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Analysing the stability of a delay differential equation involving two delays

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Abstract. Analysis of systems involving delay is a popular topic among the applied scientists. In the present work, we analyse the generalised equation $D^{\alpha}x(t) = g(x(t-\tau_1), x(t-\tau_2))$ involving two delays, viz. $\tau_1 \geq 0$ and $\tau_2 \geq 0$. We use stability conditions to propose the critical values of delays. Using examples, we show that the chaotic oscillations are observed in the unstable region only. We also propose a numerical scheme to solve such equations.

Keywords. Fractional order; chaos; multiple delay; stability.

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1. Introduction

Equations involving non-local operators are crucial in modelling natural systems. The memory involved in such systems cannot be modelled properly by using local operators such as ordinary integer-order derivative. The fractional-order derivative and the time delay (lag) are proved to be suitable for this task.

If the order of the derivative involved in the model is non-integer, then it can be called a fractional derivative. The analysis of fractional derivatives can be found in [1–5]. The dynamics and applications of fractional-order differential equations is an important topic of research [6–8].

If the modelling of differential equation includes the past values of the state variable, then it is called a delay differential equation (DDE). The basic analysis and various applications of DDE are discussed in [9–12].

Stability of fractional-order delay differential equations (FDDEs) is discussed by Matignon in [13]. Bounded input bounded output (BIBO) stability of linear time-invariant FDDEs is discussed in [14,15]. A numerical algorithm is proposed by Hwang and Cheng [16] to test the stability of FDDEs. The stability of a class of FDDE is studied in [17]. In [18], the present author proposed the stability and bifurcation analysis of generalised FDDE. Chaos in various FDDEs is analysed by Bhalekar and Daftardar-Gejji [19], Daftardar-Gejji *et al* [20] and Bhalekar [21]. The application of FDDE in NMR is discussed by Bhalekar *et al* [22,23].

Equations involving single delay are extensively studied in the literature. However, the analysis of the systems with multiple delay is complicated and hence the work is relatively rare. The geometry of stable regions in differential equations with two delays is presented in [24] and references cited therein. The Hopf bifurcation in these systems is studied in [25–27] and references cited therein. Applications of systems involving multiple delay can be found in various natural systems. Bi and Ruan [28] discussed these systems in tumour and immune system interaction models. Prey–predator system with multiple delay is analysed by Gakkhar and Singh [29].

In this work, we extend our analysis in [18] to consider two-delay differential equation with fractional order. We present the stability and bifurcation analysis of these equations, propose numerical method and solve examples.

The paper is organised as follows: Basic definitions are listed in §2. The expressions for critical curves are derived in §3. Section 4 deals with the numerical method for solving the general two-delay models. Illustrated examples are presented in §5. Conclusions are summarised in §6.

2. Preliminaries

In this section, we discuss some basic definitions [1,2].

DEFINITION 1

A real function $f(t)$, $t > 0$, is said to be in space C_α, $\alpha \in \Re$, if there exists a real number p $(>\alpha)$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$.

DEFINITION 2

A real function $f(t)$, $t > 0$, is said to be in space C_α^m, $m \in I\!N \cup \{0\}$, if $f^{(m)} \in C_\alpha$.

DEFINITION 3

Let $f \in C_\alpha$ and $\alpha \geq -1$, then the (left-sided) Riemann–Liouville integral of order μ, $\mu > 0$, is given by

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau)\,\mathrm{d}\tau, \quad t > 0. \tag{1}$$

DEFINITION 4

The Caputo fractional derivative of f, $f \in C_{-1}^m$, $m \in I\!N$, is defined as

$$\begin{aligned} D^\mu f(t) &= \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), \quad \mu = m \\ &= I^{m-\mu} \frac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}, \quad m-1 < \mu < m. \end{aligned} \tag{2}$$

Note that for $m - 1 < \mu \leq m$, $m \in IN$,

$$I^\mu D^\mu f(t) = f(t) - \sum_{k=0}^{m-1} \frac{\mathrm{d}^k f}{\mathrm{d}t^k}(0) \frac{t^k}{k!}, \tag{3}$$

$$I^\mu t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}. \tag{4}$$

3. Main results

We consider the following generalised delay differential equation (GDDE)

$$D^\alpha x(t) = g(x(t-\tau_1), x(t-\tau_2)), \tag{5}$$

where D^α is a Caputo fractional derivative of order $\alpha \in (0, 1]$, g is a continuously differentiable function and $\tau_1 \geq 0$, $\tau_2 \geq 0$ are delays.

A number $x^* \in \mathbb{R}$ is called an equilibrium point of (5) if

$$g(x^*, x^*) = 0. \tag{6}$$

Linearisation of system (5) in the neighbourhood of equilibrium point x^* is given by

$$D^\alpha \xi = a\xi_{\tau_1} + b\xi_{\tau_2}, \tag{7}$$

where $x_\tau(t) = x(t-\tau)$ and a, b are partial derivatives of g with respect to the first and second variables evaluated at (x^*, x^*), respectively. The characteristic equation of the system is

$$\lambda^\alpha = a\exp(-\lambda\tau_1) + b\exp(-\lambda\tau_2). \tag{8}$$

3.1 *Stability of equilibrium*

We present the stability analysis by using the same strategy as in [18]. If all the eigenvalues λ_i of the characteristic equation (8) satisfy

$$\mathrm{Re}(\lambda_i) < 0, \ \forall i, \tag{9}$$

then the equilibrium x^* is asymptotically stable.

Hence, the equation without delay (i.e. with $\tau_1 = \tau_2 = 0$) is asymptotically stable if

$$a + b < 0. \tag{10}$$

Now, we assume that $\tau_1 > 0$, $\tau_2 > 0$ and $\lambda = u + \imath v$, $u, v \in \Re$. The stability properties of equilibrium will change at $\lambda = \imath v$. In this case, the characteristic equation takes the form

$$(\imath v)^\alpha = a\exp(-\imath v\tau_1) + b\exp(-\imath v\tau_2). \tag{11}$$

Simplifying, we get

$$v^\alpha \cos\left(\frac{\alpha\pi}{2}\right) - a\cos(v\tau_1) = b\cos(v\tau_2), \tag{12}$$

$$v^\alpha \sin\left(\frac{\alpha\pi}{2}\right) + a\sin(v\tau_1) = -b\sin(v\tau_2). \tag{13}$$

Squaring and adding (12) and (13), we get the following expression:

$$v^{2\alpha} + a^2 - 2av^\alpha \cos\left(\frac{\alpha\pi}{2} + v\tau_1\right) = b^2. \tag{14}$$

This gives

$$\tau_1 = \frac{1}{v}\left(-\frac{\alpha\pi}{2} + \arccos\left[\frac{v^{2\alpha} + a^2 - b^2}{2av^\alpha}\right]\right). \tag{15}$$

Using similar arguments and rewriting systems (12) and (13), we get

$$\tau_2 = \frac{1}{v}\left(-\frac{\alpha\pi}{2} + \arccos\left[\frac{v^{2\alpha} - a^2 + b^2}{2bv^\alpha}\right]\right). \tag{16}$$

Equations (15) and (16) are parametric expressions of τ_1 and τ_2 in parameter v, respectively. These can be used to plot critical curves.

3.2 *Eigenvalue spectrum*

In this subsection, we prove that the eigenvalue spectrum of the FDDE (5) is discrete. Further, we show that the real part of all eigenvalues is bounded, i.e. there exists a real number μ such that all the eigenvalues lie on the left side of the vertical line $y = \mu$ in the complex plane.

Theorem 1. *The eigenvalue spectrum of the FDDE* (5) *is discrete.*

Proof. Consider the characteristic equation (8) of FDDE (5) in the neighbourhood of an equilibrium x^*. If λ_* is an eignevalue then

$$\lambda_*^\alpha = a\exp(-\lambda_*\tau_1) + b\exp(-\lambda_*\tau_2). \tag{17}$$

If the eigenvalue spectrum is not discrete (i.e. continuous), then there exists a neighbourhood $B(\lambda_*)$ of λ_* such that for all sufficiently small ϵ, the numbers $\lambda_* + \epsilon \in B(\lambda_*)$ are eigenvalues. Therefore,

$$(\lambda_* + \epsilon)^\alpha = a\exp(-\lambda_*\tau_1)\exp(-\epsilon\tau_1) + b\exp(-\lambda_*\tau_2)\exp(-\epsilon\tau_2). \tag{18}$$

However, eq. (18) shows that the value of ϵ is fixed for given values of parameters a, b, τ_1 and τ_2. Thus, eq. (18) cannot be satisfied identically for all ϵ such that $\lambda_* + \epsilon \in B(\lambda_*)$.

For example, if we take $a = b = 1$, $\tau_1 = 0.5$, $\tau_2 = 0.25$ and $\alpha = 0.8$ then we get $\lambda_* = 1.3092$. With these values, eq. (18) produces $\epsilon = -8.3914 + 97.707\iota$. However, eq. (18) is not satisfied by the values between λ_* and $\lambda_* + \epsilon$.

This proves the result. □

The following result and its proof are analogous to Lemma 4.1 (p. 18) in ref. [30].

Theorem 2. *If there is a sequence* $\{\lambda_j\}$ *of solutions of characteristic eq.* (8) *such that* $|\lambda_j| \longrightarrow \infty$ *as* $j \longrightarrow \infty$, *then* $\mathrm{Re}(\lambda_j) \to -\infty$ *as* $j \to \infty$. *Thus, there is a real number* μ *such that all the eigenvalues* λ *satisfy* $\mathrm{Re}(\lambda) < \mu$.

4. Numerical method

In this section, we present a numerical scheme based on [31] to solve GDDE (5) with initial function $x(t) = \phi(t)$, $t \le 0$. This initial value problem is equivalent to the integral equation

$$x(t) = \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f(x(\zeta-\tau_1), x(\zeta-\tau_2))\mathrm{d}\zeta + \phi(0), \quad t \in [0, T]. \tag{19}$$

We consider the uniform grid $\{t_n = nh : n = -k, -k+1, \ldots, -1, 0, 1, \ldots, N\}$, where k and N are the chosen integers such that $h = T/N = \tau_1/k_1 = \tau_2/k_2$ and $k = \max\{k_1, k_2\}$. Let x_j be an approximation to $x(t_j)$. Note that $x_j = \phi(t_j)$ if $j \le 0$ and $x(t_j - \tau_1) = x_{j-k_1}$, $x(t_j - \tau_2) = x_{j-k_2}$.

Evaluating (19) at $t = t_{n+1}$ and using the product trapezoidal formula, we get

$$x_{n+1} = \phi(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \times \sum_{j=0}^{n+1} a_{j,n+1} f(x_{j-k_1}, x_{j-k_2}), \tag{20}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha & \text{if } j=0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} \\ \quad -2(n-j+1)^{\alpha+1} & \text{if } 1 \le j \le n, \\ 1 & \text{if } j = n+1. \end{cases}$$

5. Illustrative examples

In this section, we analyse the stability and solve some examples using the numerical scheme presented in the preceding section.

Example 1 *(Uçar system).* The fractional-order Uçar system [21] is described by $D^\alpha x(t) = \delta x(t-\tau) - \epsilon[x(t-\tau)]^3$. The stability analysis and numerical computations of this system with $\epsilon = \delta = 1$ are described in [21]. If we take $\alpha = 0.9$ then the system is stable for $\tau \le 0.8$. The stable orbit is shown in figure 1 for $\tau = 0.7$. The limit cycles are observed for $0.8 < \tau \le 1.9$ (see figure 2 with $\tau = 1.8$). The chaotic oscillator can be seen for the delay $\tau = 2.0$, as shown in figure 3.

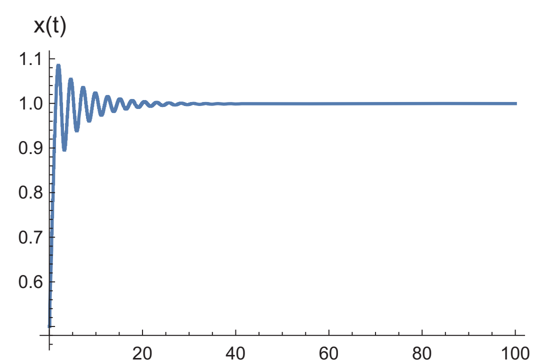

Figure 1. Stable orbit of Uçar system with one delay for $\alpha = 0.9$ and $\tau = 0.7$.

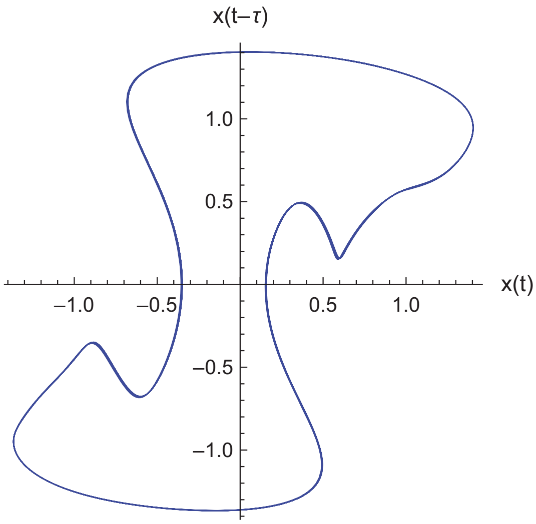

Figure 2. Limit cycle of Uçar system with one delay for $\alpha = 0.9$ and $\tau = 1.8$.

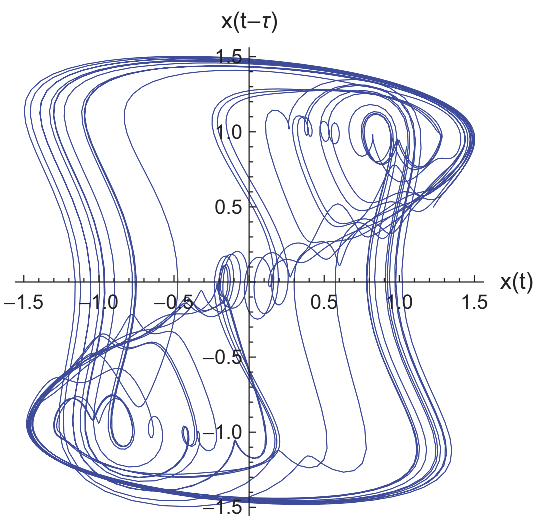

Figure 3. Chaos in Uçar system with one delay for $\alpha = 0.9$ and $\tau = 2.0$.

Now, we consider the fractional-order generalisation of Uçar system [21,32] involving two delays:

$$D^{\alpha}x(t) = \delta x(t - \tau_1) - \epsilon[x(t - \tau_2)]^3, \tag{21}$$

where δ and ϵ are positive parameters.

There are three equilibrium points of system (21), viz. $0, \pm\sqrt{\delta/\epsilon}$. If $\tau_1 = \tau_2 = 0$ then 0 is unstable whereas $\pm\sqrt{\delta/\epsilon}$ are stable. We take $\delta = 1$.

The characteristic equation of system (21) at $x^* = \pm\sqrt{\delta/\epsilon}$ is

$$\lambda^{\alpha} - \exp(-\lambda\tau_1) + 3\exp(-\lambda\tau_2) = 0. \tag{22}$$

Note that the characteristic equation and hence stability do not depend on ϵ. We take $\epsilon = 1$.

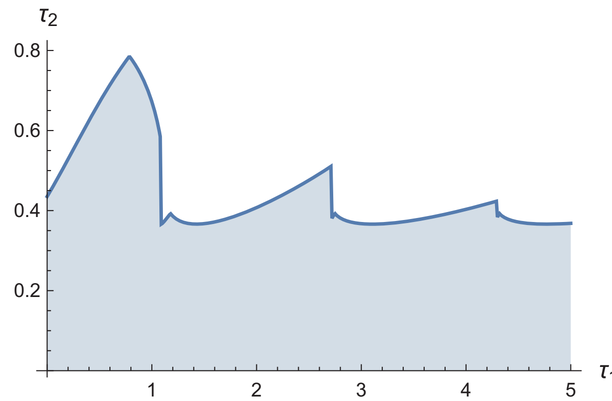

Figure 4. Extended stability region of Uçar system for $\alpha = 1$.

Using (15) and (16) we obtain the critical values of delay as

$$\tau_1 = \frac{1}{v}\left(-\frac{\alpha\pi}{2} + \arccos\left[\frac{v^{2\alpha} - 8}{2v^{\alpha}}\right]\right) \tag{23}$$

and

$$\tau_2 = \frac{1}{v}\left(-\frac{\alpha\pi}{2} + \arccos\left[\frac{v^{2\alpha} + 8}{-6v^{\alpha}}\right]\right). \tag{24}$$

The following stability result based on [26] for the integer-order case $\alpha = 1$ is proposed in [32].

Theorem 5.1 [32]. *The system $\dot{x}(t) = \delta x(t - \tau_1) - \epsilon[x(t - \tau_2)]^3$ is locally stable if one of the following conditions hold*:

(i) $\tau_2 \in [0, 0.366667/\delta)$,
(ii) $\tau_2 \in [0.366667/\delta, 0.43521/\delta)$ *and* $\tau_1 \in [0, 0.0697127/\delta)$.

It can be verified that the stability region described in this theorem can be readily obtained by using expressions (23) and (24). Further, as shown in figure 4, we can extend this region for higher values of delay also.

Figure 5 shows the critical curves for different values of α.

Now, we verify these stability results using numerical computations. In figure 6, we have taken $\tau_1 = 0.4$, $\tau_2 = 0.5$ and $\alpha = 0.9$. These values are in the stable region and the figure shows the orbit converging to an equilibrium state. In figure 7, we take the parameter values $\tau_1 = 1.6$, $\tau_2 = 1.4$ and $\alpha = 0.9$ in unstable region. In this case, the system exhibits chaotic oscillations.

In figures 8 and 9, we have taken $\tau_2 = 0$ and kept all other parameters the same as in figures 6 and 7, respectively. These one-delay cases produce stable orbits.

Example 2 *(Ikeda system)*. The detailed analysis of the fractional-order Ikeda system $D^{\alpha}x(t) = -3x(t - \tau) + 24\sin(x(t - \tau))$ with single delay τ is presented by

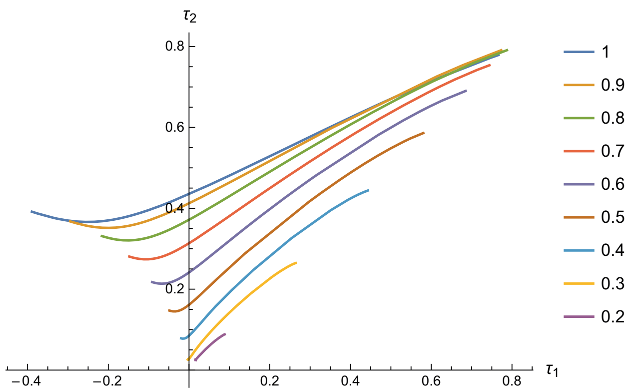

Figure 5. Critical curves of Uçar system for different values of α.

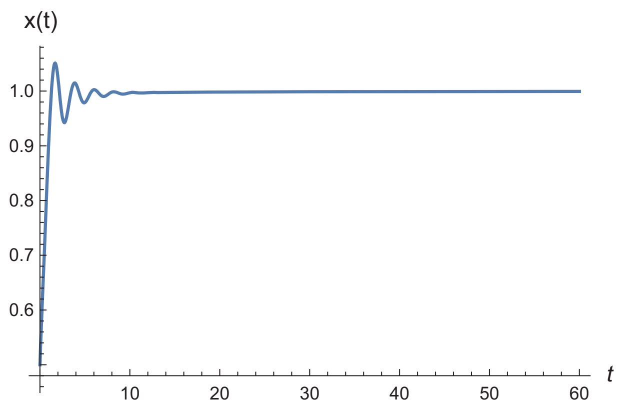

Figure 6. Stable orbit of Uçar system for $\alpha = 0.9$, $\tau_1 = 0.4$ and $\tau_2 = 0.5$.

Bhalekar in [18]. If we take $\alpha = 0.9$ then the system generates stable solutions in the range $0 \le \tau < 0.09$. Limit cycles are observed for $0.09 < \tau < 0.17$. The chaotic attractor in this system for $\tau = 0.17$ is given in figure 10.

Let us consider the generalisation of fractional-order Ikeda equation [18,33] with two delays

$$D^{\alpha}x(t) = -3x(t-\tau_1) + 24\sin(x(t-\tau_2)),$$
$$0 < \alpha \le 1. \tag{25}$$

As discussed in [18], there are seven equilibrium points. The points $x_0 = 0$ and $x_{3,4} = \pm 7.49775$ are unstable, whereas $x_{1,2} = \pm 2.7859$ and $x_{5,6} = \pm 7.95732$ are stable at $\tau_1 = \tau_2 = 0$. The characteristic equation of system (25) at the equilibrium point x^* is

$$\lambda^{\alpha} = -3\exp(-\lambda\tau_1) + 24\cos(x^*)\exp(-\lambda\tau_2). \tag{26}$$

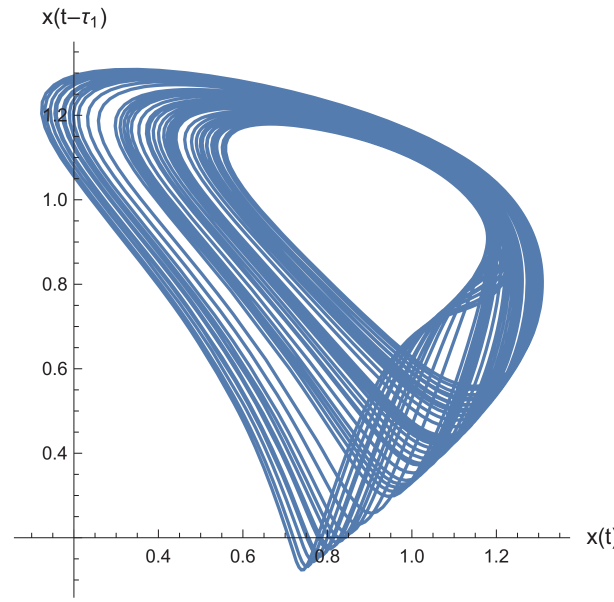

Figure 7. Chaotic attractor of Uçar system for $\alpha = 0.9$, $\tau_1 = 1.6$ and $\tau_2 = 1.4$.

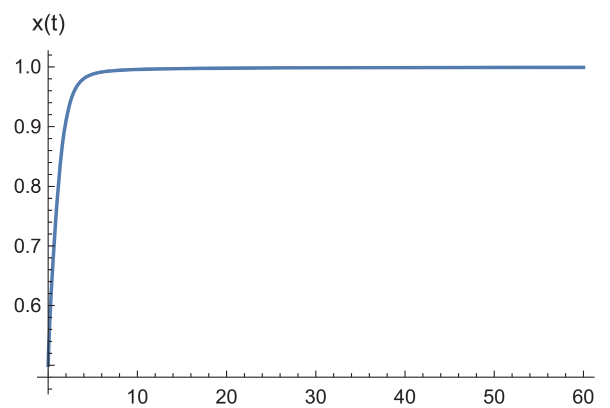

Figure 8. Stable orbit of Uçar system for $\alpha = 0.9$, $\tau_1 = 0.4$ and $\tau_2 = 0$.

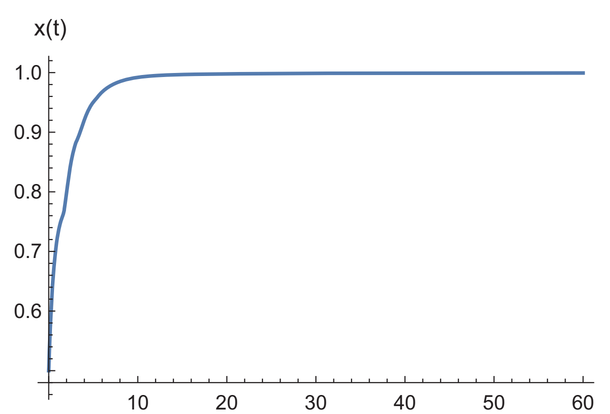

Figure 9. Stable orbit of Uçar system for $\alpha = 0.9$, $\tau_1 = 1.6$ and $\tau_2 = 0$.

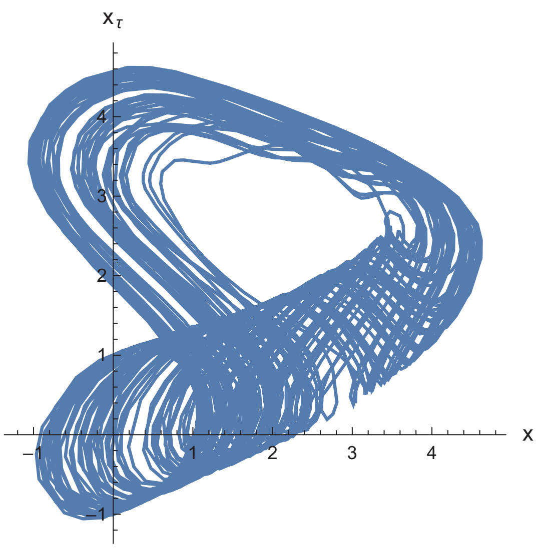

Figure 10. Chaotic attractor of Ikeda system with single delay $\tau = 0.17$ and fractional-order $\alpha = 0.9$.

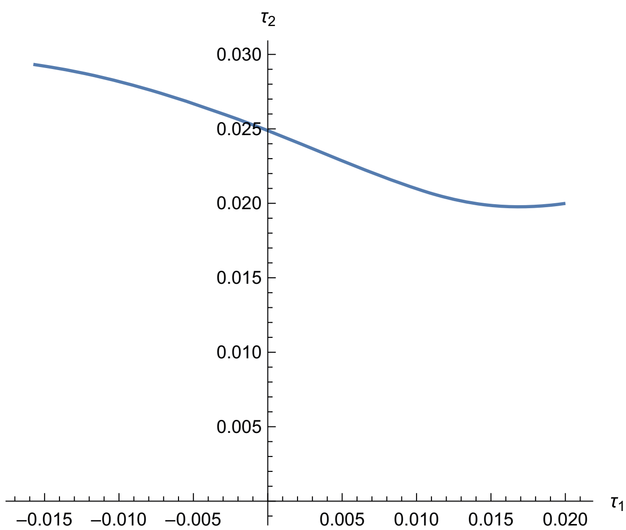

Figure 11. Critical curve of Ikeda system for $\alpha = 0.7$.

If we take $x^* = x_{1,2}$ then the characteristic equation becomes

$$\lambda^\alpha = -3\exp(-\lambda\tau_1) - 22.4977\exp(-\lambda\tau_2). \qquad (27)$$

This gives $a = -3$ and $b = -22.4977$. The critical curve in this case is given in figure 11 for $\alpha = 0.7$. In figure 12, we have presented the stable solution of system (25) with $\tau_1 = 0.02$ and $\tau_2 = 0.01$ in the stable region. If we consider the values $\tau_1 = 0.01$ and $\tau_2 = 0.1$ in the unstable region, then we get the chaotic attractor (see figure 13).

In figures 14 and 15, we have taken $\tau_2 = 0$ and kept all other parameters the same as in figures 12 and 13, respectively. These one-delay cases also produce stable orbits.

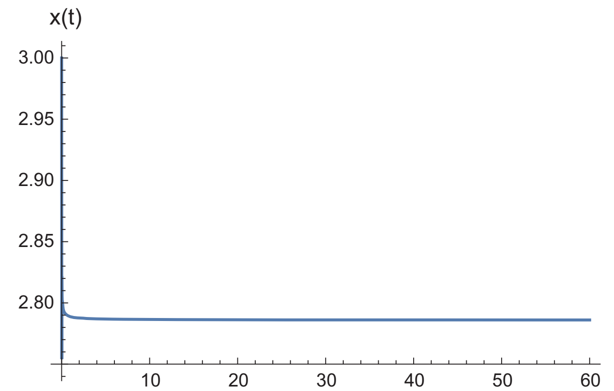

Figure 12. Stable orbit of Ikeda system for $\alpha = 0.7$, $\tau_1 = 0.02$ and $\tau_2 = 0.01$.

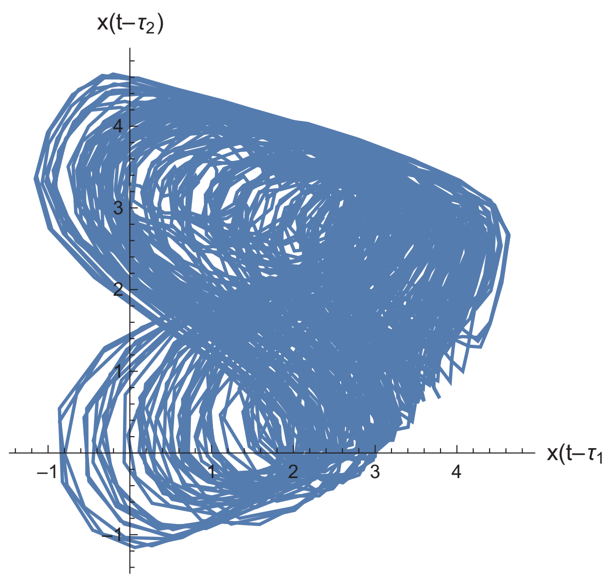

Figure 13. Chaotic attractor shown by Ikeda system for $\alpha = 0.7$, $\tau_1 = 0.01$ and $\tau_2 = 0.1$.

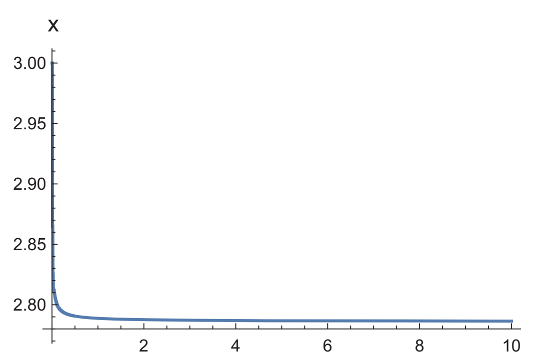

Figure 14. Stable orbit of Ikeda system for $\alpha = 0.7$, $\tau_1 = 0.02$ and $\tau_2 = 0$.

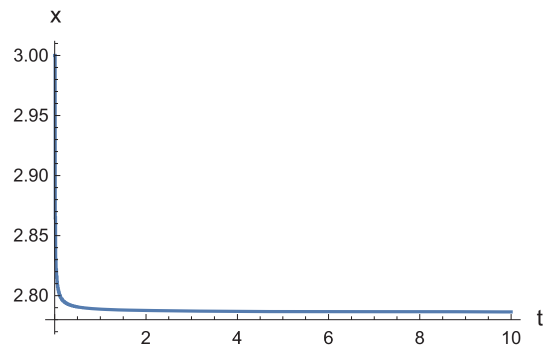

Figure 15. Stable orbit of Ikeda system for $\alpha = 0.7$, $\tau_1 = 0.01$ and $\tau_2 = 0$.

6. Conclusion

In [18], we have presented the complete analysis of the generalised equation involving delay. However, the analysis of the systems with multiple delay is not as simple as that of a single delay.

In this paper, we have presented the expressions for critical values of the generalised two-delay system. We obtained the parametric relation between the delays τ_1 and τ_2. The region bounded by horizontal axis and the critical curve is the stable region if the eigenvalues are stable at $\tau_1 = \tau_2 = 0$. The nonlinear system may exhibit chaotic oscillations in the unstable region. We also have presented numerical scheme to solve these systems. Further, the results are illustrated with two examples. The behaviour of the system with one delay τ and a counterpart with two delays τ_1, τ_2 is the same if $\tau_1 = \tau_2 = \tau$. However, it can be seen from the illustrated examples that the stability properties of the system depend on the delay τ_1 as well as τ_2.

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