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New analytical study of water waves described by coupled fractional variant Boussinesq equation in fluid dynamics

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Abstract. The main objective of this paper is to introduce an analytical study for the water wave solutions of coupled fractional variant Boussinesq equation, which is modelled to investigate the waves in fluid dynamics. Wave transformation in fractional form is applied to convert the original fractional-order nonlinear partial differential equation into another nonlinear ordinary differential equation. The strategy here is to use the unified method to obtain a variety of exact solutions. The unified method works well and reveals distinct exact solutions which are classified into two different types, namely polynomial function and rational function solutions. The results are also depicted graphically for different values of fractional parameter. These findings are highly encouraging and have significant importance for some special physical phenomena in fluid dynamics

Keywords. Coupled fractional variant Boussinesq equation; conformable derivatives; unified method.

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1. Introduction

Fractional differential equations (FDEs) are getting the attention of the researcher community since last decay. The FDEs provide excellent tools for describing hereditary and memory characteristics of different materials and processes [1–3]. It can also be observed that the obtained solutions from classical calculus are based on individual level perspective. In fact, most of the phenomena in different branches of science such as fluid dynamics, signal processing, biology, systems identification, control theory, chemical and physical systems possess memory-based concepts [4–8]. In order to know the behaviour of such systems, we require solutions from the community level and the fractional calculus possesses the property to determine such solutions. These equations have the potential to accomplish what the integer-order calculus cannot do. The most suitable way to understand the dynamics of many physical models related to FDEs is to find their exact solutions. The explicit solutions of these equations, if available, facilitate the verification of numerical researchers and aid in studying the stability analysis.

Different approaches are used in the literature for calculating exact solutions for FDEs. These methods include the fractional transformation method [9], homotopy perturbation method [10,11], new extended trial equation method [12], Kudryashov method and its extended form [13,14], the generalised unified method (UM) [15–19], first integral method [20] and improved fractional subequation method [21].

The main objective of this paper is to investigate the UM [22–25] to find new types of exact solutions for the coupled fractional variant Boussinesq equation (FBE) [26,27].

The FBE is given by

$$D_t^{\alpha} u + \lambda D_x^{\alpha}(u\, v) + \mu D_x^{3\alpha} v = 0,$$

$$D_t^{\alpha} v + \nu D_x^{\alpha} u + \sigma v D_x^{\alpha} v = 0, \tag{1}$$

where λ, μ, ν and σ are arbitrary constants and D_t^{α}, D_x^{α} are conformable derivative operators of order α with respect to t and x, respectively, $0 < \alpha \leq 1$. This is a model for water waves in fluid dynamics where $u(x,t)$ is the total depth deviating from the equilibrium position of water waves and $v(x,t)$ is the horizontal velocity of water waves under investigation.

The paper is arranged as follows: in §2, the conformable derivative of order α is explained with some of its properties. The UM is introduced in §3. The application of the UM to the FBE is given in §4. The physical meaning of some of the obtained solutions is presented in §5. Finally, conclusions are provided in §6.

2. Conformable derivative and its properties

Conformable derivative is proposed in [28] to cover the deficiencies of the existing definitions.

DEFINITION 1

Let $g\colon (0,\infty) \to R$ be a function, then its conformable derivative of order α is defined as

$$T_\alpha(g)(z) = \lim_{\sigma\to 0} \frac{g(z+\sigma z^{1-\alpha}) - g(z)}{\sigma}$$

for all $z > 0$ and $\alpha \in (0,1]$.

The following are the important properties of conformable derivative [29,30].

Suppose g and h are α-conformable differentiables for all $z > 0$ and c_1 and c_2 are real constants. Then we have the following:

(i) $T_\alpha(c_1 g + c_2 h) = c_1 T_\alpha(g) + c_2 T_\alpha(h)$.
(ii) $T_\alpha(t^\omega) = \omega\, t^{\omega-\alpha}$ for all $\omega \in R$.
(iii) $T_\alpha(gh) = hT_\alpha(g) + gT_\alpha(h)$.
(iv) $T_\alpha\left(\frac{g}{h}\right) = \frac{hT_\alpha(g) - gT_\alpha(h)}{h^2}$.
(v) If g is differentiable, then

$$T_\alpha(g)(z) = z^{1-\alpha}\frac{\mathrm{d}g}{\mathrm{d}z}(z).$$

(vi) Suppose $g\colon (0,\infty) \to R$ is differentiables and α-conformable differentiable function and h is another differentiable function which is defined on the range of the function g. Then $T_\alpha(goh)(z) = z^{1-\alpha}\, h'(z)\, g'(h(z))$.

3. Mathematical formulation for the UM

Consider a space–time FDE

$$F(u, D_x^\alpha u, D_t^\alpha u, \ldots) = 0, \tag{2}$$

where F is a polynomial in its arguments.

Apply the transformation $u(x,t) = v(\xi)$, $\xi = (x^\alpha/\alpha) - k(t^\alpha/\alpha)$, where k is a constant. It will reduce eq. (2) to another nonlinear ordinary differential equation of integer order as follows:

$$F_1(v, v', v'', \ldots) = 0, \tag{3}$$

where the prime denotes the derivative with respect to ξ.

Here, the UM is applied to find a variety of travelling wave solutions for eq. (3) [22–25]. These solutions will be classified into two forms, polynomial function and rational function forms. The outline of this method is presented as follows.

(I) *Polynomial solutions*: To obtain the solutions of eq. (3) in polynomial function forms, we assume that

$$U = U(\xi) = \sum_{i=0}^{n} p_i\, \Gamma^i(\xi),$$
$$(\Gamma'(\xi))^p = \sum_{j=0}^{pm} b_j\, \Gamma^j(\xi), \quad p = 1, 2, \tag{4}$$

where p_i and b_j are constants. The UM provides the balance principle technique to evaluate the relation between the two parameters n and m and satisfies the consistency condition between the arbitrary functions in the solutions given by eq. (4) (for details, see [22–25]).

It is worth mentioning that the UM solves (3) to elementary solutions or elliptic solutions when $p = 1$ or $p = 2$ respectively.

(II) *Rational solutions*: To get these solutions, we suppose

$$U = U(\xi) = \sum_{i=0}^{n} p_i\, \Gamma^i(\xi) \Big/ \sum_{i=0}^{k} q_i\, \Gamma^i(\xi), \quad n \geq k,$$
$$(\Gamma'(\xi))^p = \sum_{j=0}^{pm} b_j\, \Gamma^j(\xi), \quad p = 1, 2, \tag{5}$$

where p_i, q_i and b_j are constants. Similarly, the UM provides the balance principle technique to evaluate the relation between the parameters n, k and m and satisfies the consistency condition between arbitrary functions in the solutions given by eq. (5) (for details, see [22–25]). Furthermore, the values of p give different types for these solutions by the same criteria described in (I).

4. Analytical water wave solutions of the FBE using the UM

Applying the transformations $u(x,t) = u_1(\xi)$ and $v(x,t) = v_1(\xi)$ in eq. (1), it generates the following ordinary differential equations:

$$-\alpha_1 u_1' + \lambda(u_1 v_1)' + \mu v_1''' = 0, \tag{6}$$
$$-\nu u_1' - \alpha_1 v_1' + \sigma v_1 v_1' = 0, \tag{7}$$

where $u_1' = u_1/\mathrm{d}\xi$, $v_1' = v_1/\mathrm{d}\xi$, $\xi = (x^\alpha/\alpha) - \alpha_1(t^\alpha/\alpha)$ and α_1 is a constant.

Integrating eqs (6) and (7) while taking the constants of integration as zero, it gives

$$-\alpha_1 u_1 + \lambda\,(u_1 v_1) + \mu v_1'' = 0, \tag{8}$$

$$\nu u_1 - \alpha_1 v_1 + \frac{\sigma}{2} v_1^2 = 0. \tag{9}$$

From eq. (9), we have

$$u_1 = \frac{\alpha_1}{\nu} v_1 - \frac{\sigma}{2\nu} v_1^2. \tag{10}$$

Substituting u_1 from eq. (10) into eq. (8), we get

$$-2\alpha_1^2 v_1 + (2\lambda + \sigma)\,\alpha_1 v_1^2 - \lambda\sigma v_1^3 + 2\mu\nu v_1'' = 0. \tag{11}$$

In the next two subsections, we use the UM technique to find the travelling wave solutions of eq. (11) that are classified into two categories, namely the polynomial and the rational function solutions.

4.1 *Polynomial function solutions of the FBE*

To find the polynomial function solutions of the FBE, we assume that

$$v_1(\xi) = \sum_{i=0}^{n} p_i \Gamma^i(\xi),$$

$$(\Gamma'(\xi))^p = \sum_{j=0}^{pm} b_j \Gamma^j(\xi), \quad p = 1, 2, \tag{12}$$

where p_i and b_j are constants. By considering the homogeneous balance relation between v_1'' and v_1^3 in eq. (11), we get $n = m - 1$ and $m = 2, 3, \ldots$. Here, we confine ourselves to find these solutions when $m = 2$ and $p = 1$ or $p = 2$.

4.1.1 *Solitary wave solution.*

To obtain this solution, we consider $p = 1$ in the auxiliary equation given by (12). From eq. (12), we have

$$v_1(\xi) = p_0 + p_1\,\Gamma(\xi),$$

$$\Gamma'(\xi) = b_2\,\Gamma^2(\xi) + b_1\,\Gamma(\xi) + b_0. \tag{13}$$

Substituting eq. (13) into eq. (11) and equating the coefficients of all powers of $\Gamma(\xi)$ to zero, we obtain a set of algebraic equations. By means of a symbolic computations package, we get the following two sets of algebraic equations:

Set 1:

$$p_0 = \frac{2\alpha_1 + 2b_1\sqrt{\mu\nu}}{\sigma}, \quad p_1 = \frac{4b_2\sqrt{\mu\nu}}{\sigma},$$
$$b_0 = -\frac{\alpha_1^2 - \mu\nu b_1^2}{4\mu\nu b_2}, \quad \lambda = \frac{\sigma}{4}, \quad \mu\nu > 0 \tag{14}$$

or

Set 2:

$$p_0 = \frac{b_1\sqrt{-2\mu\nu}}{\sigma}, \quad p_1 = \frac{2b_2\sqrt{-2\mu\nu}}{\sigma},$$
$$b_0 = \frac{2\alpha_1^2 + \mu\nu b_1^2}{4\mu\nu b_2}, \quad \lambda = -\frac{\sigma}{4}, \quad \mu\nu < 0. \tag{15}$$

Now the solutions of eq. (1) are as follows:
From Set 1:

$$u_1(x,t) = -\frac{2\alpha_1^2}{\nu\sigma}\left(-\tanh\left(\frac{\alpha_1\xi}{2\sqrt{\mu\nu}}\right) + \tanh^2\left(\frac{\alpha_1\xi}{2\sqrt{\mu\nu}}\right)\right),$$
$$v_1(x,t) = \frac{2\alpha_1}{\sigma}\left(1 - \tanh\left(\frac{\alpha_1\xi}{2\sqrt{\mu\nu}}\right)\right), \tag{16}$$

where

$$\mu\nu > 0 \quad \text{and} \quad \xi = \frac{x^\alpha}{\alpha} - \alpha_1\left(\frac{t^\alpha}{\alpha}\right), \quad 0 < \alpha \le 1.$$

From Set 2:

$$u_2(x,t) = -\frac{2\alpha_1^2}{\nu\sigma}\left(-\tanh\left(\frac{\alpha_1\xi}{\sqrt{-2\mu\nu}}\right) + \tanh^2\left(\frac{\alpha_1\xi}{\sqrt{-2\mu\nu}}\right)\right),$$
$$v_2(x,t) = \frac{2\alpha_1}{\sigma}\,\tanh\left(\frac{\alpha_1\xi}{2\sqrt{-2\mu\nu}}\right), \tag{17}$$

where

$$\mu\nu < 0 \quad \text{and} \quad \xi = \frac{x^\alpha}{\alpha} - \alpha_1\left(\frac{t^\alpha}{\alpha}\right), \quad 0 < \alpha \le 1.$$

4.1.2 *Soliton wave solution.*

Here, we find the soliton wave solution. Consider $p = 2$ in the auxiliary equation given by (12). From eq. (12), we have

$$v_1(\xi) = p_0 + p_1\,\Gamma(\xi),$$
$$\Gamma'(\xi) = \Gamma(\xi)\sqrt{b_2\,\Gamma^2(\xi) + b_1\,\Gamma(\xi) + b_0}. \tag{18}$$

Substituting eq. (18) into eq. (11), we get a set of algebraic equations that yields

Set 3:

$$p_0 = \frac{2\alpha_1}{\sigma}, \quad p_1 = \frac{4\alpha_1 b_2(4\lambda - \sigma)}{3\sigma\lambda b_1},$$
$$b_0 = \frac{9\lambda(2\lambda - \sigma)b_1^2}{4b_2(4\lambda - \sigma)^2}, \quad \mu = \frac{4b_2(4\lambda - \sigma)^2\,\alpha_1^2}{9\lambda\nu\sigma b_1^2} \tag{19}$$

or

Set 4:

$$p_0 = 0, \quad p_1 = -\frac{4\alpha_1 b_2(2\lambda+\sigma)}{3\sigma\lambda b_1},$$
$$b_0 = \frac{9\lambda\sigma b_1^2}{4b_2(2\lambda+\sigma)^2}, \quad \mu = \frac{4b_2(2\lambda+\sigma)^2\alpha_1^2}{9\lambda\nu\sigma b_1^2} \tag{20}$$

or

Set 5:

$$p_0 = \frac{\alpha_1}{\lambda}, \quad p_1 = -\frac{8\alpha_1 b_2(\lambda-\sigma)}{3\sigma\lambda b_1},$$
$$b_0 = -\frac{9\sigma(2\lambda-\sigma)b_1^2}{32b_2(\lambda-\sigma)^2}, \quad \mu = \frac{16b_2(\lambda-\sigma)^2\alpha_1^2}{9\lambda\nu\sigma b_1^2}. \tag{21}$$

Now the solutions of eq. (1) are as follows:

From Set 3:

$$u_3(x,t) = \frac{12\alpha_1^2 b_1 K_1 \exp\left(\frac{A}{2}\xi\right)\left(k_3 b_1^2 \exp(A\xi) + k_2\left(-1+2b_1\exp\left(\frac{A}{2}\xi\right)\right) - 6b_1 k_1 \exp\left(\frac{A}{2}\xi\right)\right)}{\nu\sigma\left(k_3 b_1^2 \exp(A\xi) + k_2\left(-1+2b_1\exp\left(\frac{A}{2}\xi\right)\right)\right)^2},$$
$$v_3(x,t) = \frac{2\alpha_1}{\sigma}\left(1 - \frac{6b_1 K_1 \exp\left(\frac{A}{2}\xi\right)}{k_3 b_1^2 \exp(A\xi) + k_2\left(-1+2b_1\exp\left(\frac{A}{2}\xi\right)\right)}\right), \tag{22}$$

where

$K_1 = (2\lambda-\sigma)(4\lambda-\sigma)$,
$K_2 = (4\lambda-\sigma)^2$,
$K_3 = 2\lambda^2 - \lambda\sigma - \sigma^2$,
$A = \frac{3b_1\sqrt{2\lambda^2-\lambda\sigma}}{(-4\lambda+\sigma)\sqrt{b_2}}$ and $\xi = \frac{x^\alpha}{\alpha} - \alpha_1\left(\frac{t^\alpha}{\alpha}\right)$,
$b_2 > 0, \quad 0 < \alpha \le 1$.

From Set 4:

$$u_4(x,t) = -\frac{12b_1\alpha_1^2\sqrt{H_1}\exp\left(\frac{B}{2}\xi\right)\left(H_2 b_1^2\exp(B\xi) + 2b_1\sqrt{H_1}(3\sigma - \sqrt{H_1})\exp\left(\frac{B}{2}\xi\right) + H_1\right)}{\nu\left(H_2 b_1^2 \exp(B\xi) - 2b_1 H_1 \exp\left(\frac{B}{2}\xi\right) + H_1\right)^2},$$
$$v_4(x,t) = -\frac{12\alpha_1 b_1\sqrt{H_1}\exp\left(\frac{B}{2}\xi\right)}{H_2 b_1^2\exp(B\xi) + H_1\left(1 - 2b_1\exp\left(\frac{B}{2}\xi\right)\right)}, \tag{23}$$

where

$H_1 = (2\lambda+\sigma)^2,\ H_2 = 4\lambda^2 - 5\lambda\sigma + \sigma^2$,
$B = \frac{3b_1\sqrt{\lambda\sigma}}{(2\lambda+\sigma)\sqrt{b_2}}$ and $\xi = \frac{x^\alpha}{\alpha} - \alpha_1\left(\frac{t^\alpha}{\alpha}\right)$,
$\left(\frac{\lambda\sigma}{b_2}\right) > 0, \quad 0 < \alpha \le 1$.

From Set 5:

$$u_5(x,t) = -\frac{\alpha_1^2\left(8R_1 + R_2 b_1^2\exp(G\xi) + 8R_3 b_1\exp\left(\frac{G}{2}\xi\right)\right)}{2\nu\lambda^2\left(8R_1 + R_2 b_1^2\exp(G\xi) - 16R_1 b_1\exp\left(\frac{G}{2}\xi\right)\right)^2} \times\Big((-2\lambda+\sigma)(8R_1 + R_2 b_1^2\exp(G\xi)) + 8b_1(4R_1\lambda + R_3\sigma)\exp\left(\frac{G}{2}\xi\right)\Big),$$
$$v_5(x,t) = \frac{\alpha_1}{\lambda}\times\left(1 + \frac{8b_1(2R_1+R_3)\exp\left(\frac{G}{2}\xi\right)}{8R_1 + R_2 b_1^2\exp(G\xi) - 16R_1 b_1\exp\left(\frac{G}{2}\xi\right)}\right), \tag{24}$$

where

$R_1 = (\lambda-\sigma)^2,\ R_2 = (4\lambda-\sigma)(2\lambda+\sigma)$,
$R_3 = 4\lambda^2 - 5\lambda\sigma + \sigma^2$,

$G = \frac{3b_1\sqrt{\sigma^2-2\lambda\sigma}}{2(\lambda-\sigma)\sqrt{2b_2}}$ and $\xi = \frac{x^\alpha}{\alpha} - \alpha_1\left(\frac{t^\alpha}{\alpha}\right)$,
$b_2 > 0, \quad 0 < \alpha \le 1$.

4.1.3 *Elliptic wave solution.* The elliptic wave solutions are obtained by using the UM when $p = 2$ in the auxiliary equation given by (12). From eq. (12), we have

$$v_1(\xi) = p_0 + p_1\Gamma(\xi),$$
$$\Gamma'(\xi) = \sqrt{b_4\Gamma^4(\xi) + b_2\Gamma^2(\xi) + b_0}. \tag{25}$$

Substituting eq. (25) into eq. (11) and solving the obtained algebraic equations, we get the following sets of solutions:

Set 6:

$$p_0=\frac{\sqrt{2b_4}\alpha_1}{\sigma\sqrt{-b_2}},\quad p_1=\frac{\alpha_1}{\sigma},\quad \mu=-\frac{\alpha_1^2}{2b_2\nu},\quad \lambda=\sigma \tag{26}$$

or

Set 7:

$$p_0=0,\quad p_1=\frac{2\alpha_1\sqrt{2b_4}}{\sigma\sqrt{-b_2}},$$
$$\mu=\frac{\alpha_1^2}{b_2\nu},\quad \lambda=-\frac{\sigma}{2} \tag{27}$$

or

Set 8:

$$p_0=\frac{2\alpha_1}{\sigma},\quad p_1=\frac{2\alpha_1\sqrt{2b_4}}{\sigma\sqrt{-b_2}},$$
$$\mu=-\frac{\alpha_1^2}{2b_2\nu},\quad \lambda=\frac{\sigma}{4}. \tag{28}$$

We mention that in eqs (26)–(28) $b_4>0$ and $b_2<0$.

Finally, the solutions of eq. (1) are as follows.

From Set 6:

$$u_6(x,t)=\frac{\alpha_1^2(b_2+2b_4\Gamma^2(\xi))}{2b_2\nu\sigma},$$
$$v_6(x,t)=\frac{\alpha_1}{\sigma}\left(1+\frac{\sqrt{2b_4}\Gamma(\xi)}{\sqrt{-b_2}}\right), \tag{29}$$

where

$$\xi=\frac{x^\alpha}{\alpha}-\alpha_1\left(\frac{t^\alpha}{\alpha}\right),\quad 0<\alpha\le 1.$$

From Set 7:

$$u_7(x,t)=\frac{2\alpha_1^2(b_2\sqrt{2b_4}\Gamma(\xi)+2b_4\sqrt{-b_2}\Gamma^2(\xi))}{b_2\nu\sigma\sqrt{-b_2}},$$
$$v_7(x,t)=\frac{2\alpha_1\sqrt{2b_4}\Gamma(\xi)}{\sqrt{-b_2}\sigma}, \tag{30}$$

where

$$\xi=\frac{x^\alpha}{\alpha}-\alpha_1\left(\frac{t^\alpha}{\alpha}\right),\quad 0<\alpha\le 1.$$

From Set 8:

$$u_8(x,t)=\frac{2\alpha_1^2\sqrt{b_4}(\sqrt{-2b_2}\Gamma(\xi)+2\sqrt{b_4}\Gamma^2(\xi))}{b_2\nu\sigma},$$
$$v_8(x,t)=\frac{2\alpha_1}{\sigma}\left(1+\frac{\sqrt{2b_4}\Gamma(\xi)}{\sqrt{-b_2}}\right), \tag{31}$$

where

$$\xi=\frac{x^\alpha}{\alpha}-\alpha_1\left(\frac{t^\alpha}{\alpha}\right),\quad 0<\alpha\le 1.$$

For particular values of $b_i,\ i=0,2,4$, and according to the classification in [31], we get different solutions for $\Gamma(\xi)$ in Jacobi elliptic functions ($\mathrm{sn}(\xi,k)$, $\mathrm{cn}(\xi,k)$, $\mathrm{dn}(\xi,k)$) and so on where $0<k<1$ is the modulus of the Jacobi elliptic functions. When $k\to 0$, $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ and $\mathrm{dn}(\xi)$ degenerate into $\sin(\xi)$, $\cos(\xi)$ and 1, respectively. On the other hand, when $k\to 1$, $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ and $\mathrm{dn}(\xi)$ degenerate to $\tanh(\xi)$, $\mathrm{sech}(\xi)$ and $\mathrm{sech}(\xi)$, respectively.

4.2 *Rational function solutions of the FBE*

To find the rational function solutions of eq. (11), we assume that

$$v_1(\xi)=\sum_{i=0}^{n}p_i\,\Gamma^i(\xi)/\sum_{i=0}^{r}q_i\,\Gamma^i(\xi),\ n\ge r,$$
$$(\Gamma'(\xi))^p=\sum_{j=0}^{pm}b_j\,\Gamma^i(\xi),\ p=1,2, \tag{32}$$

where p_i, q_i and b_j are constants.

By considering the homogeneous balance between v_1'' and v_1^3 in eq. (11), we get $n-r=m-1$, $m=1,2,3,\ldots$. Here, these solutions are obtained when $m=1$ (so $n=r$) and $p=2$.

4.2.1 *Soliton rational solution.* This solution will take the form

$$v_1(\xi)=\frac{p_0+p_1\Gamma(\xi)}{q_0+q_1\Gamma(\xi)},$$
$$\Gamma'(\xi)=\sqrt{b_2\Gamma^2(\xi)+b_1\Gamma(\xi)+b_0}. \tag{33}$$

Substituting eq. (33) into eq. (11) and solving the obtained algebraic equations, we get

Set 9:

$$p_0=\frac{K(2q_0(\alpha_1^2-b_2\mu\nu)+3b_1q_1\mu\nu)}{H\sigma\alpha_1},$$
$$p_1=\frac{2q_1\alpha_1}{\sigma},\quad \lambda=\frac{(H-b_2\mu\nu)\sigma}{2\alpha_1^2},$$
$$b_0=\frac{4Rq_0(b_1q_1-b_2q_0)+9\mu\nu b_1^2q_1^2(H-b_2\mu\nu)}{8H^2q_1^2}, \tag{34}$$

where

$$H=\alpha_1^2+2b_2\mu\nu,\quad R=(\alpha_1^2-b_2\mu\nu)(2\alpha_1^2+b_2\mu\nu)$$

or

Set 10:

$$p_0=\frac{K\big(2q_0(\alpha_1^2-b_2\mu\nu)+3b_1q_1\mu\nu\big)}{H\sigma\alpha_1},$$

$$p_1=\frac{2q_1K}{\sigma\alpha_1},\quad \lambda=\frac{\alpha_1^2\sigma}{2K},$$
$$b_0=\frac{4Rq_0(b_1q_1-b_2q_0)+9\mu\nu b_1^2q_1^2K}{8H^2q_1^2},\tag{35}$$

where $K=\alpha_1^2+\mu\nu b_2$.

The solutions of eq. (1) are as follows.
From Set 9:

$$u_9(x,t)=-8b_2Hq_1\alpha_1^2W_2\exp(\sqrt{b_2}\xi)\times\left(\frac{2H^2q_1^2\exp(2\sqrt{b_2}\xi)-4q_1H\exp(\sqrt{b_2}\xi)(HM-b_2W_2)+RM^2}{(2H^2q_1^2\exp(2\sqrt{b_2}\xi)-4q_1MH^2\exp(\sqrt{b_2}\xi)+RM^2)^2\nu\sigma}\right),$$
$$v_9(x,t)=\frac{2\alpha_1}{\sigma}\left(1-\frac{4(2q_0W_1-3b_1q_1\mu\nu)b_2q_1H\exp(\sqrt{b_2}\xi)}{2H^2q_1^2\exp(2\sqrt{b_2}\xi)-4q_1MH^2\exp(\sqrt{b_2}\xi)+RM^2}\right),\tag{36}$$

where

$$M=b_1q_1-2b_2q_0,\quad W_1=H-\alpha_1^2+\mu\nu b_2,$$
$$W_2=-2q_0W_1+3b_1q_1\mu\nu$$

and

$$\xi=\frac{x^\alpha}{\alpha}-\alpha_1\left(\frac{t^\alpha}{\alpha}\right),\quad b_2>0,\quad 0<\alpha\le 1.$$

From Set 10:

$$u_{10}(x,t)=\frac{2K}{\nu\sigma}\left(1-\frac{4b_2Hq_1K_1\exp(\sqrt{b_2}\xi)}{-K_2+2H^2q_1(q_1\exp(2\sqrt{b_2}\xi)-2M\exp(\sqrt{b_2}\xi)+q_1b_1^2)}\right)\times\left(1+\frac{K}{\alpha_1^2}\left(-1+\frac{4b_2Hq_1K_1\exp(\sqrt{b_2}\xi)}{-K_2+2H^2q_1(q_1\exp(2\sqrt{b_2}\xi)-2M\exp(\sqrt{b_2}\xi)+q_1b_1^2)}\right)\right),$$
$$v_{10}(x,t)=\frac{2K}{\alpha_1\sigma}\left(1-\frac{4b_2Hq_1K_1\exp(\sqrt{b_2}\xi)}{-K_2+2H^2q_1(q_1\exp(2\sqrt{b_2}\xi)-2M\exp(\sqrt{b_2}\xi)+q_1b_1^2)}\right),\tag{37}$$

where

$$K_1=2q_0W_1-3b_1q_1\mu\nu,$$
$$K_2=9b_1^2b_2Kq_1^2\mu\nu+4b_2q_0R(b_1q_1-b_2q_0),$$
$$\xi=\frac{x^\alpha}{\alpha}-\alpha_1\left(\frac{t^\alpha}{\alpha}\right),\quad b_2>0,\quad 0<\alpha\le 1.$$

4.2.2 *Periodic rational solution.* According to the UM algorithms, we assume the periodic rational solution in the form

$$v_1(\xi)=\frac{p_0+p_1\Gamma(\xi)}{q_0+q_1\Gamma(\xi)},$$
$$\Gamma'(\xi)=\sqrt{b_0^2-b_2^2\Gamma^2(\xi)}.\tag{38}$$

Substituting eq. (38) into eq. (11) and solving the obtained algebraic equations, we get

Set 11:

$$p_0=\frac{2\sqrt{2}b_0q_1\alpha_1A_2}{A_1\sigma b_2},$$
$$p_1=\frac{2q_1\alpha_1}{\sigma},$$
$$\lambda=\frac{(3\alpha_1^2-2A_2)\sigma}{2\alpha_1^2},$$
$$q_0=\frac{\sqrt{2}b_0q_1(3\alpha_1^2-2A_2)}{b_2A_1},\tag{39}$$

where

$$A_1=\sqrt{2\alpha_1^2+b_2^2\mu\nu\alpha_1^2-b_2^4\mu^2\nu^2},\quad A_2=\alpha_1^2+b_2^2\mu\nu$$

or

Set 12:

$$p_0=\frac{2\sqrt{2}b_0q_1A_2A_3}{A_1\sigma b_2\alpha_1},\quad p_1=\frac{2q_1A_3}{\sigma\alpha_1},$$
$$\lambda=\frac{\alpha_1^2\sigma}{2A_3},\quad q_0=\frac{\sqrt{2}\,b_0\,q_1(3\alpha_1^2-2A_2)}{b_2A_1},\tag{40}$$

where $A_3=\alpha_1^2-b_2^2\,\mu\nu$.

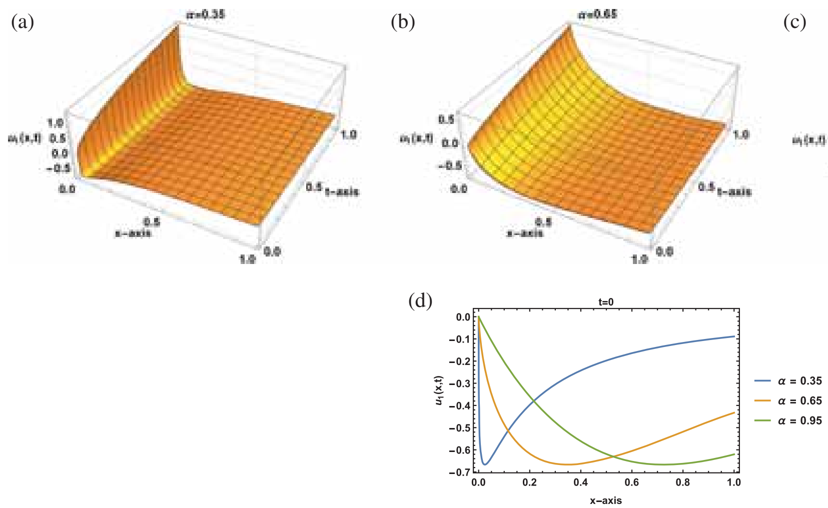

Figure 1. (**a**)–(**d**) The wave solution $u_1(x,t)$ given by (16) in 3D and 2D plots.

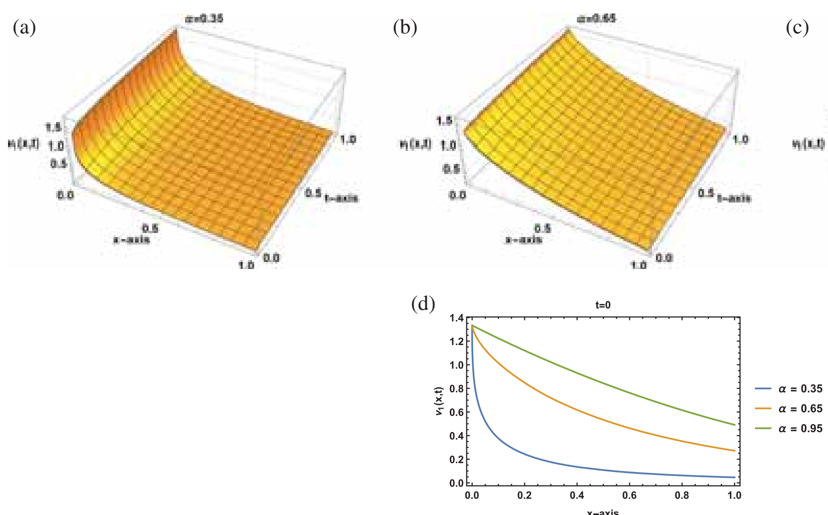

Figure 2. (**a**)–(**d**) The wave solution $v_1(x,t)$ given by (16) in 3D and 2D plots.

Using Sets 11 and 12, we obtain the solutions of eq. (1) as follows:

From Set 11:

$$u_{11}(x,t) = \frac{6\alpha_1^2(\alpha_1^2 - A_2)(2A_2 + \sqrt{2}A_1\sin(b_2\xi))}{(\sqrt{2}(3\alpha_1^2 - 2A_2) + A_1\sin(b_2\xi))^2\nu\sigma},$$

$$v_{11}(x,t) = \frac{2\alpha_1(\sqrt{2}A_2 + A_1\sin(b_2\xi))}{(\sqrt{2}(3\alpha_1^2 - 2A_2) + A_1\sin(b_2\xi))\sigma}, \quad (41)$$

where

$$\xi = \frac{x^\alpha}{\alpha} - \alpha_1\left(\frac{t^\alpha}{\alpha}\right), \quad 0 < \alpha \le 1.$$

From Set 12:

$$u_{12}(x,t) = -\frac{2A_3(\sqrt{2}A_2 + A_1\sin(b_2\xi))}{(\sqrt{2}(3\alpha_1^2 - 2A_2) + A_1\sin(b_2\xi))^2\nu\sigma\alpha_1^2}$$
$$\times(\sqrt{2}(-3\alpha_1^4 + A_2(A_3 + 2\alpha_1^2)) + A_1(A_3 - \alpha_1^2)\sin(b_2\xi)),$$

$$v_{12}(x,t) = \frac{2A_3(\sqrt{2}A_2 + A_1\sin(b_2\xi))}{(\sqrt{2}(3\alpha_1^2 - 2A_2) + A_1\sin(b_2\xi))\sigma\alpha_1}, \quad (42)$$

where

$$\xi = \frac{x^\alpha}{\alpha} - \alpha_1\left(\frac{t^\alpha}{\alpha}\right), \quad 0 < \alpha \le 1.$$

5. Physical explanation

In this section, we introduce the physical interpretation for some of the wave solutions of the FBE for different values of α, $0 < \alpha < 1$.

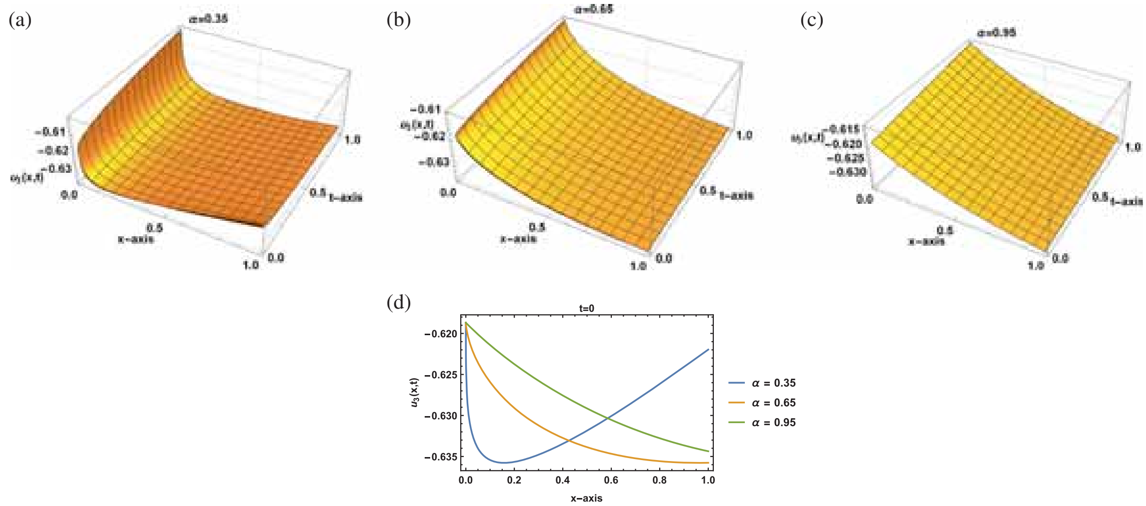

Figure 3. (**a**)–(**d**) The wave solution $u_3(x, t)$ given by (22) in 3D and 2D plots.

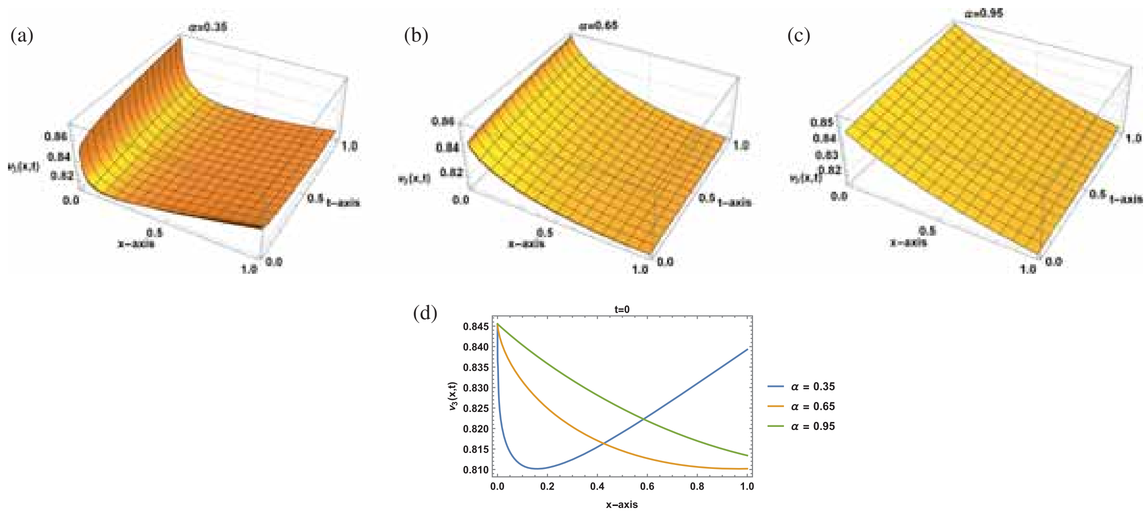

Figure 4. (**a**)–(**d**) The wave solution $v_3(x, t)$ given by (22) in 3D and 2D plots.

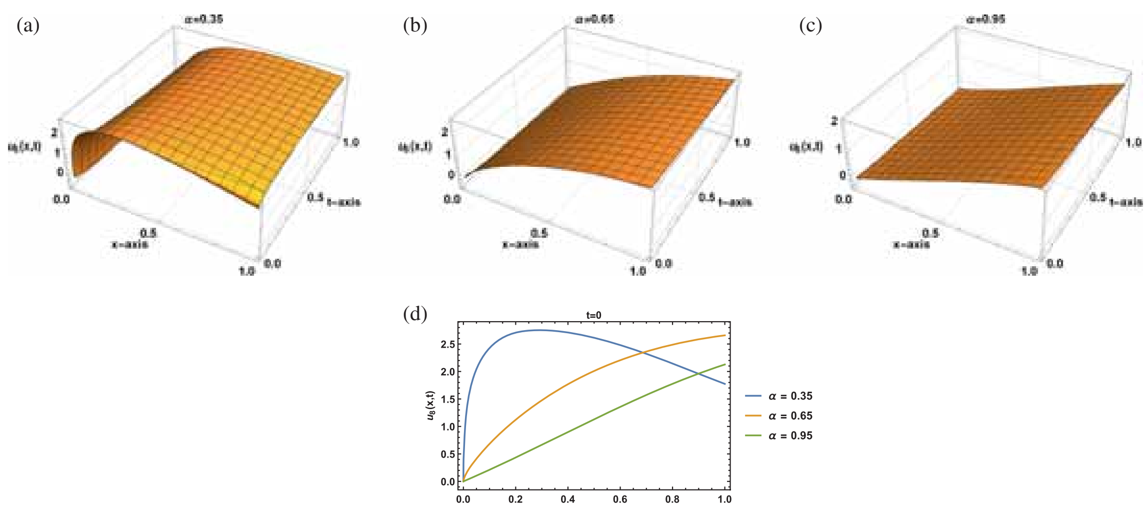

Figure 5. (**a**)–(**d**) The wave solution $u_8(x, t)$ given by (31) when $k = 0.5$ in 3D and 2D plots.

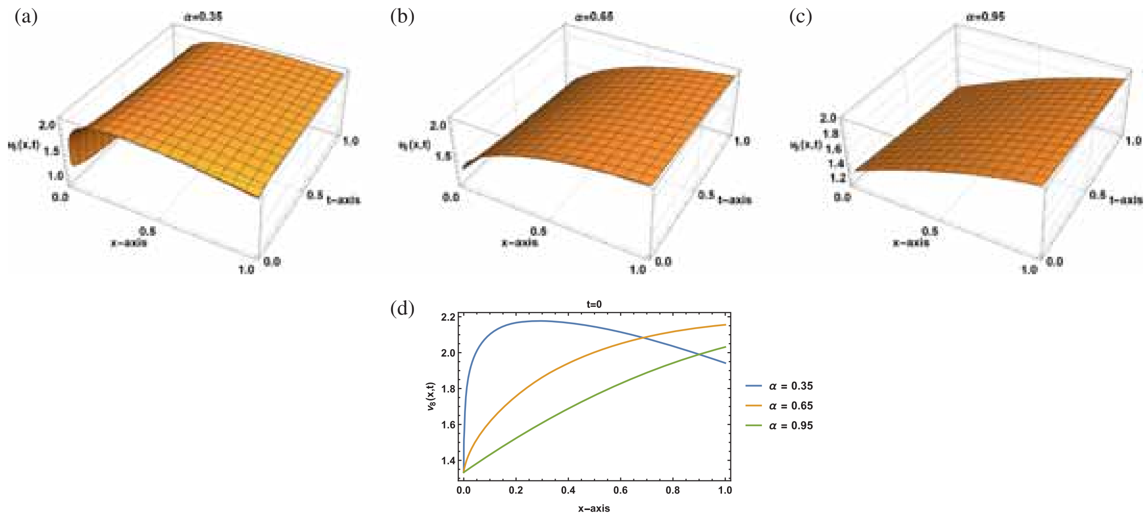

Figure 6. (**a**)–(**d**) The wave solution $v_8(x, t)$ given by (31) when $k = 0.5$ in 3D and 2D plots.

Figures 1 and 2 show the 3D and 2D charts of the solution given by $u_1(x, t)$ and $v_1(x, t)$, respectively with the parameters $\sigma = 0.3$, $\mu = -0.2\, \nu = -0.1$ and $\alpha_1 = 0.2$.

Figures 3 and 4 show the 3D and 2D charts of the solution given by $u_3(x, t)$ and $v_3(x, t)$, respectively, with the parameters $\sigma = 0.3$, $\lambda = 0.2$, $\nu = -0.1$, $b_1 = b_2 = 1$ and $\alpha_1 = 0.2$.

Figures 5 and 6 show the 3D and 2D charts of the solution given by $u_8(x, t)$ and $v_8(x, t)$, respectively, with the parameters $\sigma = 0.3$, $\lambda = 0.2$, $\nu = -0.1$ and $\alpha_1 = 0.2$. According to the classification in [31], we take $b_0 = 1$, $b_2 = -(k^2+1)$ and $b_4 = k^2$ that yield $\Gamma(\xi) = \mathrm{sn}(\xi, k)$.

6. Conclusion

The investigation has been made on the FBE in fluid dynamics. Using the UM (which is described by the wave transformation in fractional form) and symbolic computations, we have constructed different types of exact solutions for the FBE which are classified into two categories: the polynomial function and the rational function solutions. The polynomial function solutions emerged as solitary, soliton and elliptic wave solutions while the rational function solutions include periodic and hyperbolic rational solutions. The UM is direct and simple, and so it will undoubtedly be useful for studying other FDEs. The obtained results of the FBE via this method and the analysis presented in this work are significantly important in fluid dynamics where this equation is used to describe some important physical phenomena. Some of these solutions are represented graphically and the UM seems to possess great advantage and reliability in determining the exact solutions.

References

[1] D Maolin, Z Wang and H Hu, *Sci. Rep.* **3**, 3431 (2013)
[2] S Rana, S Bhattacharya, J Pal, G N Guerekata and J Chattopadhyay, *Physica A* **392(17)**, 3610 (2013)
[3] J Banerjee, U Ghosh, S Sarkar and S Das, *Pramana – J. Phys.* **88(4)**, 70 (2017)
[4] K S Miller and B Ross, *An introduction to the fractional calculus and fractional differential equations* (Wiley, New York, 1993)
[5] M Mirzazadeh, *Pramana – J. Phys.* **85**, 17 (2015)
[6] M Mirzazadeh, *Nonlinear Dyn.* **85**, 2569 (2016)
[7] I Podlubny, *Fractional differential equations* (Academic Press, California, 1999)
[8] M Mirzazadeh, M Eslami and A Biswas, *Pramana – J. Phys.* **82**, 465 (2014)
[9] S Al-Shara, *Appl. Math. Sci.* **8(116)**, 5751 (2014)
[10] K A Gepreel and S Omran, *Chin. Phys. B* **21(11)**, 110204 (2012)
[11] O S Iyiola and G O Ojo, *Pramana – J. Phys.* **85(4)**, 567 (2015)
[12] Y Pandir and Y Gurefe, *Life Sci. J.* **10(2)**, 2701 (2012)
[13] S S Ray, *Chin. Phys. B* **25(4)**, 040204 (2016)
[14] K Hosseini and R Ansari, *Waves Random Complex* **27(4)**, 628 (2017)
[15] M S Osman and A M Wazwaz, *Appl. Math. Comput.* **321**, 282 (2018)
[16] M S Osman, *Nonlinear Dyn.* **89(3)**, 2283 (2017)
[17] M Osman, *Pramana – J. Phys.* **88(4)**, 67 (2017)
[18] M S Osman, *Wave Random Complex* **26(4)**, 434 (2016)
[19] M S Osman, *Nonlinear Dyn.* **87(2)**, 1209 (2017)
[20] B Lu, *J. Math. Anal. Appl.* **395(2)**, 684 (2012)
[21] S Sahoo and S S Ray, *Comput. Math. Appl.* **70(2)**, 158 (2015)
[22] M S Osman, *Optik* **156**, 169 (2018)
[23] M S Osman, J A T Machado and D Baleanu, *Opt. Quantum Electron.* **50(73)**, 1 (2018)

[24] H I Abdel-Gawad and M S Osman, *Kyungpook Math. J.* **53(4)**, 661 (2013)
[25] H I Abdel-Gawad and M Osman, *Indian J. Pure Appl. Math.* **45(1)**, 1 (2014)
[26] L Yan, *Int. J. Numer. Method H* **25(1)**, 33 (2015)
[27] Z Yan and H Zhang, *Phys. Lett. A* **252(6)**, 291 (1999)
[28] R Khalil, M Al Horani, A Yousef and M Sababheh, *J. Comput. Appl. Math.* **264**, 65 (2014)
[29] T Abdeljawad, *J. Comput. Appl. Math.* **279**, 57 (2015)
[30] M Eslami and H Rezazadeh, *Calcolo* **53(3)**, 475 (2016)
[31] H L Zhang, *Appl. Math. Comput.* **208(1)**, 144 (2009)