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Symmetry analysis of some nonlinear generalised systems of space–time fractional partial differential equations with time-dependent variable coefficients

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Abstract. In this paper, the Lie group analysis method is applied to carry out the Lie point symmetries of some space–time fractional systems including coupled Burgers equations, Ito's system, coupled Korteweg–de-Vries (KdV) equations, Hirota–Satsuma coupled KdV equations and coupled nonlinear Hirota equations with time-dependent variable coefficients with the Riemann–Liouville derivative. Symmetry reductions are constructed using Lie symmetries of the systems. To the best of our knowledge, nobody has so far derived the invariants of space–time nonlinear fractional partial differential equations with time-dependent coefficients.

Keywords. Fractional differential equations with time-dependent variable coefficients; Lie symmetry analysis; Erdèlyi–Kober operators; Riemann–Liouville fractional derivative.

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1. Introduction

Nonlinear phenomena consistently appear in the study of applied mathematics, engineering and many related scientific fields [1–3]. Solving nonlinear systems is a great task in mathematical analysis and applications. In daily life, we come across many real-life phenomena which can be described by mathematical structure of nonlinear partial differential equations (NLPDEs) of integer or non-integer order. In order to interpret the nonlinear phenomena better, in place of NLPDEs of integer order, nonlinear fractional partial differential equations (NLFPDEs) can be used. There are many methods for solving NLFPDEs [4–9]. Numerical methods give approximate solutions [10–13]. But the study of exact solution gives proper understanding of NLFPDEs. Exact solution of NLFPDEs facilitates the authentication of numerical solvers and also supports the stability analysis of the solution. Various methods have been used by many researchers for solving NLPDEs in the last two or three decades. Some of the very important methods are: the exp-function method [14,15], the fractional sub-equation method [16], the homotopy method [17,18], the inverse scattering method, the adomian decomposition method [19], the Laplace decomposition method [20], the variational iteration method [21], the Lie symmetry method [22–26], etc.

The application of the Lie symmetry method to NLFPDEs is novel. This method is basically introduced by the Norwegian mathematician Sophus Lie at the end of the 19th century. The Lie symmetry analysis is one of the very effective methods for obtaining exact solutions of NLFPDEs. This method has been introduced, firstly, for solving ordinary differential equations. After that, it has been demonstrated that by the Lie symmetry method, systems of differential equations can be converted to other equivalent systems of differential equations in reduced form. After using the one-parameter Lie group of the infinitesimal transformations with compliance of the invariant conditions, the solution space of the differential equation remains invariant into another space. There have been many approaches for solving NLFPDEs [3,27–35], yet not much work has been performed on the Lie symmetry analysis of space–time NLFPDEs.

The main aim of this paper is to extend the application of the Lie symmetry approach from systems of space–time NLFPDEs with constant coefficients to the systems of NLFPDEs with time-dependent variable coefficients. The invariance of space–time NLFPDEs with constant coefficients under the Lie group of scaling transformations has been studied by only a few researchers [36,37]. The principal aim of this paper is to deliberate the space–time fractional systems with variable coefficients. In this study, we propose appropriate prolongation operators and used them to analyse some space–time fractional nonlinear systems with time-dependent variable coefficients. The considered systems of NLFPDEs are reduced to systems of nonlinear fractional ordinary differential equations (NLFODEs) containing the left- and right-hand side Erdélyi–Kober fractional differential operators [37,38].

This paper is divided into four sections. The sections are arranged as follows: Section 1 is introductory. In §2, the Lie symmetry method is introduced to deal with systems of space–time fractional PDEs. The Lie symmetry analysis of five systems of NLFPDEs with time-dependent variable coefficients with space–time derivatives of fractional order is presented in §3. Section 4 contains conclusion.

2. Lie symmetry analysis for a system of fractional partial differential equations

In this section, the Lie symmetry method has been introduced [39,40] for a system of space–time NLFPDEs with two independent $\mathbf{x} = (x, t)$ and p-dependent variables $\mathbf{v}= (v^1, v^2, \ldots, v^p)$. Some basic definitions and formulas are given below.

2.1 *Basic definitions*

2.1.1 *Riemann–Liouville fractional derivative.* The formal definition of Riemann–Liouville fractional derivative [41–43] is given as follows.

Let f: $[a, b] \subseteq \mathcal{R} \to \mathcal{R}$ such that $\partial^m f/\partial x^m$ is continuous and integrable $\forall m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $m \leq [\gamma] + 1$. Then, the Riemann–Liouville fractional derivative of order $\gamma > 0$ is defined as

$$\frac{\partial^\gamma f(x,t)}{\partial t^\gamma} = \begin{cases} \dfrac{1}{\Gamma(1+[\gamma]-\gamma)} \dfrac{\partial^{1+[\gamma]}}{\partial t^{1+[\gamma]}} \displaystyle\int_0^t (t-s)^{[\gamma]-\gamma} f(x,s)\mathrm{d}s, \\ \quad t>0, \quad [\gamma] < \gamma < [\gamma]+1, \\ \dfrac{\partial^m f(t)}{\partial t^m}, \quad \gamma = [\gamma] = m \in \mathbb{N}, \end{cases} \tag{1}$$

where $\Gamma(\gamma)$ is the Euler's gamma function.

2.1.2 *Erdèlyi–Kober operator.* The left-hand side Erdèlyi–Kober fractional differential operator $(\mathcal{T}_\varrho^{\vartheta,\alpha})$ is defined as

$$(\mathcal{T}_\varrho^{\vartheta,\alpha} g)(y) = \prod_{k=0}^{r-1} \left(\vartheta + k - \frac{1}{\varrho} y \frac{\mathrm{d}}{\mathrm{d}y}\right) (\mathcal{M}_\varrho^{\vartheta+\alpha, r-\alpha} g)(y)), \quad y>0,\ \varrho>0,\ \alpha>0,$$
$$r = \begin{cases} [\alpha]+1 & \text{if } \alpha \notin \mathbb{N}, \\ \alpha & \text{if } \alpha \in \mathbb{N}, \end{cases} \tag{2}$$

where

$$(\mathcal{M}_\varrho^{\vartheta,\alpha} g)(y) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^\infty (\rho-1)^{\alpha-1} \rho^{-(\vartheta+\alpha)} g(y\rho^{1/\varrho}) \mathrm{d}\rho \\ \quad \text{if } \alpha > 0, \\ g(y) \quad \text{if } \alpha = 0 \end{cases} \tag{3}$$

is the left-hand side Erdèlyi–Kober fractional integral operator.

The right-hand side Erdèlyi–Kober fractional differential operator $(\mathcal{R}_\varrho^{\vartheta,\beta})$ is defined as

$$(\mathcal{R}_\varrho^{\vartheta,\beta} f)(y) := \prod_{k=1}^{r} \left(\vartheta + k + \frac{1}{\varrho} y \frac{\mathrm{d}}{\mathrm{d}y}\right) (\mathcal{I}_\varrho^{\vartheta+\beta, r-\beta} f)(y)), \quad y>0,\ \varrho>0,\ \beta>0,$$
$$r = \begin{cases} [\beta]+1 & \text{if } \beta \notin \mathbb{N}, \\ \beta & \text{if } \beta \in \mathbb{N}, \end{cases} \tag{4}$$

where

$$(\mathcal{I}_\varrho^{\vartheta,\beta} f)(y) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^1 (1-\rho)^{\beta-1} \rho^{\vartheta} f(y\rho^{1/\varrho}) \mathrm{d}\rho \\ \quad \text{if } \beta > 0, \\ f(y) \quad \text{if } \beta = 0 \end{cases} \tag{5}$$

is the right-hand side Erdèlyi–Kober fractional integral operator.

2.2 *Symmetry analysis*

Consider a system of space–time fractional PDEs having the following form:

$$F_h\left(\mathbf{x}, \mathbf{v}, \frac{\partial^\gamma \mathbf{v}}{\partial t^\gamma}, \frac{\partial^\beta \mathbf{v}}{\partial x^\beta}, \frac{\partial \mathbf{v}}{\partial x}, \frac{\partial^2 \mathbf{v}}{\partial x^2}, \ldots, \frac{\partial^n \mathbf{v}}{\partial x^n}\right) = 0, \quad h = 1, 2, \ldots, \tag{6}$$

where $\mathbf{x} = (x, t)$ is the independent variable, $\mathbf{v} = (v^1, v^2, \ldots, v^p)$ are the dependent variables, $\partial\mathbf{v}/\partial x$ and $\partial^2\mathbf{v}/\partial x^2$ denote the integer-order derivatives of orders 1 and 2 and $\partial^\gamma\mathbf{v}/\partial t^\gamma$, $\partial^\beta\mathbf{v}/\partial x^\beta$ are the Riemann–Liouville fractional derivatives of orders $\gamma, \beta > 0$. Let us consider the one-parameter Lie group of transformations

$$
\begin{aligned}
x^* &= x + \varepsilon\xi(x,t,\mathbf{v}) + O(\varepsilon^2),\\
t^* &= t + \varepsilon\tau(x,t,\mathbf{v}) + O(\varepsilon^2),\\
v^{r*} &= v^r + \varepsilon\eta^{(r)}(x,t,v^r) + O(\varepsilon^2),\\
\frac{\partial^\gamma v^{r*}}{\partial t^\gamma} &= \frac{\partial^\gamma v^r}{\partial t^\gamma} + \varepsilon\eta^{(r)\gamma,t} + O(\varepsilon^2),\\
\frac{\partial^\beta v^{r*}}{\partial x^\beta} &= \frac{\partial^\beta v^r}{\partial x^\beta} + \varepsilon\eta^{(r)\beta,x} + O(\varepsilon^2),\\
\frac{\partial v^{r*}}{\partial x} &= \frac{\partial v^r}{\partial x} + \varepsilon\eta^{(r)x} + O(\varepsilon^2),\\
\frac{\partial^2 v^{r*}}{\partial x^2} &= \frac{\partial^2 v^r}{\partial x^2} + \varepsilon\eta^{(r)xx} + O(\varepsilon^2),\\
&\vdots
\end{aligned}
\tag{7}
$$

where $r = 1, 2, \ldots, p$, ε is the group parameter, $\xi, \tau, \eta^{(r)}$ are the infinitesimals and $\eta^{(r)\gamma,t}$ are the extended infinitesimals of order γ. $\eta^{(r)\beta,x}$ are the extended infinitesimals of order β. $\eta^{(r)x}$ and $\eta^{(r)xx}$ are the integer-order extended infinitesimals, which leave system (6) invariant. The infinitesimal transformations (7) are such that if v^r, $r = 1, 2, \ldots, p$ are solutions of (6), then so are v^{r*}, $r = 1, 2, \ldots, p$. The associated symmetry generator is given by the following form:

$$
V = \xi(x,t,\mathbf{v})\frac{\partial}{\partial x} + \tau(x,t,\mathbf{v})\frac{\partial}{\partial t} + \sum_{r=1}^{p}\eta^{(r)}(x,t,\mathbf{v})\frac{\partial}{\partial v^r}. \tag{8}
$$

The corresponding prolonged symmetry generator is given by

$$
\begin{aligned}
\mathrm{pr}^{(\gamma,\beta,n)}V = V &+ \sum_{r=1}^{p}\eta^{(r)\gamma,t}\partial_{\partial_t^\gamma v^r} + \sum_{r=1}^{p}\eta^{(r)\beta,x}\partial_{\partial_x^\beta v^r}\\
&+ \sum_{r=1}^{p}\eta^{(r)x}\partial_{v_x^r} + \sum_{r=1}^{p}\eta^{(r)xx}\partial_{v_{xx}^r} + \cdots,
\end{aligned}
\tag{9}
$$

where n is the order of system (6) and $\partial_t^\gamma v^r = \partial^\gamma v^r/\partial t^\gamma$, $\partial_x^\beta v^r = \partial^\beta v^r/\partial x^\beta$, $v_x^r = \partial v^r/\partial x$, $v_{xx}^r = \partial^2 v^r/\partial x^2$ for $r = 1, 2, \ldots, p$.

Also, the lower limit of the Riemann–Liouville fractional derivative (1) is fixed, and so the invariance condition yields

$$
\xi(x,t,\mathbf{v})|_{x=0} = 0, \quad \tau(x,t,\mathbf{v})|_{t=0} = 0. \tag{10}
$$

The γth-order extended infinitesimal $\eta^{(k)\gamma,t}$ related to the Riemann–Liouville fractional derivative is given by

$$
\begin{aligned}
\eta^{(k)\gamma,t} = {} & D_t^\gamma(\eta^{(k)}) + \xi D_t^\gamma(v_x^k) - D_t^\gamma(\xi v_x^k)\\
&+ \tau D_t^\gamma(v_t^k) - D_t^\gamma(\tau v_t^k).
\end{aligned}
\tag{11}
$$

The generalised Leibnitz rule [41,42] for $\gamma \in \mathbb{R}$ is

$$
D_z^\gamma(f(z)g(z)) = \sum_{n=0}^{\infty}\binom{\gamma}{n}D_z^n f(z) D_z^{\gamma-n}g(z), \tag{12}
$$

where

$$
D_z^n f = \frac{\mathrm{d}^n f}{\mathrm{d}z^n},
$$

$$
\binom{\gamma}{n} = \frac{\Gamma(\gamma+1)}{\Gamma(n+1)\Gamma(\gamma+1-n)}, \quad n \in \mathbb{N}.
$$

Using the Leibnitz rule (12), eq. (11) can be expanded as

$$
\begin{aligned}
\eta^{(k)\gamma,t} = {} & D_t^\gamma(\eta^{(k)}) - \gamma D_{\mathrm{t}}\tau\frac{\partial^\gamma v^k}{\partial t_\gamma}\\
&- \sum_{n=1}^{\infty}\binom{\gamma}{n}D_t^n(\xi)D_t^{\gamma-n}(v_x^k)\\
&- \sum_{n=1}^{\infty}\binom{\gamma}{n+1}D_t^{n+1}(\tau)D_t^{\gamma-n}(v^k),
\end{aligned}
\tag{13}
$$

where D_{t} represents the total derivative operator given as

$$
D_{\mathrm{t}} = \partial_t + v_t^k\partial_{v^k} + v_{tt}^k\partial_{v_t^k} + \cdots. \tag{14}
$$

Using the generalised chain rule [42,44] and generalised Leibnitz rule (12), the first term in (13) can be written as

$$
\begin{aligned}
D_t^\gamma(\eta^{(k)}) = {} & \frac{\partial^\gamma\eta^{(k)}}{\partial t^\gamma} + \sum_{r=1}^{p}\sum_{n=1}^{\infty}\binom{\gamma}{n}\frac{\partial^n\eta_{v^r}^{(k)}}{\partial t^n}D_t^{\gamma-n}(v^r)\\
&+ \sum_{r=1}^{p}\mu_{\eta^{(k)},\gamma,r}\\
&+ \sum_{r=1}^{p}\left(\eta_{v^r}^{(k)}\frac{\partial^\gamma v^r}{\partial t^\gamma} - v^r\frac{\partial^\gamma\eta_{v^r}^{(k)}}{\partial t^\gamma}\right),
\end{aligned}
\tag{15}
$$

where

$$\mu_{\eta^{(k)},\gamma,r} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{j=2}^{m}\sum_{q=2}^{j-1}\binom{\gamma}{n}\binom{n}{m}\binom{j}{q}\frac{1}{j!} \times \frac{t^{n-\gamma}}{\Gamma(n-\gamma+1)}(-v^{(r)})^q \times \frac{\partial^m}{\partial t^m}((v^{(r)})^{j-q})\frac{\partial^{n-m+j}\eta}{\partial t^{n-m}\partial(v^{(r)})^j}, \quad r = 1, 2, \ldots, p. \tag{16}$$

Here, $\eta_{v^r}^{(k)} = \partial\eta^{(k)}/\partial v^r$.

Therefore, the γth-order extended infinitesimal $\eta^{(k)\gamma,t}$ for the system of fractional partial differential equations (6) is

$$\eta^{(k)\gamma,t} = \frac{\partial^\gamma \eta^{(k)}}{\partial t^\gamma} + (\eta_{v^{(k)}}^{(k)} - \gamma D_t(\tau))\frac{\partial^\gamma v^k}{\partial t^\gamma} - v^k\frac{\partial^\gamma \eta_{v^k}}{\partial t^\gamma} + \sum_{r=1}^{p}\mu_{\eta^{(k)},\gamma,r} + \sum_{r\neq k, r=1}^{p}\left(\eta_{v^r}\frac{\partial^\gamma v^r}{\partial t^\gamma} - v^r\frac{\partial^\gamma \eta_{v^r}}{\partial t^\gamma}\right) - \sum_{n=1}^{\infty}\binom{\gamma}{n}D_t^n(\xi)D_t^{\gamma-n}(v^k, x) + \sum_{n=1}^{\infty}\left[\binom{\gamma}{n}\frac{\partial^n \eta_{v^k}^{(k)}}{\partial t^n} - \binom{\gamma}{n+1}D_t^{n+1}(\tau)\right] \times D_t^{\gamma-n}(v^k) + \sum_{r\neq k, r=1}^{p}\sum_{n=1}^{\infty}\binom{\gamma}{n}\frac{\partial^n \eta_{v^r}^{(k)}}{\partial t^n}D_t^{\gamma-n}(v^r), \tag{17}$$

where $\mu_{\eta^{(k)},\gamma,r}$ is given by (16).

Similarly, the βth-order extended infinitesimal $\eta^{(k)\beta,x}$ for $\beta > 0$ is proposed by

$$\eta^{(k)\beta,x} = \frac{\partial^\beta \eta^{(k)}}{\partial x^\beta} + (\eta_{v^k}^{(k)} - \beta D_x(\tau))\frac{\partial^\beta v^k}{\partial x^\beta} - v^k\frac{\partial^\beta \eta_{v^k}}{\partial x^\beta} + \sum_{r=1}^{p}\mu_{\eta^{(k)},\beta,r} + \sum_{r\neq k, r=1}^{p}\left(\eta_{v^r}\frac{\partial^\beta v^r}{\partial x^\beta} - v^r\frac{\partial^\beta \eta_{v^r}}{\partial x^\beta}\right) - \sum_{n=1}^{\infty}\binom{\beta}{n}D_x^n(\xi)D_x^{\beta-n}(v_t^k) + \sum_{n=1}^{\infty}\left[\binom{\beta}{n}\frac{\partial^n \eta_{v^k}^{(k)}}{\partial x^n} - \binom{\beta}{n+1}D_x^{n+1}(\tau)\right] \times D_x^{\beta-n}(v^k) + \sum_{r\neq k, r=1}^{p}\sum_{n=1}^{\infty}\binom{\beta}{n}\frac{\partial^n \eta_{v^r}^{(k)}}{\partial x^n}D_x^{\beta-n}(v^r), \tag{18}$$

where

$$\mu_{\eta^{(k)},\beta,r} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{j=2}^{m}\sum_{q=2}^{j-1}\binom{\beta}{n}\binom{n}{m}\binom{j}{q}\frac{1}{j!} \times \frac{x^{n-\beta}}{\Gamma(n-\beta+1)}(-v^r)^q\frac{\partial^m}{\partial x^m}((v^{(k)})^{j-q}) \times \frac{\partial^{n-m+j}\phi}{\partial x^{n-m}\partial(v^{(k)})^j}, \quad r = 1, 2, \ldots, p. \tag{19}$$

3. Application to some system of NLFPDEs with time-dependent variable coefficients

In this section, the proposed symmetry approach is applied to investigate the Lie point symmetries and reductions for some well-known nonlinear system of fractional partial differential equations with time-dependent variable coefficients.

3.1 *Space–time fractional coupled Burgers system with time-dependent variable coefficients*

In this subsection, we consider the space–time fractional coupled Burgers system [28,33] with time-dependent variable coefficients:

$$\begin{aligned} \frac{\partial^\gamma u}{\partial t^\gamma} &= A_1(t)u\frac{\partial^\beta u}{\partial x^\beta} + A_2(t)u\frac{\partial^\beta v}{\partial x^\beta} + A_3(t)v\frac{\partial^\beta u}{\partial x^\beta} + A_4(t)\frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial^\gamma v}{\partial t^\gamma} &= B_1(t)v\frac{\partial^\beta v}{\partial x^\beta} + B_2(t)u\frac{\partial^\beta v}{\partial x^\beta} + B_3(t)v\frac{\partial^\beta u}{\partial x^\beta} + B_4(t)\frac{\partial^2 v}{\partial x^2}. \end{aligned} \tag{20}$$

The invariance criteria for system (20) under one-parameter Lie group of transformation are obtained as

$$\begin{aligned} \eta^{\gamma,t} &= A_4(t)\eta^{xx} + (\tau A_1'(t)u + A_1(t)\eta + \tau A_3'(t)v + A_3(t)\phi)\partial_x^\beta u + A_2(t)u\phi^{\beta,x} + \tau A_4'(t)u_{xx} + (A_1(t)u + A_3(t))\eta^{\beta,x} + (\tau A_2'(t)u + A_2(t)\eta)\partial_x^\beta v, \\ \phi^{\gamma,t} &= B_4(t)\phi^{xx} + \tau B_4'(t)v_{xx} + (\tau B_1'(t)v + B_1(t)\phi + \tau B_2'(t)u + B_2(t)\eta)\partial_x^\beta v + (B_1(t)v + B_2(t)u)\phi^{\beta,x} + b_3(t)v\eta^{\beta,x} + (\tau B_3'(t)v + B_3(t)\phi)\partial_x^\beta u, \end{aligned} \tag{21}$$

where η^{xx}, ϕ^{xx} are the extended infinitesimals of order 2, $\eta^{\gamma,t}$, $\phi^{\gamma,t}$ are the extended infinitesimals of order γ and $\eta^{\beta,x}$, $\phi^{\beta,x}$ are the extended infinitesimals of order β.

Now substituting the values of the prolongations and equating the coefficient of various linearly independent variables to zero, for $0 < \gamma, \beta \leq 1$, the determining equations are obtained as

$$
\begin{aligned}
&\xi_t = \xi_u = \xi_v = 0,\\
&\tau_x = \tau_u = \tau_v = 0,\\
&\eta_{vv} = \eta_{uu} = \phi_{vv} = \phi_{uu} = 0,\\
&A_4'(t)\tau + A_4(t)(\gamma\tau_t - 2\xi_x) = 0,\\
&B_4'(t)\tau + B_4(t)(\gamma\tau_t - 2\xi_x) = 0,\\
&\tau B_3'(t)v + B_3(t)\phi + B_3(t)v(\gamma\tau_t - \beta\xi_x)\\
&\quad +(B_1(t)v + B_2(t)u)\phi_u - (A_1(t)u + A_3(t)v)\phi_u = 0,\\
&\tau B_1'(t)v + B_1(t)\phi + \tau B_2'(t)u + B_3(t)v\eta_v + (B_1(t)v\\
&\quad +B_2(t)u)(\gamma\tau_t - \beta\xi_x) + B_2(t)\eta - A_2(t)u\phi_u = 0,\\
&\tau A_2'(t)u + A_2(t)\eta + (A_1(t)u + A_2(t)u(\gamma\tau_t - \beta\xi_x)\\
&\quad +A_3(t)\eta_v = 0,\\
&\tau A_1'(t)u + A_1(t)\eta + \tau A_3'(t)v + A_3(t)\phi + (A_1(t)u\\
&\quad +A_3(t)v)(\gamma\tau_t - \beta\xi_x) + A_2(t)u\phi_u = 0,\\
&A_4'(t)\eta_{xx} + (A_1(t)u + A_3(t)v)(\partial_x^\beta\eta - u\partial_x^\beta\eta_u\\
&\quad -v\partial_x^\beta\eta_v) + u\partial_t^\gamma\eta_u + v\partial_t^\gamma\eta_v\\
&\quad +A_2(t)u(\partial_x^\beta\phi - v\partial_x^\beta\phi_v - u\partial_x^\beta\phi_u) - \partial_t^\gamma\eta = 0,\\
&B_4'(t)\phi_{xx} + (B_1(t)+B_2(t)u)(\partial_x^\beta\phi - u\partial_x^\beta\phi_u - v\partial_x^\beta\phi_v)\\
&\quad +u\partial_t^\gamma\phi_u + v\partial_t^\gamma\phi_v + B_3(t)v(\partial_x^\beta\eta - u\partial_x^\beta\eta_u\\
&\quad -v\partial_x^\beta\eta_v) - \partial_t^\gamma\phi = 0,\\
&\binom{\gamma}{n}\partial_t^n\eta_u - \binom{\gamma}{n+1}D_{\mathrm{t}}^{n+1}\tau = 0, \quad n \in \mathbb{N},\\
&\binom{\gamma}{n}\partial_t^n\phi_v - \binom{\gamma}{n+1}D_{\mathrm{t}}^{n+1}\tau = 0, \quad n \in \mathbb{N},\\
&\binom{\beta}{n}\partial_x^n\eta_u - \binom{\beta}{n+1}D_x^{n+1}\xi = 0, \quad n \in \mathbb{N},\\
&\binom{\beta}{n}\partial_x^n\phi_v - \binom{\beta}{n+1}D_x^{n+1}\xi = 0, \quad n \in \mathbb{N}.
\end{aligned}
\tag{22}
$$

In a particular case, take $A_4(t) = at^m$ and $B_4(t) = bt^s$, where a, b are arbitrary constants and m, s are real numbers.

Solving these PDEs and FPDEs (22) together, and also using (10), we obtain the infinitesimals given as

$$\xi = \frac{c_1}{2}x, \quad \tau = \frac{c_1}{\gamma + m}t, \quad \eta = \frac{(\gamma - 1)c_1}{2(\gamma + m)}u,$$

$$\phi = \frac{(\gamma - 1)c_1}{2(\gamma + m)}v, \tag{23}$$

where c_1 is an arbitrary constant and variable coefficients are governed by the following conditions:

$$
\begin{aligned}
&A_1(t) = M_1 t^{\frac{1-3\gamma}{2} + \frac{\beta(\gamma+m)}{2}}, \quad A_2(t) = K_1A_1(t),\\
&A_3(t) = K_2A_1(t), \quad B_1(t) = K_3A_1(t),\\
&B_2(t) = K_4A_1(t), \quad B_3(t) = K_5A_1(t),
\end{aligned}
\tag{24}
$$

where M_1, K_1, K_2, K_3, K_4 and K_5 are arbitrary constants.

For the symmetry generator, given by

$$
\begin{aligned}
X = {}& \frac{x}{2}\frac{\partial}{\partial x} + \frac{t}{\gamma + m}\frac{\partial}{\partial t} + \frac{(\gamma - 1)u}{2(\gamma + m)}\frac{\partial}{\partial u}\\
&+\frac{(\gamma - 1)v}{2(\gamma + m)}\frac{\partial}{\partial v},
\end{aligned}
\tag{25}
$$

the auxiliary equations are written as

$$\frac{\mathrm{d}x}{\frac{x}{2}} = \frac{\mathrm{d}t}{\frac{t}{\gamma+m}} = \frac{\mathrm{d}u}{\frac{\gamma-1}{2(\gamma+m)}u} = \frac{\mathrm{d}v}{\frac{\gamma-1}{2(\gamma+m)}v}. \tag{26}$$

Solving (26), the similarity variable of the considered system (20) is obtained as

$$y = xt^{-\frac{\gamma+m}{2}} \tag{27}$$

and the corresponding similarity transformations are

$$
\begin{aligned}
&u(x,t) = t^{\frac{\gamma-1}{2}}g(y),\\
&v(x,t) = t^{\frac{\gamma-1}{2}}h(y)
\end{aligned}
\tag{28}
$$

and the admissible coefficients given by (24), with (27) and (28), reduce the system of NLFPDEs (20) for $\gamma, \beta > 0$ to the system of NLFODEs.

Let us assume $n - 1 < \gamma < n$; $n \in \mathbb{N}$ then by the definition of Riemann–Liouville fractional differentiation (1), we have

$$
\begin{aligned}
\frac{\partial^\gamma u}{\partial t^\gamma} = {}& \frac{\partial^n}{\partial t^n}\left(\frac{1}{\Gamma(n-\gamma)}\int_0^t (t-\rho)^{n-\gamma-1}\rho^{\frac{\gamma-1}{2}}\right.\\
&\left.\times\, g\left(x\rho^{-\frac{\gamma+m}{2}}\right)\mathrm{d}s\right).
\end{aligned}
\tag{29}
$$

Let $\rho = t/s$ and using the similarity variable (27), we get the following equation:

$$
\begin{aligned}
\frac{\partial^\gamma u}{\partial t^\gamma} = {}& \frac{\partial^n}{\partial t^n}\left(\frac{t^{n-\frac{\gamma+1}{2}}}{\Gamma(n-\gamma)}\right.\\
&\left.\times\int_1^\infty (s-1)^{n-\gamma-1}s^{-(n-\gamma+\frac{\gamma+1}{2})}\; g(ys^{\frac{\gamma+m}{2}})\mathrm{d}s\right).
\end{aligned}
$$

By using the left-hand side Erdèlyi–Kober fractional integral operator (3), we have

$$\frac{\partial^\gamma u}{\partial t^\gamma}=\frac{\partial^n}{\partial t^n}\left(t^{n-\frac{\gamma+1}{2}}\left(\mathcal{M}_{\frac{2}{\gamma+m}}^{\frac{\gamma+1}{2},n-\gamma}g\right)(y)\right). \quad (30)$$

Now for more simplification, let $\mu(y)$ be the continuously differentiable function for $y=xt^{-((\gamma+m)/2)}$ from (27). Then

$$\begin{aligned}t\frac{\partial}{\partial t}\mu(y)&=tx\left(-\frac{\gamma+m}{2}\right)t^{-(\frac{\gamma+m}{2})-1}\mu'(y)\\&=\left(-\frac{\gamma+m}{2}\right)y\frac{\mathrm{d}}{\mathrm{d}y}\mu(y).\end{aligned}$$

Therefore, (30) becomes

$$\begin{aligned}\frac{\partial^\gamma u}{\partial t^\gamma}&=\frac{\partial^n}{\partial t^n}\left(t^{n-\frac{\gamma+1}{2}}\left(\mathcal{M}_{\frac{2}{\gamma+m}}^{\frac{\gamma+1}{2},n-\gamma}g\right)(y)\right)\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left(\frac{\partial}{\partial t}t^{n-\frac{\gamma+1}{2}}\left(\mathcal{M}_{\frac{2}{\gamma+m}}^{\frac{\gamma+1}{2},n-\gamma}g\right)(y)\right)\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left(t^{n-\frac{\gamma+1}{2}-1}\left(n-\frac{\gamma+1}{2}-\frac{\gamma+m}{2}y\frac{\mathrm{d}}{\mathrm{d}y}\right)\right.\\&\quad\left.\times\left(\mathcal{M}_{\frac{2}{\gamma+m}}^{\frac{\gamma+1}{2},n-\gamma}g\right)(y)\right).\end{aligned}$$

Continuing in this way, we get

$$\begin{aligned}\frac{\partial^\gamma u}{\partial t^\gamma}&=\frac{\partial^n}{\partial t^n}\left(t^{n-\frac{\gamma+1}{2}}\left(\mathcal{M}_{\frac{2}{\gamma+m}}^{\frac{\gamma+1}{2},n-\gamma}g\right)(y)\right)\\&=t^{-\frac{\gamma+1}{2}}\prod_{j=0}^{n-1}\left(1-\frac{\gamma+1}{2}+j-\left(\frac{\gamma+m}{2}\right)y\frac{\mathrm{d}}{\mathrm{d}y}\right)\\&\quad\times\left(\mathcal{M}_{\frac{2}{\gamma+m}}^{\frac{\gamma+1}{2},n-\gamma}g\right)(y). \quad (31)\end{aligned}$$

Using the left-hand side Erdèlyi–Kober fractional differential operator (2) in (31), we have

$$\frac{\partial^\gamma u}{\partial t^\gamma}=t^{-\frac{\gamma+1}{2}}\left(\mathcal{T}_{\frac{2}{\gamma+m}}^{\frac{1-\gamma}{2},\gamma}g\right)(y). \quad (32)$$

Analogous to the result obtained above, it can also be concluded that

$$\frac{\partial^\gamma v}{\partial t^\gamma}=t^{-\frac{\gamma+1}{2}}\left(\mathcal{T}_{\frac{2}{\gamma+m}}^{\frac{1-\gamma}{2},\gamma}h\right)(y). \quad (33)$$

Similarly, the partial fractional derivatives of order $\beta>0$, $\partial^\beta u/\partial x^\beta$ and $\partial^\beta v/\partial x^\beta$ are obtained by

$$\frac{\partial^\beta u}{\partial x^\beta}=t^{\frac{\gamma-1}{2}}x^{-\beta}(\mathcal{R}_1^{-\beta,\beta}g)(y), \quad (34)$$

$$\frac{\partial^\beta v}{\partial x^\beta}=t^{\frac{\gamma-1}{2}}x^{-\beta}(\mathcal{R}_1^{-\beta,\beta}h)(y), \quad (35)$$

where $(\mathcal{R}_1^{-\beta,\beta})$ is the right-hand side Erdèlyi–Kober fractional differential operator by (4).

Using (32)–(35) and the similarity variable (27), the reduced NLFODEs of the coupled Burger's system (20) is

$$\begin{aligned}&\left(\mathcal{T}_{\frac{2}{\gamma+m}}^{\frac{1-\gamma}{2},\gamma}g\right)(y)\\&\quad=y^{-\beta}(M_1g(y)+M_3h(y))(\mathcal{R}_1^{-\beta,\beta}g)(y))\\&\qquad+M_2y^{-\beta}g(y)(\mathcal{R}_1^{-\beta,\beta}h)(y)+ag''(y),\\&\left(\mathcal{T}_{\frac{2}{\gamma+m}}^{\frac{1-\gamma}{2},\gamma}h\right)(y)\\&\quad=y^{-\beta}(M_4h(y)+M_5g(y))(\mathcal{R}_1^{-\beta,\beta}h)(y)\\&\qquad+M_6y^{-\beta}h(y)(\mathcal{R}_1^{-\beta,\beta}g)(y)+bh''(y), \quad (36)\end{aligned}$$

where $(\mathcal{T}_{\frac{2}{\gamma+m}}^{\frac{1-\gamma}{2},\gamma})$ and $(\mathcal{R}_1^{-\beta,\beta})$ are given by (2) and (4), respectively, and arbitrary constants are given by $M_2=K_2M_1$, $M_3=K_1M_1$, $M_4=K_3M_1$, $M_5=K_4M_1$, $M_6=K_5M_1$.

Remark 1. In the obtained symmetries of the considered system (20), if we take values $A_1(t)=2$, $A_2(t)=-1$, $A_3(t)=-1$, $A_4(t)=1$, $B_1(t)=2$, $B_2(t)=-1$, $B_3(t)=-1$ and $B_4(t)=1$, we can find the symmetries of the system considered in [33]. Also, if we take $\beta=1$ along with these values, then we can find symmetries of the system considered in [36].

3.2 *Space–time fractional Ito's system with time-dependent variable coefficients*

Consider the space–time fractional Ito's system with time-dependent variable coefficients [33] in the following form:

$$\begin{aligned}\frac{\partial^\gamma u}{\partial t^\gamma}&=A_1(t)u\frac{\partial^\beta u}{\partial x^\beta}+A_2(t)v\frac{\partial^\beta u}{\partial x^\beta}+A_3(t)\frac{\partial^3 u}{\partial x^3},\\\frac{\partial^\gamma v}{\partial t^\gamma}&=B_1(t)u\frac{\partial^\beta v}{\partial x^\beta}+B_2(t)v\frac{\partial^\beta u}{\partial x^\beta}. \quad (37)\end{aligned}$$

The invariance criteria for system (37) under the one-parameter Lie group of transformation is obtained as

$$\begin{aligned}\eta^{\gamma,t}=&A_3(t)\eta^{xxx}+\tau A_3'(t)u_{xxx}+(\tau A_1'(t)u+A_1(t)\eta\\&+\tau A_2'(t)v+A_2(t)\phi)\partial_x^\beta u\\&+(A_1(t)u+A_2(t)v)\eta^{\beta,x},\\\phi^{\gamma,t}=&B_1(t)u\phi^{\beta,x}+(\tau B_1'(t)u+B_1(t)\eta)\partial_x^\beta v\\&+b_2(t)v\eta^{\beta,x}+(\tau B_2'(t)v+B_2(t)\phi)\partial_x^\beta u, \quad (38)\end{aligned}$$

where η^{xxx} is the extended infinitesimal of order 3, $\eta^{\gamma,t}$, $\phi^{\gamma,t}$ are the extended infinitesimals of order γ and $\eta^{\beta,x}$, $\phi^{\beta,x}$ are the extended infinitesimals of order β.

System (37) for $A_3(t)=at^m$ (a is an arbitrary constant and m is a real number) admits the group of transformations (7). Then the obtained Lie symmetries are

$$\xi = \frac{c_1 x}{3}, \quad \tau = \frac{c_1 t}{\gamma + m},$$
$$\eta = \frac{\gamma - 1}{2(\gamma + m)} c_1 u, \quad \phi = \frac{\gamma - 1}{2(\gamma + m)} c_1 v, \tag{39}$$

where c_1 is an arbitrary constant and variable coefficients are governed by the following conditions:

$$A_1(t) = M_1 t^{\frac{1-3\gamma}{2} + \frac{\beta(\gamma+m)}{3}},$$
$$A_2(t) = K_1 A_1(t),$$
$$B_1(t) = K_2 A_1(t), \quad B_2(t) = K_3 A_1(t), \tag{40}$$

where M_1, K_1, K_2 and K_3 are arbitrary constants.

The Lie symmetry generator of system (37) is

$$X = \frac{x}{3}\frac{\partial}{\partial x} + \frac{t}{\gamma + m}\frac{\partial}{\partial t} + \frac{(\gamma - 1)u}{2(\gamma + m)}\frac{\partial}{\partial u} + \frac{(\gamma - 1)v}{2(\gamma + m)}\frac{\partial}{\partial v}. \tag{41}$$

The solution of the auxiliary equations of the infinitesimal generator (41) gives the symmetry variable

$$y = x t^{-\frac{\gamma+m}{3}} \tag{42}$$

AND the corresponding symmetry transformations are obtained as

$$u(x,t) = t^{\frac{\gamma-1}{2}} g(y),$$
$$v(x,t) = t^{\frac{\gamma-1}{2}} h(y). \tag{43}$$

The admissible coefficients given by (40), with (42) and (43), reduce the system of NLFPDEs (37) for $\gamma, \beta > 0$ to the system of NLFODEs, given by

$$\left(\mathcal{T}^{\frac{1-\gamma}{2},\gamma}_{\frac{3}{\gamma+m}} g\right)(y) = y^{-\beta}(M_1 g(y) + M_2 h(y)) \times (\mathcal{R}_1^{-\beta,\beta} g)(y)) + a g'''(y),$$
$$\left(\mathcal{T}^{\frac{1-\gamma}{2},\gamma}_{\frac{3}{\gamma+m}} h\right)(y) = M_3 y^{-\beta} g(y)(\mathcal{R}_1^{-\beta,\beta} h)(y) + M_4 y^{-\beta} h(y)(\mathcal{R}_1^{-\beta,\beta} g)(y), \tag{44}$$

where $(\mathcal{T}^{\frac{1-\gamma}{2},\gamma}_{\frac{3}{\gamma+m}})$ is the left-hand side Erdèlyi–Kober fractional differential operator given by (2) and $(\mathcal{R}_1^{-\beta,\beta})$ is the right-hand side Erdèlyi–Kober fractional differential operator given by (4) and $M_2 = K_1 M_1$, $M_3 = K_3 M_1$, $M_4 = K_2 M_1$ are arbitrary constants.

Remark 2. In the obtained symmetries of the considered system (45), if we put $A_1(t) = 3$, $A_2(t) = 1$, $A_3(t) = 1$, $B_1(t) = 1$ and $B_2(t) = 1$, we can find the symmetries of the system considered in [33]. Also, if we take $\beta = 1$ along with these values, then we can find symmetries of the system considered in [36].

3.3 *Space–time fractional coupled Korteweg–de Vries (KdV) equations with time-dependent variable coefficients*

Various methods have been applied to study the time fractional coupled KdV equations [45] such as homotopy decomposition method [46]. Also, the space–time fractional coupled KdV equations have been studied by the Lie symmetry method [33] with constant variables. We consider the space–time fractional coupled KdV equations with time-dependent variable coefficients

$$\frac{\partial^\gamma u}{\partial t^\gamma} = A_1(t) u \frac{\partial^\beta u}{\partial x^\beta} + A_2(t) v \frac{\partial^\beta v}{\partial x^\beta} + A_3(t)\frac{\partial^3 u}{\partial x^3},$$
$$\frac{\partial^\gamma v}{\partial t^\gamma} = B_1(t) u \frac{\partial^\beta v}{\partial x^\beta} + B_2(t)\frac{\partial^3 v}{\partial x^3}. \tag{45}$$

The invariance criteria for system (37) under the one-parameter Lie group of transformation is obtained as

$$\eta^{\gamma,t} = A_3(t)\eta^{xxx} + \tau A_3'(t) u_{xxx} + (\tau A_1'(t) u + A_1(t)\eta)\partial_x^\beta u + A_2(t) v \phi^{\beta,x} + (\tau A_2'(t) v + A_2(t)\phi)\partial_x^\beta v + A_1(t) u \eta^{\beta,x},$$
$$\phi^{\gamma,t} = B_2(t)\eta^{xxx} + \tau B_2'(t) v_{xxx} + B_1(t) u \phi^{\beta,x} + (\tau B_1'(t) u + B_1(t)\eta)\partial_x^\beta v, \tag{46}$$

where η^{xxx}, ϕ^{xxx} are the extended infinitesimals of order 3, $\eta^{\gamma,t}$, $\phi^{\gamma,t}$ are the extended infinitesimals of order γ and $\eta^{\beta,x}$, $\phi^{\beta,x}$ are the extended infinitesimals of order β.

System (45) for $A_3(t) = a t^m$ and $B_2(t) = b t^s$ (a, b are arbitrary constants and m, n are real numbers) admits the group of transformations (7). Then the obtained Lie symmetries are

$$\xi = \frac{c_1 x}{3}, \quad \tau = \frac{c_1 t}{\gamma + m},$$
$$\eta = \frac{\gamma - 1}{2(\gamma + m)} c_1 u, \quad \phi = \frac{\gamma - 1}{2(\gamma + m)} c_1 v, \tag{47}$$

where c_1 is an arbitrary constant and variable coefficients are governed by the following conditions:

$$A_1(t) = M_1 t^{\frac{1-3\gamma}{2} + \frac{\beta(\gamma+m)}{3}},$$
$$A_2(t) = K_1 A_1(t),$$
$$B_1(t) = K_2 A_1(t), \tag{48}$$

where M_1, K_1 and K_2 are arbitrary constants.

The Lie symmetry generator of system (37) is obtained as

$$X = \frac{x}{3}\frac{\partial}{\partial x} + \frac{t}{\gamma+m}\frac{\partial}{\partial t} + \frac{(\gamma-1)u}{2(\gamma+m)}\frac{\partial}{\partial u} + \frac{(\gamma-1)v}{2(\gamma+m)}\frac{\partial}{\partial v}. \tag{49}$$

The solution of the auxiliary equations of infinitesimal generator (49) gives the similarity variable

$$y = xt^{-\frac{\gamma+m}{3}} \tag{50}$$

and the corresponding similarity transformations are given as

$$u(x,t) = t^{\frac{\gamma-1}{2}} g(y),$$
$$v(x,t) = t^{\frac{\gamma-1}{2}} h(y). \tag{51}$$

Also, the admissible coefficients given by (48), with (50) and (51), reduce the system of NLFPDEs (37) for $\gamma, \beta > 0$ to the system of NLFODEs, given as

$$\begin{aligned}\left(\mathcal{T}^{\frac{1-\gamma}{2},\gamma}_{\frac{3}{\gamma+m}} g\right)(y) &= y^{-\beta}(M_1 g(y)(\mathcal{R}_1^{-\beta,\beta} g)(y)) \\ &\quad + y^{-\beta}(M_2 h(y)(\mathcal{R}_1^{-\beta,\beta} h)(y)) \\ &\quad + a g'''(y), \\ \left(\mathcal{T}^{\frac{1-\gamma}{2},\gamma}_{\frac{3}{\gamma+m}} h\right)(y) &= M_3 y^{-\beta} g(y)(\mathcal{R}_1^{-\beta,\beta} h)(y) + b h'''(y),\end{aligned} \tag{52}$$

where $(\mathcal{T}^{\frac{1-\gamma}{2},\gamma}_{\frac{3}{\gamma+m}})$ is the left-hand side Erdèlyi–Kober fractional differential operator given by (2) and $(\mathcal{R}_1^{-\beta,\beta})$ is the right-hand side Erdèlyi–Kober fractional differential operator given by (4) and $M_2 = K_1 M_1$, $M_3 = K_2 M_1$ are arbitrary constants.

Remark 3. In the obtained symmetries of the considered system (45), if we take values $A_1(t) = -6a$, $A_2(t) = 6$, $A_3 = -a$, $B_1(t) = -3a$ and $B_2(t) = -a$, we can find symmetries of the system considered in [33]. Also, if we take $\beta = 1$ along with these values, then we can find symmetries of the system considered in [36].

3.4 *Space–time fractional generalised Hirota–Satsuma coupled KdV equations with time-dependent variable coefficients*

Space–time fractional generalised Hirota–Satsuma coupled KdV equations have been studied using many methods such as subequation method [47] and the Lie symmetry method with constant coefficients [33]. Consider the space–time fractional generalised Hirota–Satsuma coupled KdV equations with time-dependent variable coefficients as

$$\begin{aligned}\frac{\partial^\gamma u}{\partial t^\gamma} &= A_1(t) u \frac{\partial^\beta u}{\partial x^\beta} + A_2(t) v \frac{\partial^\beta v}{\partial x^\beta} \\ &\quad + A_3(t)\frac{\partial^\beta w}{\partial x^\beta} + A_4(t)\frac{\partial^3 u}{\partial x^3}, \\ \frac{\partial^\gamma v}{\partial t^\gamma} &= A_5(t) u \frac{\partial^\beta v}{\partial x^\beta} + A_6(t)\frac{\partial^3 v}{\partial x^3}, \\ \frac{\partial^\gamma w}{\partial t^\gamma} &= A_7(t) u \frac{\partial^\beta w}{\partial x^\beta} + A_8(t)\frac{\partial^3 w}{\partial x^3}.\end{aligned} \tag{53}$$

The invariance criteria for system (53) under one-parameter Lie group of transformation is obtained as

$$\begin{aligned}\eta^{\gamma,t} &= A_4(t)\eta^{xxx} + (\tau A_1'(t) u \\ &\quad + A_1(t)\eta)\partial_x^\beta u + A_3(t)\psi^{\beta,x} \\ &\quad + (\tau A_2'(t) v + A_2(t)\phi)\partial_x^\beta v + \tau A_4'(t) u_{xxx} \\ &\quad + \tau A_3'(t)\partial_x^\beta w + A_1(t) u \eta^{\beta,x} + A_2(t) v \phi^{\beta,x}, \\ \phi^{\gamma,t} &= A_6(t)\phi^{xxx} + \tau A_6'(t) v_{xxx} + A_5(t) u \phi^{\beta,x} \\ &\quad + (\tau A_5'(t) u + A_5(t)\eta)\partial_x^\beta v, \\ \psi^{\gamma,t} &= A_8(t)\psi^{xxx} + \tau A_8'(t) w_{xxx} + A_7(t) u \psi^{\beta,x} \\ &\quad + (\tau A_7'(t) u + A_7(t)\eta)\partial_x^\beta w,\end{aligned} \tag{54}$$

where η^{xxx}, ψ^{xxx}, ϕ^{xxx} are the extended infinitesimals of order 3, $\eta^{\gamma,t}$, $\phi^{\gamma,t}$, $\psi^{\gamma,t}$ are the extended infinitesimals of order γ and $\eta^{\beta,x}$, $\phi^{\beta,x}$, $\psi^{\beta,x}$ are the extended infinitesimals of order β.

System (53) for $A_4(t) = K_4 t^m$, $A_6(t) = K_6 t^s$ and $A_8(t) = K_8 t^r$ (where K_4, K_6, K_8 are arbitrary constants and m, s, r are real numbers) admits the group of transformations (7). Then the obtained Lie symmetries are

$$\xi = \frac{c_1 x}{3}, \quad \tau = \frac{c_1 t}{\gamma+m}, \quad \eta = \frac{\gamma-1}{2(\gamma+m)} c_1 u,$$
$$\phi = \frac{\gamma-1}{2(\gamma+m)} c_1 v, \quad \psi = \frac{\gamma-1}{2(\gamma+m)} c_1 w, \tag{55}$$

where c_1 is an arbitrary constant and variable coefficients are governed by the following conditions:

$$\begin{aligned}&A_1(t) = K_1 t^{\frac{1-3\gamma}{2}+\frac{\beta(\gamma+m)}{3}}, \\ &A_2(t) = K_2 A_1(t), \quad A_3(t) = K_3 A_1(t), \\ &A_5(t) = K_5 A_1(t), \quad A_7(t) = K_7 A_1(t),\end{aligned} \tag{56}$$

where M_1, K_2, K_3, K_5 and K_7 are arbitrary constants.

The Lie symmetry generator of system (53) is

$$X = \frac{x}{3}\frac{\partial}{\partial x} + \frac{t}{\gamma+m}\frac{\partial}{\partial t} + \frac{(\gamma-1)u}{2(\gamma+m)}\frac{\partial}{\partial u} + \frac{(\gamma-1)v}{2(\gamma+m)}\frac{\partial}{\partial v}$$

$$+\frac{(\gamma-1)w}{2(\gamma+m)}\frac{\partial}{\partial w}. \quad (57)$$

The solution of the auxiliary equations of the infinitesimal generator (57) gives the similarity variable

$$y = xt^{-\frac{\gamma+m}{3}} \quad (58)$$

and the corresponding similarity transformations are obtained as

$$u(x,t) = t^{\frac{\gamma-1}{2}} g(y),$$
$$v(x,t) = t^{\frac{\gamma-1}{2}} h(y),$$
$$w(x,t) = t^{\frac{\gamma-1}{2}} f(y). \quad (59)$$

Therefore, (56), (58) and (59) reduce the system of NLFPDEs (53) for $\gamma, \beta > 0$ to the system of NLFODEs, given as

$$\left(\mathcal{T}^{\frac{1-\gamma}{2},\gamma}_{\frac{3}{\gamma+m}} g\right)(y) = y^{-\beta}\bigg(M_1 g(y)(\mathcal{R}_1^{-\beta,\beta} g)(y) + (M_2 h(y)(\mathcal{R}_1^{-\beta,\beta} h)(y)) + (M_3(\mathcal{R}_1^{-\beta,\beta} f)(y))\bigg) + K_4 g'''(y),$$
$$\left(\mathcal{T}^{\frac{1-\gamma}{2},\gamma}_{\frac{3}{\gamma+m}} h\right)(y) = K_6 h'''(y) + M_4 y^{-\beta} \times g(y)(\mathcal{R}_1^{-\beta,\beta} h)(y),$$
$$\left(\mathcal{T}^{\frac{1-\gamma}{2},\gamma}_{\frac{3}{\gamma+m}} f\right)(y) = K_8 f'''(y) + M_5 y^{-\beta} \times g(y)(\mathcal{R}_1^{-\beta,\beta} f)(y), \quad (60)$$

where $(\mathcal{T}^{\frac{1-\gamma}{2},\gamma}_{\frac{3}{\gamma+m}})$ is the left-hand side Erdèlyi–Kober fractional differential operator given by (2), $(\mathcal{R}_1^{-\beta,\beta})$ is the right-hand side Erdèlyi–Kober fractional differential operator given by (4) and $M_2 = K_2 M_1$, $M_3 = K_3 M_1$, $M_4 = K_5 M_1$, $M_5 = K_7 M_1$ are arbitrary constants.

Remark 4. In the obtained symmetries of the considered system (45), if we put values of $A_1(t) = 3$, $A_2(t) = -6$, $A_3(t) = 3$, $A_4(t) = \frac{1}{4}$, $A_5(t) = -3$, $A_6(t) = -\frac{1}{2}$, $A_7(t) = -3$ and $A_8(t) = -\frac{1}{2}$, we can find the symmetries of the system considered in [36]. Also, if we choose $\beta = 1$ along with these values, then we can find symmetries of the system considered in [36].

3.5 *Space–time fractional coupled nonlinear Hirota equations with time-dependent variable coefficients*

The time fractional coupled nonlinear Hirota equations have been studied by various methods such as the Lie symmetry method [36] and homotopy method [48]. Also, space–time fractional coupled nonlinear Hirota equations with constant coefficients have been studied by the Lie symmetry method [36]. Now, we consider the space–time fractional coupled nonlinear Hirota equations with time-dependent variable coefficients as

$$\frac{\partial^\gamma P}{\partial t^\gamma} = A_1(t)(|P|^2+|Q|^2)\frac{\partial^\beta P}{\partial x^\beta} + A_2(t)\frac{\partial^3 P}{\partial x^3},$$
$$\frac{\partial^\gamma Q}{\partial t^\gamma} = A_3(t)(|P|^2+|Q|^2)\frac{\partial^\beta Q}{\partial x^\beta} + A_4(t)\frac{\partial^3 Q}{\partial x^3}, \quad (61)$$

where $P(x,t)$ and $Q(x,t)$ are complex valued functions.

Consider $P(x,t) = u(x,t) + \iota v(x,t)$ and $Q(x,t) = w(x,t) + \iota z(x,t)$ $(\iota = \sqrt{-1})$, then system (61) transfers into the following set of equations:

$$\frac{\partial^\gamma u}{\partial t^\gamma} = A_1(t)(u^2+v^2+w^2+z^2)\frac{\partial^\beta u}{\partial x^\beta} A_2(t)\frac{\partial^3 u}{\partial x^3},$$
$$\frac{\partial^\gamma v}{\partial t^\gamma} = A_1(t)(u^2+v^2+w^2+z^2)\frac{\partial^\beta v}{\partial x^\beta} + A_2(t)\frac{\partial^3 v}{\partial x^3},$$
$$\frac{\partial^\gamma w}{\partial t^\gamma} = A_3(t)(u^2+v^2+w^2+z^2)\frac{\partial^\beta w}{\partial x^\beta} + A_4(t)\frac{\partial^3 w}{\partial x^3},$$
$$\frac{\partial^\gamma z}{\partial t^\gamma} = A_3(t)(u^2+v^2+w^2+z^2)\frac{\partial^\beta z}{\partial x^\beta} + A_4(t)\frac{\partial^3 z}{\partial x^3}. \quad (62)$$

The invariance criteria for system (62) under the one-parameter Lie group of transformation is obtained as

$$\eta^{\gamma,t} = \big(2(A_1(t)(u\eta + v\phi + w\psi + z\delta)) + \tau A_1'(t)(u^2+v^2+w^2+z^2)\big)\partial_x^\beta u + (u^2+v^2+w^2+z^2)A_1(t)\eta^{\beta,x} + A_2(t)\eta^{xxx} + \tau A_2'(t)u_{xxx},$$
$$\phi^{\gamma,t} = \big(2(A_1(t)(u\eta + v\phi + w\psi + z\delta)) + \tau A_1'(t)(u^2+v^2+w^2+z^2)\big)\partial_x^\beta v + (u^2+v^2+w^2+z^2)A_1(t)\phi^{\beta,x} + A_2(t)\phi^{xxx} + \tau A_2'(t)v_{xxx},$$
$$\psi^{\gamma,t} = \big(2(A_3(t)(u\eta + v\phi + w\psi + z\delta)) + \tau A_3'(t)(u^2+v^2+w^2+z^2)\big)\partial_x^\beta w + (u^2+v^2+w^2+z^2)A_3(t)\psi^{\beta,x}$$

$$+ A_4(t)\eta^{xxx} + \tau A_4'(t)w_{xxx},$$

$$\delta^{\gamma,t} = \big(2(A_3(t)(u\eta + v\phi + w\psi + z\delta)) + \tau A_3'(t)(u^2 + v^2 + w^2 + z^2)\big)\partial_x^\beta z + (u^2 + v^2 + w^2 + z^2)A_3(t)\delta^{\beta,x} + A_4(t)\delta^{xxx} + \tau A_4'(t)z_{xxx}, \quad (63)$$

where η^{xxx}, ϕ^{xxx}, ψ^{xxx}, δ^{xxx} are the extended infinitesimals of order 3, $\eta^{\gamma,t}$, $\phi^{\gamma,t}$, $\psi^{\gamma,t}$, $\delta^{\gamma,t}$ are the extended infinitesimals of order γ and $\eta^{\beta,x}$, $\phi^{\beta,x}$, $\psi^{\beta,x}$, $\delta^{\beta,x}$ are the extended infinitesimals of order β.

System (62) for $A_2(t) = K_1 t^m$ and $A_4(t) = K_2 t^s$ (for K_1, K_2 being arbitrary constants and m, s are real numbers) admits the group of transformations (7), then the obtained Lie symmetries are as follows:

$$\xi = \frac{c_1 x}{3}, \quad \tau = \frac{c_1 t}{\gamma + m},$$

$$\eta = \frac{\gamma - 1}{2(\gamma + m)} c_1 u + c_2 v + c_3 w + c_4 z,$$

$$\phi = - c_2 u + \frac{\gamma - 1}{2(\gamma + m)} c_1 v + c_5 w + c_6 z,$$

$$\psi = - c_3 u - c_5 v + \frac{\gamma - 1}{2(\gamma + m)} c_1 + c_7 z,$$

$$\delta = - c_4 u - c_6 v - c_7 w + \frac{\gamma - 1}{2(\gamma + m)} c_1 z, \quad (64)$$

where c_1, c_2, c_3, c_4, c_5, c_6 and c_7 are arbitrary constants and variable coefficients are governed by the following conditions:

$$A_1(t) = M_1 t^{1-2\gamma+\frac{\beta(\gamma+m)}{3}},$$
$$A_3(t) = K A_1(t), \quad (65)$$

where M_1 and K are arbitrary constants.

The Lie symmetry generator of system (62) is given as

$$X_1 = \frac{x}{3}\frac{\partial}{\partial x} + \frac{t}{\gamma + m}\frac{\partial}{\partial t} + \frac{(\gamma - 1)u}{2(\gamma + m)}\frac{\partial}{\partial u} + \frac{(\gamma - 1)v}{2(\gamma + m)}\frac{\partial}{\partial v} + \frac{(\gamma - 1)w}{2(\gamma + m)}\frac{\partial}{\partial w} + \frac{(\gamma - 1)z}{2(\gamma + m)}\frac{\partial}{\partial z},$$

$$X_2 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v},$$

$$X_3 = w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w},$$

$$X_4 = z\frac{\partial}{\partial u} - u\frac{\partial}{\partial z},$$

$$X_5 = w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w},$$

$$X_6 = z\frac{\partial}{\partial v} - v\frac{\partial}{\partial w},$$

$$X_7 = z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}. \quad (66)$$

The solution of auxiliary equations of the infinitesimal generator X_1 gives the similarity variable

$$y = x t^{-\frac{\gamma+m}{3}} \quad (67)$$

and the corresponding similarity transformations are as follows:

$$u(x,t) = t^{\frac{\gamma-1}{2}} g(y),$$
$$v(x,t) = t^{\frac{\gamma-1}{2}} h(y),$$
$$w(x,t) = t^{\frac{\gamma-1}{2}} f(y),$$
$$z(x,t) = t^{\frac{\gamma-1}{2}} k(y). \quad (68)$$

Thus, the admissible coefficients given by (65), with (67) and (68), reduce the system of NLFPDEs (62) for $\gamma, \beta > 0$ to the system of NLFODEs, given as

$$\left(\mathcal{T}_{\frac{3}{\gamma+m}}^{\frac{1-\gamma}{2},\gamma} g\right)(y) = K_1 g'''(y) + y^{-\beta} \times (M_1(g^2 + h^2 + f^2 + k^2) \times (\mathcal{R}_1^{-\beta,\beta} g)(y)),$$

$$\left(\mathcal{T}_{\frac{3}{\gamma+m}}^{\frac{1-\gamma}{2},\gamma} h\right)(y) = K_1 h'''(y) + y^{-\beta} \times (M_1(g^2 + h^2 + f^2 + k^2) \times (\mathcal{R}_1^{-\beta,\beta} h)(y)),$$

$$\left(\mathcal{T}_{\frac{3}{\gamma+m}}^{\frac{1-\gamma}{2},\gamma} f\right)(y) = K_2 f'''(y) + y^{-\beta} \times (M_2(g^2 + h^2 + f^2 + k^2) \times (\mathcal{R}_1^{-\beta,\beta} f)(y)),$$

$$\left(\mathcal{T}_{\frac{3}{\gamma+m}}^{\frac{1-\gamma}{2},\gamma} k\right)(y) = K_2 k'''(y) + y^{-\beta} \times (M_2(g^2 + h^2 + f^2 + k^2) \times (\mathcal{R}_1^{-\beta,\beta} k)(y)), \quad (69)$$

where $(\mathcal{T}_{\frac{3}{\gamma+m}}^{\frac{1-\gamma}{2},\gamma})$ is the left-hand side Erdèlyi–Kober fractional differential operator given by (2), $(\mathcal{R}_1^{-\beta,\beta})$ is the right-hand side Erdèlyi–Kober fractional differential operator given by (4) and $M_2 = KM_1$ is an arbitrary constant.

Remark 5. By putting the values of $A_1(t) = -6$, $A_2(t) = -1$, $A_3(t) = -6$ and $A_4(t) = -1$ in the obtained symmetries of the considered system (45), we can obtain the symmetries of the system with constant coefficients considered in [33]. Also, if we take $\beta = 1$ along with these values, then we can find symmetries of the system considered in [36].

4. Conclusion

In the present study, we investigated the Lie symmetry for some systems of space–time FPDEs with time-dependent variable coefficients. The fractional Lie symmetry method is considered for application to space–time fractional coupled Burgers equations, Ito's system, coupled KdV equations, Hirota–Satsuma coupled KdV equations and coupled nonlinear Hirota equations with time-dependent variable coefficients with Riemann–Liouville derivative. As an application of the infinitesimal symmetries, we have shown that these space–time NLFPDE systems can be transformed into system of NLFODEs.

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