. \tag{A.2}$$

During the Earth's rotation around the Sun, the time variations of r are insignificant. With regard to (A.2) and noticing that the constant Z_h has the same order of magnitude as r, one can neglect the time dependency of θ too. Therefore, to solve eq. (A.1), we can consider the term $\xi \sin\theta$ as a constant.

Define the variable q as

$$q = r - r_{\rm eq}, \tag{A.3}$$

where $r_{\rm eq}$ represents the equilibrium distance, and eq. (A.1) yields

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \xi \sin\theta \sqrt{\frac{2\mu}{r_{\rm eq}(\frac{q}{r_{\rm eq}} + 1)} - \frac{\mu}{a}}. \tag{A.4}$$

Hence, the variable q represents the deviation of r from the equilibrium distance $r_{\rm eq} \simeq a$. The term $q/r_{\rm eq}$ is small, so that one can expand $(1 + q/r_{\rm eq})^{-1}$ to obtain

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \xi \sin\theta \sqrt{\frac{\mu}{r_{\rm eq}}\left(2 - \frac{q}{r_{\rm eq}}\right) - \frac{\mu}{a}}. \tag{A.5}$$

By expanding the radical term, one finally gets

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \xi \sin\theta \left[B - \frac{\mu}{2Br_{\rm eq}^2}q - \frac{\mu^2}{8B^3 r_{\rm eq}^4}q^2\right], \tag{A.6}$$

where B is already defined in eq. (22). Equation (A.6) is a linear differential equation which can be solved easily. Consequently, eq. (20) is obtained as an answer from (A.6), followed by relations (21) and (22) as definitions in (20).

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