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Quantum Hamilton–Jacobi route to exceptional Laguerre polynomials and the corresponding rational potentials

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Abstract. A method to construct the multi-indexed exceptional Laguerre polynomials using the isospectral deformation technique and quantum Hamilton–Jacobi (QHJ) formalism is presented. For a given potential, the singularity structure of the quantum momentum function, defined within the QHJ formalism, allows us to find its solutions. We show that this singularity structure can be exploited to construct the generalised superpotentials, which lead to rational potentials with exceptional polynomials as solutions. We explicitly construct such rational extensions of the radial oscillator and their solutions, which involve exceptional Laguerre orthogonal polynomials having two indices. The weight functions of these polynomials are also presented. We also discuss the possibility of constructing more rational potentials with interesting solutions.

Keywords. Exceptional orthogonal polynomials; exactly solvable models; rational potentials; shape invariance; isospectral deformation; quantum Hamilton–Jacobi formalism.

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1. Introduction

The previous decade saw a lot of interesting work in the areas related to Sturm–Loiville's theory and orthogonal polynomials owing to the discovery of exceptional orthogonal polynomials (EOPs) [1]. The subsequent construction of rational potentials with these polynomials as solutions [2] has led to a renewed interest in exactly solvable quantum mechanics. Various studies delved into the different aspects of these EOP systems like their classification [3], spectral analysis [4], the structure of zeros and other interesting properties [5–9]. Simultaneously, different methods were developed to construct these polynomials and rational potentials [10–21]. Over the years, these new polynomials appeared in connection with nonlinear oscillators [22,23] and superintegrability [24]. They can also be found in the context of quantum information theory [25], discrete quantum mechanics [26] and the Schrödinger equation with position-dependent mass and other studies [27–29].

Among the well-studied polynomials in the exceptional class, we have generalised families of exceptional Hermite, three families of exceptional Laguerre and two families of exceptional Jacobi polynomials [5–17]. These polynomials are characterised by the codimension index m, which gives the number of gaps in the sequence of polynomials and can take values $1, 2, 3, \dots$. Therefore, X_m EOP sequence implies a polynomial sequence with m number of gaps and different m gives a different order of exceptional polynomials. For the review of these EOPs and exactly solvable quantum mechanics, we refer to [30].

Another significant development has been the construction of multi-indexed EOPs using methods like Krein–Adler transformations, multistep Darboux transformation and higher-order supersymmetry (SUSY) [31–34]. The explicit construction of two indexed Laguerre polynomials has been presented in [33]. These polynomials have a complex structure and are currently being studied [31,34]. The construction of these generalised families of polynomials is more complicated owing to the weight regularity problem [35]. The existence of the EOPs has extended the class of orthogonal polynomials and widened the scope of the Bochner's theorem [3,36] on classical orthogonal polynomials (COPs) [37,38]. The differential equations of the EOPs have rational coefficients and the polynomials form a complete set with respect to a rational weight function. Therefore, for a well-defined spectral problem, these weight functions should be well behaved and should not have any singularities in the domain of orthonormality.

In a recent paper [21], we used the technique of isospectral deformation [18,39] to construct rational potentials having X_m EOPs as solutions. In order to apply the same technique to construct potentials with multi-indexed EOPs as solutions, more information is required.

In the present study, we couple this method with the simple, yet powerful techniques of the quantum Hamilton–Jacobi formalism (QHJF) to obtain the additional information required to construct rational potentials having multi-indexed EOPs in their solutions. Within the framework of the QHJF, the singularity structure of the exactly solvable (ES) potentials and the corresponding quantum momentum functions (QMFs) are well understood. To obtain the required results, we exploit the facts that (i) for the zero energy state, the superpotential is equivalent to the QMF of the given potential, (ii) for the nth energy state, the moving poles of the QMF correspond to the zeros of the wave function and (iii) the fixed poles of the QMF correspond to the singularities of the potential.

Here, for a given ES potential using isospectral deformation, we construct a generalised superpotential such that one of the partners is the original potential and the other is its rational extension. The complete form of the new superpotential is obtained by doing the singularity structure analysis. This process gives us all the possible generalised superpotentials associated with the original potential. We use these to construct the rational extensions of the potential and analyse their solutions. For demonstration, we consider the radial oscillator potential and explicitly construct the rational potentials with solutions in terms of the two indexed exceptional Laguerre polynomials. We also discuss the construction of a hierarchy of rational potentials with EOPs as solutions, by iteratively applying this method.

The presentation of the paper is as follows. In §2, we give a brief introduction to the supersymmetric quantum mechanics (SUSYQM), followed by a description of isospectral deformation technique for the first and the second iterations in §3. In §4, we summarise the QHJF and its connection with the SUSYQM. In §5, we discuss the construction of the rational potentials. In §5.1, we summarise our results on the three generalised families of rational potentials associated with the radial oscillator and their solutions involving X_m exceptional Laguerre polynomials. In §5.2, we perform the second iteration explicitly and construct rational potentials and their solutions involving two indexed EOPs. This is followed by a discussion of the results in §6 and concluding remarks in §7. For convenience, we have put $\hbar = 2m = 1$ throughout this paper.

2. Supersymmetric quantum mechanics

In SUSYQM [39,40], we have a pair of supersymmetric partner potentials

$$V^{\pm}(x) = W^2(x) \pm \partial_x W(x), \tag{1}$$

where

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_0^-(x) \tag{2}$$

is the superpotential associated with the pair. Here $\psi_0^-(x)$ is the ground-state wave function of $V^-(x)$. If $\psi_0^-(x)$ is normalisable with $E_0^- = 0$ and $A\psi_0^-(x) = 0$, then SUSY is known to be exact between the corresponding partners. In this case, $\psi_0^+(x)$ is non-normalisable and the partners are isospectral except for the ground states. In this case the wave functions, $\psi_n^{\pm}(x)$, of the partners are related by

$$\psi_{n+1}^-(x) = A^{\dagger}\psi_n^+(x), \quad \psi_n^+(x) = A\psi_{n+1}^-(x), \tag{3}$$

where $n = 0, 1, 2, \ldots$ and

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x), \quad A^{\dagger} = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \tag{4}$$

are the intertwining operators. For all the ES models, the wave functions are of the form

$$\psi_n^-(x) = \psi_0^-(x)P_n(x), \tag{5}$$

where $P_n(x)$ is a COP. Using (2), we obtain

$$\psi_n^-(x) = \exp\left(-\int W(x)\,\mathrm{d}x\right)P_n(x). \tag{6}$$

In contrast, SUSY is said to be broken if $W(x)$ leads to non-normalisable ground states for both the partners. Here, both the partners are isospectral including the ground states. The relation between the eigenfunctions of the partners is

$$\psi_n^-(x) = A^{\dagger}\psi_n^+(x), \quad \psi_n^+(x) = A\psi_n^-(x). \tag{7}$$

Using the above equations, we can construct an hierarchy of ES potentials and their solutions from $V^-(x)$ and its solutions.

2.1 *Shape invariance*

The partners $V^{\pm}(x)$ are known to be translationally shape invariant potentials (SIPs) if

$$V^+(x, a_0) = V^-(x, a_1) + R(a_0), \tag{8}$$

where a_0 is the potential parameter with a_1 and $R(a_0)$ being the functions of a_0. In these cases, the solutions of $V^+(x, a_0)$ can be obtained in a simple way using

$$\psi_n^+(x) = \psi_n^-(x, a_0 \to a_1), \tag{9}$$

without having to use the intertwining operators.

It is also well known that a given potential can have more than one superpotential associated with it. Each superpotential, $W_i(x)$, leads to partner potentials $V_i^{\pm}(x)$, where every $V_i^-(x) = V^-(x) + C_i$ with C_i being the constants in terms of the potential parameters. SUSY may be exact or broken between the partners depending on whether $W_i(x)$ leads to normalisable ground state or not. Moreover, the solutions of all the partners, $V_i^+(x)$, associated with $V_i^-(x)$ can be obtained from (9), where $\psi_n^-(x)$ is substituted from (6) with suitable values of a_1 in each case. In the discussion of the rational potentials and the EOPs, various superpotentials associated with $V^-(x)$, the corresponding partners and their solutions play crucial roles in the isospectral deformation method.

3. Isospectral deformation

Given the supersymmetric partners $V_i^{\pm}(x)$ and the corresponding superpotential $W_i(x)$, we construct a general superpotential

$$\tilde{W}_i(x) = W_i(x) + \phi_1(x), \tag{10}$$

by demanding

$$\tilde{V}_i^+(x) = V_i^+(x) + R_1, \tag{11}$$

where R_1 is a constant to be determined. The partner potentials associated with $\tilde{W}(x)$ are

$$\tilde{V}_i^{\pm}(x) = \tilde{W}_i^2(x) \pm \partial_x \tilde{W}_i(x). \tag{12}$$

We can determine $\phi_1(x)$ by writing (11) in terms of the two superpotentials,

$$\tilde{W}_i^2(x) + \partial_x \tilde{W}_i(x) = W_i^2(x) + \partial_x W_i(x) + R_1. \tag{13}$$

Making use of (10) gives

$$\phi_1^2(x) + 2W_i(x)\phi_1(x) + \partial_x \phi_1(x) - R_1 = 0, \tag{14}$$

which is nothing but a Riccati equation. Linearising it using the Kole–Hopf transformation,

$$\phi_1(x) = \frac{\partial_x P(x)}{P(x)}, \tag{15}$$

gives the second-order differential equation

$$\partial_x^2 P(x) + 2W_i(x)\partial_x P(x) - R_1 P(x) = 0. \tag{16}$$

For all the ES models studied, by demanding that $P(x)$ be an mth degree polynomial, with $m = 0, 1, 2, \ldots$, a suitable point canonical transformation (PCT) reduces (16) to a second-order differential equation of one of the COP, with a condition on R_1. Thus, $P(x)$ coincides with a COP denoted by $P_m^{\alpha_i}(r)$, where the index i in α_i gives correspondence to the index i in $W_i(r)$. Thus, we obtain

$$\tilde{W}_i(x) = W_i(x) + \frac{\partial_x P_m^{\alpha_i}(x)}{P_m^{\alpha_i}(x)}. \tag{17}$$

By substituting different $W_i(x)$ associated with $V^-(x)$ in (16), we obtain different $P_m^{\alpha_i}(x)$ leading to different $\tilde{W}_i(x)$.

3.1 *Construction of rational potentials and their solutions*

The partners $\tilde{V}_i^{\pm}(x)$ associated with $\tilde{W}_i(x)$ can be constructed using

$$\tilde{V}_i^{\pm}(x) = \tilde{W}_i(x) \pm \partial_x \tilde{W}_i(x). \tag{18}$$

From eq. (11), $\tilde{V}_i^+(x)$ is the same as $V_i^+(x)$, but shifted by R_1, but its partner $\tilde{V}_i^-(x)$ turns out to be a distinct new potential of the form

$$\tilde{V}_i^-(x) = V_i^-(x) - 2\partial_x \phi_1(x) + R_1. \tag{19}$$

This, in terms of $P_m^{\alpha_i}(x)$, becomes

$$\tilde{V}_i^-(x) = V_i^-(x) - 2\partial_x \left(\frac{\partial_x P_m^{\alpha_i}(x)}{P_m^{\alpha_i}(x)} \right) + R_1. \tag{20}$$

From the above equations, it is clear that for each value of m, we get a different family of potentials and these $\tilde{V}_i^-(x)$ are rational extensions of the original potentials $V_i^-(x)$. We call these the first generation rational potentials. With the limit $m \to 0$, $\tilde{W}_i(x) \to W_i(x)$ and $\tilde{V}_i^-(x) \to V_i^-(x)$, because for all COP sequences $P_0^{\alpha_i}(x) = 1$. The available supersymmetry machinery allows us to construct eigenvalues and eigenfunctions for $\tilde{V}_i^-(x)$, without having to solve the Schrödinger equation. $\tilde{V}^+(x) \equiv V^+(x)$ implies that their solutions are also equivalent. Therefore, the nth excited state of $\tilde{V}^+(r)$ is

$$\tilde{\psi}_n^+(x) = \psi_n^+(x), \tag{21}$$

apart from a normalisation constant. We make use of the intertwining operators

$$\tilde{A} = \frac{\mathrm{d}}{\mathrm{d}x} + \tilde{W}(x), \quad \tilde{A}^{\dagger} = -\frac{\mathrm{d}}{\mathrm{d}x} + \tilde{W}(x), \tag{22}$$

defined for $\tilde{W}(r)$ and $\tilde{V}^{\pm}(r)$ to construct $\tilde{\psi}_n^-(x)$. Using (7), we have

$$\tilde{\psi}_n^-(x) = \tilde{A}^\dagger \tilde{\psi}_n^+(x), \quad n = 1, 2, \ldots. \tag{23}$$

For all the rational extensions constructed, the wave functions are of the form

$$\tilde{\psi}_n^-(x) = \frac{\psi_0^-(x)}{P_m^{\alpha_i}(x)} \mathcal{P}_{m,n}^{\alpha_i}(x), \tag{24}$$

where $\mathcal{P}_{m,n}^{\alpha_i}(x)$ are the X_m EOPs of codimension m, the degree of $P_m^{\alpha_i}(x)$. Here $\psi_0^-(x)$ is the normalisable ground state of $V^-(x)$. From the above equation it can be clearly seen that for $\tilde{\psi}_n^-(x)$ to be an acceptable wave function, $P_m^{\alpha_i}(x)$ should not have any zeros in the domain of orthonormality. The choice of suitable $W_i(x)$ in (16) takes care of this.

The eigenvalues of $\tilde{V}^-(x)$ can also be obtained in a simple way. From (11), it is obvious that $\tilde{E}_n^+ = E_n^+ + R_1$ and as $\tilde{V}^\pm(x)$ are strictly isospectral, the eigenvalues of $\tilde{V}^-(x)$ will be

$$\tilde{E}_n^- = E_n^+ + R_1. \tag{25}$$

In [21], we constructed different generalised rational potentials and their solutions for the radial oscillator and trigonometric Pöshl–Teller potentials using this method. This can be used to rationally extend all known shape-invariant potentials in one dimension and show that their solutions are in terms of the single indexed EOPs.

3.2 *Second iteration of isospectral deformation*

For the second iteration of isopectral deformation, we begin with $\tilde{W}_i(x)$ and construct a new generalised superpotential

$$\bar{W}_i(x) = \tilde{W}_i(x) + \phi_2(x), \tag{26}$$

by demanding that

$$\bar{V}^-(x) = \tilde{V}^-(x) + R_2. \tag{27}$$

Here the constant R_2 and $\phi_2(x)$ need to be determined. Proceeding as in the first iteration, the equation for $\phi_2(x)$ turns out to be

$$\phi_2^2(x) + 2\tilde{W}_i(x)\phi_2(x) - \partial_r\phi_2(x) - R_2 = 0, \tag{28}$$

which again is a Riccati equation for each $\tilde{W}_i(x)$. The simple Kole–Hopf transformation, used in the first iteration, is not sufficient to give us the complete structure of $\phi_2(x)$. We show that the singularity structure analysis followed within the QHJF provides the necessary inputs to fix $\phi_2(x)$. Once $\phi_2(x)$ is determined, we can construct $\bar{W}_i(x)$, which in turn can be used to construct the partners $\bar{V}_i^\pm(x)$. Again from eq. (27), $\bar{V}_i^-(x)$ are the same as $\tilde{V}_i^-(x)$, but $\bar{V}_i^+(x)$ will be distinct potentials of the form

$$\bar{V}_i^+(x) = \tilde{V}_i^+(x) + 2\partial_x\phi_2(x) + R_2, \tag{29}$$

which is a rational extension of $\tilde{V}_i^+(x)$. Using (11), we can write

$$\bar{V}_i^+(x) = V_i^+(x) + 2\partial_x\phi_2(x) + R_2 + R_1, \tag{30}$$

which shows that it is a rational extension of $V_i^+(r)$, obtained after a second iteration of isospectral deformation. We show that for these extensions, the rational terms are a combination of logarithmic derivatives of the COPs and X_m EOPs. As in the first iteration, we can make use of the fact that $\bar{\psi}_n^-(x) = \tilde{\psi}_n^-(x)$ apart from a normalisation constant and construct the solutions of $\bar{V}_i^+(x)$ using

$$\bar{\psi}_n^+(x) = \bar{A}\bar{\psi}_n^-(x), \tag{31}$$

where

$$\bar{A} = \frac{\mathrm{d}}{\mathrm{d}x} + \bar{W}(x), \quad \bar{A}^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + \bar{W}(x) \tag{32}$$

are the intertwining operators connecting the solutions of $\bar{V}^\pm(x)$. Substituting (24) into (31), we obtain eigenfunctions of the form

$$\bar{\psi}_n^+(x) = \frac{\psi_0^-(x)}{\mathcal{P}_{m,n'}^{\alpha_i}(x)} \mathcal{Q}_N(x) \tag{33}$$

and we show that $\mathcal{Q}_N(x)$ is an EOP with two indices. It should be noted that for the eigenfunctions to be well behaved, the polynomial appearing in the denominator should not have any zeros in the orthonormality interval. In the next section, we show that the singularity structure analysis of $\bar{W}(x)$ ensures such behaviour. The eigenvalues in this case will be

$$\bar{E}_n^+ = \tilde{E}_n^- + R_2. \tag{34}$$

Using (25), we obtain

$$\bar{E}_n^+ = E_n^- + R_1 + R_2. \tag{35}$$

From the above discussion, it is clear that we can further rationally extend $\bar{V}^+(x)$ and, in fact, continue to repeat the process and construct a hierarchy of rational potentials with solutions involving multi-indexed EOPs. This method allows us to obtain explicit expressions for all these potentials and the multi-indexed EOPs in each iteration easily. Before proceeding to the radial oscillator problem, we give a brief description of the QHJF.

4. The quantum Hamilton–Jacobi (QHJ) connection

In the QHJF, the singularity structure analysis of the QMF in the complex plane allows us to calculate the eigenvalues and eigenfunctions for a given potential $V^-(x)$ [41–43]. The QMF is defined as

$$q(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log \psi_m(x), \tag{36}$$

where $\psi_m(x)$ is the mth excited state of the potential $V^-(x)$ with eigenvalue E_m. Comparing with (2), we can straight away see that

$$\lim_{m\to 0} q(x) \to -W(x). \tag{37}$$

For a potential, the QMF again is not unique and different QMFs, $q_i(x)$, will lead to different $W_i(x)$. Among the different QMFs only one will lead to physically acceptable solutions. The QMF consists of fixed and moving singularities. The knowledge of these singularities and their residues allows us to write the QMF in a meromorphic form, in terms of its singular and analytical parts. The location of the fixed singularities can be obtained from the QHJ equation, a Riccati equation,

$$q^2(x) + q'(x) + E - V^-(x) = 0, \tag{38}$$

and coincides with the singularities of the potential. Note that substitution of (36) into (38) gives the Schrödinger equation for the given potential. The moving singularities are first-order poles. For all ES models, it has been seen that the point at infinity is an isolated singularity, implying that the QMF has finite number of moving poles in the complex domain. As seen from (36), these correspond to the nodes of the mth excited state. The quadratic nature of the QHJ equation (38), results in the residues at all the poles being dual valued. Different residue combinations give rise to different QMFs. For all the conventional ES models, the choice of residue at m moving poles turns out to be unity. For all these models, the QMF can be cast in the form

$$q_i(x) = Q_i(x) + \frac{\partial_x P_m(x)}{P_m(x)}. \tag{39}$$

Here

$$\frac{\partial_x P_m(x)}{P_m(x)} = \sum_{i=0}^{m} \frac{1}{x - x_i}$$

is the sum of all the principal parts of the individual Laurent expansions of $q_i(x)$ around m moving poles. Similarly, $Q_i(x)$ is the sum of all the principal parts of the individual Laurent expansions of $q_i(x)$ around each fixed pole, plus its behaviour at infinity. Different combinations of residues at the fixed poles and at the isolated singularity at infinity will lead to different $Q_i(x)$ and hence different QMFs, $q_i(x)$. From (37) and (39), we can see that with the limit $m \to 0$,

$$Q_i(x) = -W_i(x), \tag{40}$$

which gives us all the possible superpotentials associated with the potential $V^-(x)$. Thus, (39) can be written as

$$q_i(x) = -W_i(x) + \frac{\partial_x P_m(x)}{P_m(x)} \tag{41}$$

substitution of which in (38) gives a second-order differential equation for $P_m(x)$:

$$\partial_x^2 P_m(x) - 2W_i(x)\partial_x P_m(x) + E_m P_m(x) = 0, \tag{42}$$

which reduces to a COP differential equation after a suitable PCT and we obtain $P_m(x) = P_m^{\alpha_i}(x)$, a COP. Substituting (41) into (36) gives expression for the wave function as

$$\psi_m^-(x) = \exp\left(-\int W_i(x)\,\mathrm{d}x\right) P_m^{\alpha_i}(x). \tag{43}$$

Thus, different $W_i(x)$ will lead to different differential equations leading to different solutions. For physically acceptable solutions, appropriate boundary conditions are applied to fix the values of residues at the fixed poles. This ensures that the wave function, obtained using (36), is well behaved and does not diverge at the end points. One such boundary condition is that with the limit $m \to 0$, the above equation reduces to

$$\psi_0^-(x) = \exp\left(-\int W_i(x)\,\mathrm{d}x\right). \tag{44}$$

Thus, one combination of residues leads to physically acceptable solutions and the other combinations lead to unphysical solutions related to the deconjugacy of the Schrödinger equation [35].

Among the different $W_i(x)$ available from (40), $W_i(x)$ which leads to normalisable ground state from (44), is the superpotential which keeps SUSY exact. The same $W_i(x)$ in (42) and (43) will lead to acceptable solutions of the potential $V^-(x)$. The other combinations lead to non-normalisable ground states and hence give us superpotentials, which break SUSY between the corresponding partners. Thus for any given potential, we can construct all the possible superpotentials from the QMFs.

Interestingly for all the cases studied, it turns out that the exponential term is also the weight function, $w(x)$, i.e.

$$\exp\left(-\int W_i(x)\,\mathrm{d}x\right) = \psi_0^-(x) = w(x), \tag{45}$$

with respect to which the orthogonal polynomials, $P_m^{\alpha_i}(x)$, are orthonormal. Thus, the residue combination leading to physically acceptable solutions also gives a well-behaved weight function in the orthonormality interval. The corresponding polynomials have only real zeros in the orthonormality interval which are governed by the oscillation theorem [37]. The other combinations lead to weight functions, which do not have the right asymptotic behaviour at one or both the end points and therefore do not lead to well-defined spectral problems. These solutions are used for the construction of rational potentials [42]. Thus, we have a neat connection between the QMFs, superpotentials and the weight functions, which can be used to overcome the weight regularity problem, especially when performing higher iterations of isospectral deformation. The QHJ analysis works for the ES rational potentials too and for more details please see [42].

4.1 *QHJF and the isospectral deformation*

It is clear from (1) that substituting $W_i(x) = -W_i(x)$ in the expression for $V_i^-(x)$ gives $V_i^+(x)$. This also implies

$$\psi_0^+(x) = \exp\left(\int W_i(x)\,\mathrm{d}x\right). \tag{46}$$

The same substitution in (41) gives the QMF of $V_i^+(x)$ as

$$q_i(x) = W_i(x) + \frac{\partial_x P_m(x)}{P_m(x)}, \tag{47}$$

which is nothing but $\tilde{W}_i(r)$ given in (17). In addition, substituting $W_i(x) = -W_i(x)$ in (42) gives (16), with $R_1 = E_m$. Thus, by isospectrally deforming $V_i^-(x)$ and demanding that the polynomial $P(x)$ in (15) is an mth degree polynomial, we are, in fact, constructing the QMFs associated with the corresponding translational shape-invariant partner, $V^+(x) \equiv \tilde{V}^+(x)$.

5. Rational potentials associated with the radial oscillator

Before proceeding further, we present the results related to the rational potentials, belonging to the radial oscillator family, having X_m EOPs as solutions studied in [21]. We perform the next iteration of isospectral deformation and rationally extend these potentials and also obtain their solutions.

The radial oscillator potential is given by

$$V(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+1)}{r^2}, \quad r \in (0, \infty). \tag{48}$$

The eigenfunctions and the eigenvalues are

$$\psi_n(r) = r^{\ell+1} \exp\left(-\frac{1}{4}\omega r^2\right) L_n^{\ell+1/2}(r), \quad E_n^- = 2n\omega, \tag{49}$$

respectively, with $n = 0, 1, 2, \ldots$. Here $L_n^{\ell+1/2}(r)$ are the classical associated Laguerre polynomials which are orthonormalwith respect to the weight function

$$w_{\mathrm{cop}}(r) = r^{\ell+1} \exp\left(-\frac{1}{4}\omega r^2\right) \tag{50}$$

in the orthonormality interval $[0, \infty)$.

5.1 *First iteration of isospectral deformation of the radial oscillator*

The QHJ analysis of the radial oscillator leads to four superpotentials $W_i(r)$ $(i = 1, 2, 3, 4)$ presented in table 1. Note that the solutions of all $V_i^-(r)$ are the same as (49) apart from the normalising constant. Using (8) and a_0, a_1 listed in table 1, we can construct the corresponding shape-invariant partners $V_i^+(r)$. In addition, $\psi^+(r) = \psi^-(r, a_0 \to a_1)$ gives eigenfunctions for each partner potential $V_i^+(r)$.

By substituting $W_i(r)$ in (2) we can construct the ground-state wave functions for all $V_i^-(r)$. We can see that $W_1(r)$ leads to a normalisable ground state for $V_1^-(r)$ and a non-normalisable ground state for $V_1^+(r)$. The superpotentials $W_i(r)$ with $i = 2$ and 3 lead to a non-normalisable ground states for both the partners. The last superpotential $W_4(r)$ leads to non-normalisable ground state of $V_4^-(r)$, but gives normalisable ground state for $V_4^+(r)$, because $W_4(r) = -W_1(r, \ell \to \ell - 1)$. For the construction of $\tilde{W}_i(r)$, we need $W_i(r)$ which lead to $P_m^{\alpha_i}(r)$ with no zeros in the interval $[0, \infty)$. Therefore, we use $W_i(r)$ with $i = 1, 2, 3$ in (16) and this takes care of the weight regularity problem. In table 2, we list out $\tilde{W}_i(r)$ which lead to ES rational potentials with physically acceptable solutions. For convenience, we write $(1/2)\omega r^2 = y$ from here on.

The explicit expressions for $P_m^{\alpha_i}(r)$ obtained by substituting different $W_i(r)$ in (16) along with the condition on R_1 are also given. These in turn give different $\tilde{W}_i(r)$

Table 1. Superpotentials of the radial oscillator.

k	$W_i(r)$	$V_i^-(r) = W_i^2(r) - \partial_r W_i(r)$	a_0	a_1
1	$\frac{1}{2}\omega r - [(\ell+1)/r]$	$V(r) - \omega(\ell + 3/2)$	ℓ	$\ell + 1$
2	$\frac{1}{2}\omega r + (\ell/r)$	$V(r) + \omega(\ell - 1/2)$	ℓ	$\ell - 1$
3	$-\frac{1}{2}\omega r - [(\ell+1)/r]$	$V(r) + \omega(\ell + 3/2)$	ℓ	$\ell + 1$
4	$-\frac{1}{2}\omega r + (\ell/r)$	$V(r) - \omega(\ell - 1/2)$	ℓ	$\ell - 1$

from (17). In addition, explicit expressions for the three families of the rational potentials $\tilde{V}_i^-(r)$, obtained using (20), are given in table 3, along with their solutions, constructed using (23) and (25). These involve the generalised X_m exceptional Laguerre polynomials, whose expressions are given in table 4. For details of the calculations, please see [21].

From table 3, it is clear that the normalisable wave functions for each $\tilde{V}_i^-(r)$, obtained using (23), are of the form given in (6):

$$\tilde{\psi}_n^-(r) = \frac{\exp\left(-\int W_1(r)\,\mathrm{d}r\right)}{P_m^{\alpha_i}(r)} L_{m,n}^{j,\alpha_i}(r), \tag{51}$$

where $L_{m,n}^{j,\alpha_i}(r)$ represent the three exceptional Laguerre EOPs, with $j =$ I, II and III representing the $L1$, $L2$ and $L3$ type polynomials, respectively, given in table 4. Our notation coincides with the notation given in [5,11] and hence follows the classification given there for our study. For a more recent classification of EOPs, please see [8]. Here, $\exp(-\int W_1(r)\,\mathrm{d}r)$ is nothing but the normalisable ground state $\psi_0^-(x)$ of $V^-(x)$. Moreover,

$$\frac{\exp\left(-\int W_1(r)\,\mathrm{d}r\right)}{P_m^{\alpha_i}(r)} = w_{i,\mathrm{eop}}(r) \tag{52}$$

are the rational weight functions associated with each family of the X_m exceptional Laguerre polynomials. Substituting $W_1(r)$ from table 1, we can see that

$$\exp\left(-\int W_1(r)\,\mathrm{d}r\right) = w_{\mathrm{cop}}(r) \tag{53}$$

given in (50), which gives the relation between the weight functions of the COPs and X_m EOPs. As it is already ensured that $P_m^{\alpha_i}(r)$ for $i = 1, 2, 3$ do not have zeros on the positive real line, the weight functions for all the three cases are well behaved in the domain of orthonormality.

5.2 *Second iteration of isospectral deformation and construction of rational potentials and EOPs with two indices*

In this section, we do a second iteration of isospectral deformation and construct $\bar{W}_i(r)$ using different $\tilde{W}_i(r)$ given in table 2. Substituting each $\tilde{W}_i(r)$ in eqs (26) and (28) we obtain equations for $\phi_2(r)$ in each case. The details of the calculations are given as follows.

5.2.1 *Construction of* $\bar{V}_1^+(r)$.

For the first case, we use

$$\tilde{W}_1(r) = \frac{1}{2}\omega r - \frac{(\ell+1)}{r} + \frac{\partial_r L_m^{\alpha_1}(-y)}{L_m^{\alpha_1}(-y)} \tag{54}$$

with $\alpha_1 = -\ell - 3/2$. Substituting this in (26) gives

$$\bar{W}_1(r) = \frac{1}{2}\omega r - \frac{(\ell+1)}{r} + \frac{\partial_r L_m^{\alpha_1}(-y)}{L_m^{\alpha_1}(-y)} + \phi_2(r) \tag{55}$$

and the equation for $\phi_2(r)$ from (28) is

Table 2. The superpotentials obtained after the first isospectral deformation. Here, $m = 1, 2, \ldots$ and $y = (1/2)\omega r^2$.

$W_i(r)$	$P_m^{\alpha_i}(r)$	$\tilde{W}_i(r)$	α_i	R_1
$W_1(r)$	$L_m^{\alpha_1}(-y)$	$\frac{1}{2}\omega r - \frac{(\ell+1)}{r} + \frac{\partial_r L_m^{\alpha_1}(-y)}{L_m^{\alpha_1}(-y)}$	$-\ell-\frac{3}{2}$	$2m\omega$
$W_2(r)$	$L_m^{\alpha_2}(-y)$	$\frac{1}{2}\omega r + \frac{\ell}{r} + \frac{\partial_r L_m^{\alpha_2}(-y)}{L_m^{\alpha_2}(-y)}$	$\ell-\frac{1}{2}$	$2m\omega$
$W_3(r)$	$L_m^{\alpha_3}(y)$	$-\frac{1}{2}\omega r - \frac{(\ell+1)}{r} + \frac{\partial_r L_m^{\alpha_3}(y)}{L_m^{\alpha_3}(y)}$	$-\ell-\frac{3}{2}$	$-2m\omega$

Table 3. First-generation rational potentials, their eigenfunctions and eigenvalues.

i	$\tilde{V}_i^-(r)$	$\tilde{\psi}_n^-(r)$	$\tilde{E}_n$
1	$\tilde{V}_1^-(r) = V_1^-(r) - 2\partial_x^2(\ln L_m^{\alpha_1}(-y)) + R_1$	$\left(\frac{y^{(\ell+1)/2}\exp(-(y/2))}{L_m^{\alpha_1}(-y)}\right) L_{m,n}^{\mathrm{III},\alpha_1}(r)$	$2\omega(n+m)$
2	$\tilde{V}_2^-(r) = V_2^-(r) - 2\partial_x^2(\ln L_m^{\alpha_2}(-y)) + R_1$	$\left(\frac{y^{(\ell+1)/2}\exp(-(y/2))}{L_m^{\alpha_2}(-y)}\right) L_{m,n}^{\mathrm{I},\alpha_2}(r)$	$2\omega(n+m)$
3	$\tilde{V}_3^-(r) = V_3^-(r) - 2\partial_x^2(\ln L_m^{\alpha_3}(y)) + R_1$	$\left(\frac{y^{(\ell+1)/2}\exp(-(y/2))}{L_m^{\alpha_3}(y)}\right) L_{m,n}^{\mathrm{II},\alpha_3}(r)$	$2\omega(n-m)$

Table 4. Explicit expression of the exceptional Laguerre polynomials.

$\mathcal{L}^{j,\alpha_i}_{m,n}(r)$	Expression
$L^{\mathrm{I},\alpha_2}_{m,n}(r)$	$L^{\alpha_2+1}_m(-y)L^{\alpha_2}_n(y) - \frac{1}{\omega r}L^{\alpha_2}_m(-y)\partial_r L^{\alpha_2}_n(y)$
$L^{\mathrm{II},\alpha_3}_{m,n}(r)$	$\left(\ell+\frac{1}{2}\right)L^{\alpha_3+1}_m(y)L^{-\alpha_3}_n(y) + rL^{\alpha_3}_m(y)\partial_r L^{-\alpha_3}_n(y)$
$L^{\mathrm{III},\alpha_1}_{m,n}(r)$	$yL^{-\alpha_1+1}_n(y)L^{\alpha_1}_m(-y) + (m+\alpha_1)L^{\alpha_1-1}_m(-y) \times L^{-\alpha_1}_n(y)$

$$\phi_2^2(r) + 2\left(\frac{\omega r}{2} - \frac{(\ell+1)}{r} + \frac{\partial_r L^{\alpha_1}_m(-y)}{L^{\alpha_1}_m(-y)}\right)\phi_2(r) - \partial_x\phi_2(r) - R_2 = 0. \tag{56}$$

We need to solve the above Riccati equation to obtain $\phi_2(r)$. Noting that $\bar{W}(r)$ is nothing but one of the QMF of $\tilde{V}_1^-(r)$, we can obtain its complete form by fixing $\phi_2(r)$.

From (56), we can see that $\phi_2(r)$ has a fixed pole at $r = 0$ in addition to $2m$ fixed poles located at the zeros of $L^{\alpha_1}_m(-y)$. As we want to construct ES potentials, we continue with the ansatz that for $\phi_2(r)$ the point at infinity is an isolated singular point. This implies that $\phi_2(r)$ has finite number of moving poles and let there be N such moving poles. For all ES models, including the rational potentials, this ansatz turned out to be correct [42]. This information allows us to write $\phi_2(r)$ as a meromorphic function in terms of its singular and analytic parts given below:

$$\phi_2(r) = \frac{b_1}{r} + d_1\sum_{i=1}^{2m}\frac{1}{r-a_i} + d_1'\sum_{i=1}^{N}\frac{1}{r-b_j} + c_1 r + Q(r). \tag{57}$$

Here, the first term is the principal part of the Laurent expansion of $\phi_2(r)$ around the fixed pole $r = 0$, with b_1 being the residue. The summation terms describe the sum of all the principal parts of the individual Laurent expansions around the $2m$ fixed poles and the N moving poles, respectively, with d_1 and d_1' denoting the corresponding residues. The term $c_1 r$ describes the behaviour of $\phi_2(r)$ at infinity as seen from (56) and $Q(r)$ is the analytical part of $\phi_2(r)$, which from Liouville's theorem is a constant, say C. Writing

$$\sum_{i=1}^{2m}\frac{1}{r-a_i} = \frac{\partial_r L^{\alpha_1}_m(-y)}{L^{\alpha_1}_m(-y)}$$

and

$$\sum_{i=1}^{N}\frac{1}{r-b_j} = \frac{\partial_r \mathcal{P}_N(r)}{\mathcal{P}_N(r)},$$

where $\mathcal{P}_N(r)$ is an Nth degree polynomial, and the above equation becomes

$$\phi_2(r) = \frac{b_1}{r} + d_1\frac{\partial_r L^{\alpha_1}_m(-y)}{L^{\alpha_1}_m(-y)} + d_1'\frac{\partial_r \mathcal{P}_N(r)}{\mathcal{P}_N(r)} + c_1 r + C. \tag{58}$$

To find the residues at these poles, we expand $\phi_2(r)$ in a Laurent expansion around each pole individually and substitute it in (56). For example, to calculate b_1, we do a Laurent expansion of $\phi_2(r)$ around $r = 0$,

$$\phi_2(r) = \frac{b_1}{r} + a_0 + a_1 r + \cdots \tag{59}$$

and substitute it in (56). Equating the coefficients of $1/r^2$ to zero, we obtain two values for b_1, namely

$$b_1 = 0, \quad b_1 = 2\ell + 1. \tag{60}$$

Similarly, the dual values of residues at the $2m$ fixed and N moving poles are obtained as

$$d_1 = 0, \quad d_1 = -3, \tag{61}$$

$$d_1' = 0, \quad d_1' = -1, \tag{62}$$

respectively. Next, in order to calculate the behaviour of $\phi_2(r)$ at infinity, we perform a change of variable $r = 1/t$ in (56), which gives

$$\phi_2^2(t) + 2\left(\frac{\omega}{2t} - (\ell+1)t - t^2\frac{\partial_r L^{\alpha_1}_m(-t)}{L^{\alpha_1}_m(-t)}\right)\phi_2(t) - t^2\partial_t\phi_2(t) - R_2 = 0. \tag{63}$$

The residue at $t = 0$ is calculated by doing a Laurent expansion of $\phi_2(t)$ around $t = 0$. Once again we obtain two values for c_1 as

$$c_1 = 0, \quad c_1 = -\omega. \tag{64}$$

Substituting $\phi_2(r)$ into (55) gives

$$\bar{W}_1(r) = \left(c_1 + \frac{\omega}{2}\right)r + \frac{b_1-\ell-1}{r} + (d_1+1)\frac{\partial_r L^{\alpha_1}_m(-y)}{L^{\alpha_1}_m(-y)} + d_1'\frac{\partial_r \mathcal{P}_N(r)}{\mathcal{P}_N(r)} + C. \tag{65}$$

We need to choose residue values such that (65) will lead to non-normalisable solutions, more specifically to an unnormalised ground state. Thus, the choice of residues at $r = 0$ and ∞ plays an important role. Since $2m$ fixed poles do not lie in the domain of orthonormality, we can, in principle, choose any of the residue values. As for the residues at N moving poles, we choose the only non-trivial value available. For the present case, we consider the combination

$$b_1 = 2\ell + 1, \quad d_1 = 0, \quad d_1' = -1, \quad c_1 = 0 \tag{66}$$

and we can check that weight function obtained using these residues tends to infinity as $r \to \infty$. Thus, we obtain

$$\phi_2(r) = \frac{2\ell+1}{r} - \frac{\partial_r \mathcal{P}_N(r)}{P_N(r)} + C. \tag{67}$$

Substituting this into (56) gives the following second-order differential equation for $\mathcal{P}_N(r)$:

$$\partial_r^2\mathcal{P}_N(r) - 2\left(\frac{\omega r^2}{2} + \ell + r\frac{\partial_r L_m^{\alpha_1}(-y)}{L_m^{\alpha_1}(-y)}\right)\frac{1}{r}\partial_r\mathcal{P}_N(r) + \left(2(2\ell+1)\left(\frac{\omega}{2} + \frac{1}{r}\frac{\partial_r L_m^{\alpha_1}(-y)}{L_m^{\alpha_1}(-y)}\right) - R_2\right)\mathcal{P}_N(r) = 0. \tag{68}$$

Note that in the process of simplification, by comparing the coefficients of r, we can show that $C = 0$. Performing a PCT $(1/2)\omega r^2 = z$ and dividing the resultant equation by 2ω, we obtain

$$z\partial_z^2\mathcal{P}_N(z) + \left(-z - \left(\ell - \frac{1}{2}\right) - 2z\frac{\partial_z L_m^{\alpha_1}(z)}{L_m^{\alpha_1}(z)}\right) \times\partial_z\mathcal{P}_N(z) + \left((2\ell+1)\left(\frac{1}{2} + \frac{\partial_z L_m^{\alpha_1}(z)}{L_m^{\alpha_1}(z)}\right) - \frac{R_2}{2\omega}\right) \times\mathcal{P}_N(z) = 0. \tag{69}$$

Redefining $\ell = -d - 1$ and putting $m = 1$, the above equation reduces to

$$z\partial_z^2\mathcal{P}_N(z) + \left(-z + \left(d + \frac{3}{2}\right) - 2z\frac{\partial_z L_1^{d-1/2}(z)}{L_1^{d-1/2}(z)}\right) \times\partial_z\mathcal{P}_N(z) - \left((2d+1)\left(\frac{1}{2} + \frac{\partial_z L_1^{d-1/2}(z)}{L_1^{d-1/2}(z)}\right) + \frac{R_2}{2\omega}\right)\mathcal{P}_N(z) = 0. \tag{70}$$

Comparing the above equation with the X_1 exceptional Laguerre equation of $L1$ type given below [5]

$$z\partial_z^2 L_{1,n'}^{\mathrm{I},\,g-1/2}(z) + \left(-z + \left(g + \frac{3}{2}\right) - 2z\frac{\partial_z L_1^{g-1/2}(z)}{L_1^{g-1/2}(z)}\right) \times\partial_z L_{1,n'}^{\mathrm{I},\,g-1/2}(z) + \left(-2z\frac{\partial_z L_1^{g+1/2}(z)}{L_1^{g-1/2}(z)} + n' + 1\right) \times L_{1,n'}^{\mathrm{I},\,g-1/2}(z) = 0, \tag{71}$$

we can see that both the equations match, except for the last terms. As R_2 is an unknown constant, we fix R_2 by demanding that the last terms of eqs (70) and (71) match. This gives

$$R_2 = -(-n' + d + 3/2)2\omega. \tag{72}$$

Though for general m, (69) can be reduced to an X_m exceptional polynomial differential equation by suitably fixing R_2, we find that only for $m = 1$, R_2 turns out to be a constant. For any other choice of m, R_2 turns out to be a function of r, which violates the initial condition (27). Thus, the polynomial, $\mathcal{P}_N(r)$, coincides with

$$L_{1,n'}^{\mathrm{I},\,\delta}(y) = L_1^{\delta+1}(-y)L_{n'}^{\delta}(y) - \frac{1}{\omega r}L_1^{\delta}(-y)\partial_r L_{n'}^{\delta}(y), \tag{73}$$

where $\delta = d - 1/2$ and the degree of the polynomial is $N = n' + 1$. For small values of n' and ℓ, we have checked numerically that as long as $-2 < R_2 < 0$, the polynomial does not have any zeros in the interval $[0, \infty)$. A random check for bigger values of n' and ℓ showed the same pattern. Therefore, we conjecture that as long as we choose n' and ℓ values such that the above condition on R_2 is satisfied, we have polynomials $L_{1,n'}^{\mathrm{I},\,\delta}(y)$ with no zeros in the orthonormality interval. Thus, for all such values, we can write

$$\phi_2(r) = \frac{2\ell+1}{r} - \frac{\partial_r L_{1,n'}^{\mathrm{I},\,\delta}(y)}{L_{1,n'}^{\mathrm{I},\,\delta}(y)}, \tag{74}$$

which in turn gives

$$\bar{W}_1(r) = \frac{\omega r}{2} - \frac{\ell}{r} + \frac{\partial_r L_1^{\alpha_1}(-y)}{L_1^{\alpha_1}(-y)} - \frac{\partial_r L_{1,n'}^{\mathrm{I},\,\delta}(y)}{L_{1,n'}^{\mathrm{I},\,\delta}(y)}. \tag{75}$$

Thus, $\bar{W}_1(r)$ has rational terms involving the logarithmic derivative of Laguerre polynomials and X_1 exceptional Laguerre polynomials of $L1$ type X_1 EOPs. The potential $\bar{V}_1^+(r)$ is constructed by substituting $\phi_2(r)$ in (30) and we obtain

$$\bar{V}_1^+(r) = V_1^+(r) + 2\partial_r\left(\frac{2\ell+1}{r} - \frac{\partial_r L_{1,n'}^{\mathrm{I},\,\delta}((1/2)\omega r^2)}{L_{1,n'}^{\mathrm{I},\,\delta}((1/2)\omega r^2)}\right) + 2\omega\left(\ell - n' - \frac{1}{2}\right), \tag{76}$$

which is the second generation rational extension of the radial oscillator $V^+(r)$. For different values of n' we get different $\bar{V}_1^+(r)$. For each case, the eigenvalues can be calculated using (35) as

$$\bar{E}_n = 2\omega\left(n - n' + \ell + \frac{1}{2}\right). \tag{77}$$

Thus, we can construct a family of generalised rational potentials indexed by n'. Next, we calculate the solutions of $\bar{V}_1^+(r)$ using

$$\bar{\psi}_n^+(r) = \bar{A}\bar{\psi}_n^-(r), \quad n = 1, 2, \ldots, \tag{78}$$

where the intertwining operators are defined in (32). We know $\bar{\psi}_n^-(r)$, since by definition $\bar{V}_1^-(r) \equiv \tilde{V}_1^-(r)$

and therefore their solutions are equal, apart from the normalisation constant. Thus, from tables 3 and 4, we get

$$\bar{\psi}_n^-(r) = \left(\frac{y^{(\ell+1)/2}\exp(-(y/2))}{L_1^{\alpha_1}(-y)}\right) L_{1,n}^{\text{III},\alpha_1}(y). \tag{79}$$

Substituting this in (78) and operating $\bar{A}$ on it gives

$$\bar{\psi}_n^+(r) = \left(\frac{y^{\ell/2}\exp(-(y/2))}{L_1^{\alpha_1}(-y)L_{1,n'}^{\text{I},\delta}(y)}\right) \mathcal{Q}_{m=1,n',n}^{\alpha_1,\delta}(y), \tag{80}$$

where

$$\begin{aligned}\mathcal{Q}_{m=1,n',n}^{\alpha_1,\delta}(y) = {} & (2\ell+1)L_{1,n'}^{\text{I},\delta}(y)L_{1,n}^{\text{III},\alpha_1}(y) \\ & + rL_{1,n'}^{\text{I},\delta}(y)\partial_r L_{1,n}^{\text{III},\alpha_1}(y) \\ & - rL_{1,n}^{\text{III},\alpha_1}(y)\partial_r L_{1,n'}^{\text{I},\delta}(y).\end{aligned} \tag{81}$$

Here, $\mathcal{Q}_{m=1,n',n}^{\alpha_1,\delta}(y)$ is an EOP with two indices $m = 1$ and n' taking values $1, 2, \ldots,$ provided the condition on R_2 is taken care off. It can be seen that these polynomials have a complicated yet interesting structure of zeros, consisting of both the exceptional and the regular zeros. A careful investigation is needed to obtain more information regarding their distribution. As discussed in §4, the non-polynomial part in (80) gives the rational weight function

$$w(r) = \frac{y^{\ell/2}\exp(-(y/2))}{L_1^{\alpha_i}(-y)L_{1,n}^{\text{I},\delta}(y)}, \tag{82}$$

with respect to which $\mathcal{Q}_{m=1,n',n}^{\alpha_1,\delta}(y)$ are orthonormal, in the interval $[0,\infty)$. As discussed earlier, for appropriate choices of ℓ and n' values, these polynomials appearing in the denominator will not have any real zeros in this interval. This takes care of the weight regularity problem. Thus, we are led to a family of EOPs with two indices, which are explicitly written in terms of the single indexed EOPs. One can continue to iterate this process and construct a hierarchy of rational potentials with multi-index EOPs in their solutions.

5.2.2 *Construction of* $\bar{V}_2^+(r)$. The second family of rational potentials is constructed by substituting

$$\tilde{W}_2(r) = \frac{\omega r}{2} + \frac{\ell}{r} + \frac{\partial_r L_m^{\alpha_2}(-y)}{L_m^{\alpha_2}(-y)} \tag{83}$$

in eqs (26) and (28). The equation for $\phi_2(r)$ turns out to be

$$\begin{aligned}&\phi_2^2(r) + 2\left(\frac{\omega r}{2} + \frac{\ell}{r} + \frac{\partial_r L_m^{\alpha_2}(-y)}{L_m^{\alpha_2}(-y)}\right)\phi_2(r) \\ &-\partial_x\phi_2(r) - R_2 = 0\end{aligned} \tag{84}$$

with $\alpha_2 = \ell - 1/2$. As in the above case, the meromorphic form of $\phi_2(r)$ is

$$\begin{aligned}\phi_2(r) = {} & \frac{b_1}{r} + d_1\frac{\partial_r L_m^{\alpha_2}(-y)}{L_m^{\alpha_2}(-y)} + d_1'\frac{\partial_r \mathcal{P}_N(r)}{\mathcal{P}_N(r)} \\ & + c_1 r + Q(r)\end{aligned} \tag{85}$$

with b_1, d_1 and d_1' being residues at $r = 0$, $2m$ fixed poles due to of $L_m^{\alpha_2}(-y)$ and N moving poles due to $\mathcal{P}_N(y)$. Because the point at infinity is assumed to be an isolated singularity, the analytical part $Q(r)$ is a constant C. Proceeding as in the previous case, the residue values turn out to be

$$\begin{aligned}& b_1 = 0, \quad -2\ell - 1; \quad d_1 = 0, \quad -3; \\ & d_1' = 0, \quad -1; \quad c_1 = 0, \quad -\omega.\end{aligned} \tag{86}$$

We choose the combination $b_1 = 0, d_1 = 0, d_1' = -1, c_1 = -\omega$, such that $\bar{W}_2(r)$ is constructed, leading to a non-normalisable ground state. With these values, we get

$$\phi_2(r) = -\omega r - \frac{\partial_r \mathcal{P}_N(r)}{\mathcal{P}_N(r)} + C, \tag{87}$$

which when substituted in (84) gives $C = 0$ and

$$\begin{aligned}&\partial_r^2\mathcal{P}_N(r) + 2\left(\frac{\omega r^2}{2} - \ell - r\frac{\partial_r L_m^{\alpha_2}(-y)}{L_m^{\alpha_2}(-y)}\right)\frac{1}{r}\partial_r\mathcal{P}_N(r) \\ &-\left(2\omega\left(\ell - \frac{1}{2}\right) + 2\omega r\frac{\partial_r L_m^{\alpha_2}(-y)}{L_m^{\alpha_2}(-y)} + R_2\right)\mathcal{P}_N(r) \\ &= 0.\end{aligned} \tag{88}$$

Doing a PCT $(1/2)\omega r^2 = -z$ and dividing the resultant equation by 2ω reduces the above equation to

$$\begin{aligned}& z\partial_z^2\mathcal{P}_N(z) + \left(-z - \left(\ell - \frac{1}{2}\right) - 2z\frac{\partial_z L_m^{\alpha_2}(z)}{L_m^{\alpha_2}(z)}\right) \\ & \times\partial_z\mathcal{P}_N(z) + \left(\left(\ell - \frac{1}{2}\right) + 2z\frac{\partial_z L_m^{\alpha_2}(-z)}{L_m^{\alpha_2}(-z)} + \frac{R_2}{2\omega}\right) \\ & \times\mathcal{P}_N(z) = 0.\end{aligned} \tag{89}$$

Redefining the potential parameter $\ell = -a - 1$ and considering the special case $m = 1$, the above equation reduces to the differential equation of the $L2$ type, X_1 exceptional Laguerre polynomial,

$$\begin{aligned}& z\partial_z^2\mathcal{P}_N(z) + \left(-z + \left(a + \frac{3}{2}\right) - 2z\frac{\partial_z L_1^{\beta}(z)}{L_1^{\beta}(z)}\right) \\ & \times\partial_z\mathcal{P}_N(z) + \left(1 + n' - 2\left(a + \frac{1}{2}\right)\frac{\partial_z L_1^{\beta+1}(z)}{L_1^{\beta+1}(z)}\right) \\ & \times\mathcal{P}_N(z) = 0,\end{aligned} \tag{90}$$

where $\beta = -a - 3/2$. As in the previous case, only for $m = 1$, we obtain

$$R_2 = \left(a + \frac{1}{2} + n'\right)2\omega, \tag{91}$$

a constant. Therefore,

$$\mathcal{P}_N(y) \equiv L^{\mathrm{II},\beta}_{1,n'}(-y) = \left(a + \frac{1}{2}\right)L_1^{\beta+1}(-y)L_{n'}^{-\beta}(-y) + rL_1^{\beta}(-y)\partial_r L_{n'}^{-\beta}(-y) \tag{92}$$

with the degree $N = 1+n'$. Here again, we have checked numerically that for small values of n' and ℓ, these polynomials will not have any zeros in the orthonormality interval, provided the potential parameters satisfy certain conditions. We have observed that the above polynomial has no zeros in the orthonormality interval,

1. if $R_2 \geq -1.5$ for both n' and ℓ being odd and
2. if $R_2 \leq -2.5$ for both n' and ℓ being even.

Therefore, by taking suitable parameter values we can obtain $\mathcal{P}_N(z)$ with the required behaviour. Thus, we get

$$\phi_2(r) = -\omega r - \frac{\partial_r L^{\mathrm{II},\beta}_{1,n'}(-y)}{L^{\mathrm{II},\beta}_{1,n'}(-y)} \tag{93}$$

and

$$\bar{W}_2(r) = -\frac{\omega r}{2} + \frac{\ell}{r} + \frac{\partial_r L_1^{\alpha_2}(-y)}{L_1^{\alpha_2}(-y)} - \frac{\partial_r L^{\mathrm{II},\beta}_{1,n'}(-y)}{L^{\mathrm{II},\beta}_{1,n'}(-y)}. \tag{94}$$

Using $\bar{W}_2(r)$, the new rational potential can be constructed from (30) as

$$\bar{V}_2^{+}(r) = V^{+}(r) - 2\partial_r\left(\frac{\partial_r L^{\mathrm{II},\beta}_{1,n'}(-y)}{L^{\mathrm{II},\beta}_{1,n'}(-y)}\right) + 2\omega\left(n' + a + \frac{1}{2}\right), \tag{95}$$

clearly a different rational extension of the radial oscillator. The eigenvalues calculated using (35) will be

$$\bar{E}_n = \left(n + n' - \ell + \frac{1}{2}\right)2\omega. \tag{96}$$

Thus, we have another family of generalised rational potentials, indexed by n'. The eigenfunctions $\bar{\psi}_n^{+}(r)$ are obtained by applying the corresponding $\bar{A}$ on $\bar{\psi}_n^{-}(r) = \tilde{\psi}_n^{-}(r)$, obtained from tables 2 and 3. The eigenfunctions turn out to be

$$\bar{\psi}_n^{+}(r) = \left(\frac{y^{\ell/2}\exp(-(y/2))}{L_1^{\alpha_2}(-y)L^{\mathrm{II},\beta}_{1,n'}(-y)}\right)\mathcal{Q}^{\alpha_2,\beta}_{m=1,n',n}(y), \tag{97}$$

where

$$\mathcal{Q}^{\alpha_2,\beta}_{m=1,n',n}(y) = (2\ell + 1 - 2y)L^{\mathrm{I},\alpha_2}_{1,n}(y)L^{\mathrm{II},\beta}_{1,n'}(-y) - rL^{\mathrm{II},\beta}_{1,n'}(-y)\partial_r L^{\mathrm{I},\alpha_2}_{1,n}(y) - rL^{\mathrm{I},\alpha_2}_{1,n}(y)\partial_r L^{\mathrm{II},\beta}_{1,n'}(-y). \tag{98}$$

The above polynomials are orthonormal with respect to to the rational weight function

$$w(r) = \left(\frac{y^{\ell/2}\exp(-(y/2))}{L_1^{\alpha_2}(-y)L^{\mathrm{II},\beta}_{1,n'}(-y)}\right). \tag{99}$$

These polynomials again have both exceptional and regular zeros and well behaved rational weight functions as long as the conditions on the parameters are taken into account. Thus, we obtain another set of generalised rational potentials and $L1$ type-two indexed exceptional Laguerre polynomials.

5.2.3 *Construction of* $\bar{V}_3^{+}(r)$. The third family of rational potentials is constructed deforming $\tilde{V}_3^{-}(r)$ using

$$\bar{W}_3(r) = -\frac{\omega r}{2} - \frac{\ell + 1}{r} + \frac{\partial_r L_m^{\alpha_3}(y)}{L_m^{\alpha_3}(y)} \tag{100}$$

in table 2. Following the same procedure as in the previous two cases, the equation of $\phi_2(r)$ turns out to be

$$\phi_2^2(r) + 2\left(-\frac{\omega r}{2} - \frac{\ell+1}{r} + \frac{\partial_r L_m^{\alpha_3}(y)}{L_m^{\alpha_3}(y)}\right)\phi_2(r) - \partial_r\phi_2(r) - R_2 = 0, \tag{101}$$

with $\alpha_3 = -\ell - 3/2$. Again the meromorphic form of $\phi_2(r)$ is written as

$$\phi_2(r) = \frac{b_1}{r} + d_1\frac{\partial_r L_m^{\alpha_3}(y)}{L_m^{\alpha_3}(y)} + d_1'\frac{\partial_r \mathcal{P}_N(r)}{\mathcal{P}_N(r)} + c_1 r + C, \tag{102}$$

with b_1, d_1 and d_1' being the residues at $r = 0$, $2m$ fixed poles due to $L_m^{\alpha_3}(y)$ and N moving poles due to $P_N(y)$ respectively. The residue values turn out to be

$$b_1 = 0,\ 2\ell + 1;\quad d_1 = 0,\ 3;\\ d_1' = 0,\ -1;\quad c_1 = 0,\ \omega. \tag{103}$$

The suitable choice $b_1 = 2\ell + 1$, $d_1 = 0$, $d_1' = -1$, $c_1 = 0$ gives

$$\phi_2(r) = \frac{2\ell + 1}{r} - \frac{\partial_r \mathcal{P}_N(r)}{\mathcal{P}_N(r)} + C, \tag{104}$$

which when substituted in (101) gives $C = 0$ and

$$\partial_r^2\mathcal{P}_N(r) - 2\left(-\frac{\omega r^2}{2} + \ell + r\frac{\partial_r L_m^{\alpha_3}(y)}{L_m^{\alpha_3}(y)}\right)\frac{1}{r}\partial_r\mathcal{P}_N(r) + \left(\frac{2(2\ell+1)}{r}\left(-\frac{\omega r}{2} + \frac{\partial_r L_m^{\alpha_3}(y)}{L_m^{\alpha_3}(y)}\right) - R_2\right)\mathcal{P}_N(r) = 0. \tag{105}$$

A PCT $(1/2)\omega r^2 = -z$ and redefining $\ell = -b - 1$ and taking the special case of $m = 1$ reduces the above equation to the $L1$-type exceptional Laguerre differential equation

$$z\partial_z^2\mathcal{P}_N(z) + \left(-z + \left(b + \frac{3}{2}\right) - 2z\frac{\partial_z L_1^{b-1/2}(-z)}{L_1^{b-1/2}(-z)}\right) \times\partial_z\mathcal{P}_N(z) + \left(2z\frac{\partial_z L_1^{b+1/2}(-z)}{L_1^{b-1/2}(-z)} + n' - 1\right)\mathcal{P}_N(z) = 0 \tag{106}$$

and in the process fixes $R_2 = (n' + b + 3/2)2\omega$. Thus, the polynomial $\mathcal{P}_N(z)$ coincides with

$$L_{1,n'}^{\mathrm{I},\gamma}(-y) = L_1^{\gamma+1}(y)L_{n'}^{\gamma}(-y) + \frac{1}{\omega r}L_1^{\gamma}(y)\partial_r L_{n'}^{\gamma}(-y), \tag{107}$$

where $\gamma = b - 1/2$ and the degree $N = 1 + n'$. These polynomials do not have any zeros in the orthonormality interval with the following conditions placed on the parameters. For small values of n' and ℓ, we have numerically checked that the polynomial will not have zeros for a given n',

1. if ℓ is even and R_2 is greater than $3/2$,
2. if ℓ is odd R_2 is less than 0.

Therefore, for all such values, we obtain

$$\bar{W}_3(r) = -\frac{\omega r}{2} + \frac{\ell}{r} + \frac{\partial_r L_1^{\alpha_3}(y)}{L_1^{\alpha_3}(y)} - \frac{\partial_r L_{1,n'}^{\mathrm{I},\gamma}(-y)}{L_{1,n'}^{\mathrm{I},\gamma}(-y)}. \tag{108}$$

The new rational potential turns out to be

$$\bar{V}_3^+(r) = V_3^+(r) - 2\partial_r\left(\frac{\partial_r L_{1,n'}^{\mathrm{I},\gamma}(-y)}{L_{1,n'}^{\mathrm{I},\gamma}(-y)}\right) + 2\omega(n' - \ell + 1/2), \tag{109}$$

the third rational extension of the radial oscillator belonging to the second generation. The eigenvalues turn out to be

$$\bar{E}_n = \left(n + n' - \ell - \frac{1}{2}\right)2\omega, \tag{110}$$

calculated using (35). The eigenfunctions calculated using with $\bar{A} = (\mathrm{d}/\mathrm{d}r) + \bar{W}_3(r)$ are

$$\bar{\psi}_n^+(r) = \left(\frac{y^{\ell/2}\exp(-(y/2))}{L_1^{\alpha_3}(y)L_{1,n'}^{\mathrm{I},\gamma}(-y)}\right)\mathcal{Q}_{m=1,n',n}^{\alpha_3,\gamma}(y), \tag{111}$$

where

$$\mathcal{Q}_{m=1,n',n}^{\alpha_3,\gamma}(y) = (2\ell + 1 - 2y)L_{1,n'}^{\mathrm{I},\gamma}(-y)L_{1,n}^{\mathrm{II},\alpha_3}(y) + L_{1,n'}^{\mathrm{I},\gamma}(-y)r\partial_r L_{1,n}^{\mathrm{II},\alpha_3}(y) - L_{1,n}^{\mathrm{II},\alpha_3}(y)r\partial_r L_{1,n'}^{\mathrm{I},\gamma}(-y). \tag{112}$$

The well-behaved rational weight function with respect to which these polynomials are orthonormal is

$$w(r) = \left(\frac{y^{\ell/2}\exp(-(y/2))}{L_1^{\alpha_3}(y)L_{1,n'}^{\mathrm{I},\gamma}(-y)}\right). \tag{113}$$

These polynomials too have both exceptional and regular zeros. Thus, we have a third family of generalised rational potential with $L2$ type-two indexed exceptional Laguerre polynomials as solutions.

6. Discussion

6.1 *The other combinations of residues*

In all the three cases studied above, we have considered only one combination of the residues in obtaining the meromorphic form of $\phi_2(r)$. Among the remaining combinations, there will be one combination, which will lead to the superpotential $\bar{\mathcal{W}}_i(r)$ in each case. The use of the remaining combinations of residues in $\phi_2(r)$ will lead to two different scenarios.

1. The second-order differential equation for $\mathcal{P}_N(r)$ does not reduce to any of the known X_m exceptional Laguerre equations. Therefore, we need to check whether the differential equation can be solved exactly and what is the nature of the solutions in each case.
2. In this case, the differential equation for $\mathcal{P}_N(r)$ does reduce to an EOP differential equation. In these cases, R_2 turns out to be a function of r, which violates our initial condition (27). This also implies that we have a distinct $\bar{V}_i^-(r)$ too. These potentials may or may not be ES and we cannot use the supersymmetric techniques to obtain solutions of $\bar{V}^\pm(r)$.

Thus, there is a need for a careful investigation of these cases. We cannot rule out the possibility of some more rational potentials, which can be ES or quasiexactly solvable. It will be interesting to study the nature of their solutions.

6.2 *New rational potentials and EOPs*

It is clear that we can continue to rationally extend the potentials constructed in this paper and obtain new ES potentials for suitable potential parameters. These potentials will have solutions involving multi-indexed EOPs and we can explicitly construct the polynomials. In each iteration we can perform the singularity structure analysis of the Riccati equation. The appropriate choice of the residues allows us to construct the generalised superpotential. We have seen that the choice of the residues at fixed poles coinciding with the end points of the orthonormality interval is very crucial to ensure that the weight regularity problem does not arise.

In the case of other fixed poles, laying off the orthonormality interval, we can choose any value in principle. However, we see that only certain values ensure that the differential equation for $\mathcal{P}_N(r)$ reduces to a known orthogonal polynomial equation and that the resultant polynomials do not have any zeros in the orthonormality interval. Other choices may lead to new potentials, but further analysis is required to comment about the nature of these potentials and their solutions. Thus, it is clear that with proper choice of residues, we can construct a hierarchy of rational potentials using isospectral deformation. The zeroth member in the hierarchy is the radial oscillator with COPs as solutions and is followed by a rational potential with EOPs as solutions where the number of indices in the polynomials increases with each iteration.

7. Conclusions

In this study, we have explicitly constructed the rational potentials and their eigenvalues and eigenfunctions. We have also shown that the latter are in terms of the double indexed exceptional Laguerre polynomials. This method can be used to construct newer ES rational potentials with their solutions involving multi-indexed EOPs. The method is simple and makes use of the supersymmetric machinery and inputs from the QHJF, which involves the singularity structure analysis of the generalised superpotential. The fact that the isospectral deformation of a potential leads to the QMF of its shape invariant partner has been a crucial input to arrive at the complete form of the new generalised superpotentials $\bar{W}_i(r)$. We have shown that appropriate choices of residues will automatically take care of the weight regularity problem and we obtain a complete set of well-behaved solutions. This singularity structure analysis allows us to completely fix the rational terms, which extend the original potentials and makes our method different from the existing ones.

The same method can be used to rationally extend other ES models having Hermite and Jacobi COPs and first-order EOPs as solutions. This study is currently underway and will be published elsewhere. In addition, the existence of multi-indexed EOPs leads to a lot of questions about the classification of the orthogonal polynomials, their impact on the Bochner's theorem, Sturm–Liouville's theory, ES quantum mechanical models and other related areas. Therefore, a systematic study of these new polynomials and potentials is required.

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