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Extreme multistable synchronisation in coupled dynamical systems

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Abstract. A rule for designing extreme multistable synchronised systems by coupling two identical dynamical systems has been proposed in this paper. The basic idea behind the proposed scheme is the existence of chaos in the coupled system in the presence of initial condition-dependent constants of motion. A new conjecture has been introduced according to which an extreme multistable synchronised system can be designed if all states of one system will synchronise with the corresponding states of the other system (of the two coupled systems) and the basin of the synchronised state depends on the difference between the initial conditions of the corresponding states of the individual systems. The proposed scheme has been illustrated with the help of coupled Rössler systems, coupled Hénon maps and coupled logistic maps. Moreover, the existence of flip bifurcation with the variation of initial conditions has been shown analytically as well as numerically in the case of coupled Hénon maps. Numerical results are reported to show the proficiency of the proposed scheme to design extreme multistable synchronisation behaviour. This work establishes a theoretical foundation for constructing extreme multistable synchronised continuous as well as discrete dynamical systems.

Keywords. Multistability; Rössler system; Hénon map; flip bifurcation; Lyapunov exponent.

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1. Introduction

Multistability is found in almost all areas of science and nature ranging from ecological models to neuronal systems. Various studies have been reported for multistability, e.g. in visual perceptions [1], in ecosystems [2,3], in biological systems [4,5], in neuron dynamics [6,7], in climate dynamics [8], in optical systems [9], in semiconductor materials [10], in social systems [11], in chemical reactions [12], in hydrodynamics [13]. Multistability is generated when a large number of asymptotic stable states coexist for a fixed set of parameters depending on their initial conditions. In the case of extreme multistability of a system, there exist infinitely many stable attractors in the system. Designing extreme multistable systems has become a topic of great interest to theoretical and experimental researchers in the last few decades [14–22]. Hens *et al* [17] proposed a coupling scheme to obtain extreme multistable continuous dynamical systems by using the concept of partial synchronisation and existence of initial condition-dependent conserved quantities. Hens *et al* [17] reported that the coexistence of infinitely many attractors in two coupled m-dimensional continuous systems will be possible if $m-1$ of the variables of the two systems are completely synchronised and one of them obeys a constant difference between them. Pal *et al* [23] generalised the condition for extreme multistability and noted that multistable nature can be obtained if i number of variables of the two systems are completely synchronised and j number of variables keep a constant difference between them, where $i + j = m$ and $1 \leq i, j \leq m-1$.

According to Abarbanel *et al* [24], the states $x(t)$ and $y(t)$ are in a state of generalised synchronisation if $|y(t) - \phi(x(t))| \to 0$ as $t \to \infty$ for a time-independent continuous function ϕ. If $x_i(t) - y_i(t) = C_i, \forall t$ $(i = 1, 2, 3, \ldots, m)$ where C_i's are constants, then one can say that the states are synchronised in the generalised sense. Therefore, it is clear from the above discussions that according to the existing theory, an extreme multistable system can be designed if i number of variables of the two systems are synchronised identically and j number of variables are synchronised in generalised sense where $i + j = m$.

Multistability of synchronised states has been reported by many researchers [25–27]. Extreme multistability of synchronised state means the existence of infinitely many synchronised attractors in the coupled system. Therefore, there are infinitely many basins of attraction of the synchronised states in the coupled systems. In this paper, our basic motivation is to show that an extreme multistable synchronised system can be designed if all the m states of one system will synchronise with the m states of the other system in the presence of m number of initial condition-dependent constants of motion. In this case, synchronisation means $\lim_{t\to\infty}|x_i(t) - y_i(t)| \to C_i$, where C_i's are initial condition-dependent constants of motion. This is definitely a kind of generalisation of the existing theory of designing extreme multistable systems [17,23]. The infinite number of initial condition-dependent conserved quantities is present in the phase space which generates the possibilities of obtaining infinitely many synchronised attractors in the coupled systems. The existence of extreme multistable synchronisation opens up new areas of research, namely the control of synchronised states in nonlinear dynamics similar to chaos control. Schemes for constructing extreme multistable synchronised system have been proposed for continuous and discrete dynamical systems. In particular, coupled Rössler systems, coupled Hénon maps and coupled logistic maps have been considered for the discussion. Numerical simulation results are provided in support of the theoretical predictions.

The paper is organised as follows: in §2, the scheme for designing extreme multistable synchronised systems has been proposed. Section 3 illustrates the proposed scheme with the help of two coupled Rössler systems and two-dimensional Hénon maps. In §4, the existence of period doubling bifurcation with the variation of initial conditions is shown analytically with the help of normal form in the case of Hénon map. The numerical simulation results are presented and analysed in §5. Finally, a conclusion is drawn in §6.

2. Design of extreme multistable synchronised systems

2.1 *Continuous system*

Consider two identical n ($n \geq 3$)-dimensional dynamical systems of the following type:

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}) \quad \text{and} \quad \dot{\mathbf{Y}} = \mathbf{f}(\mathbf{Y}), \tag{1}$$

where the state variables $\mathbf{X}$ and $\mathbf{Y}$ are n-dimensional vectors, i.e. $\mathbf{X},\ \mathbf{Y} \in \mathbb{R}^n$, $\mathbf{f} = (f_1, f_2, \ldots, f_n)^{\mathrm{t}}$ is a vector function with n components.

Now, couple the two identical dynamical systems of the above type in the following way:

$$\begin{aligned}\dot{x}_i &= f_i(x_1, x_2, x_3, \ldots, x_n)\\ &\quad + u_{1i}(x_1, x_2, x_3, \ldots, x_n; y_1, y_2, y_3, \ldots, y_n),\\ \dot{y}_i &= f_i(y_1, y_2, y_3, \ldots, y_n)\\ &\quad + u_{2i}(x_1, x_2, x_3, \ldots, x_n; y_1, y_2, y_3, \ldots, y_n),\end{aligned} \tag{2}$$

where u_{1i} and u_{2i} ($i = 1, 2, 3, \ldots, n$) are the controllers. The synchronisation error between the state variables of system (2) is defined as $e_i = y_i - x_i$, $i = 1, 2, \ldots, n$. The error dynamical system becomes

$$\begin{aligned}\dot{e}_i =& f_i(y_1, y_2, y_3, \ldots, y_n) - f_i(x_1, x_2, x_3 \ldots, x_n)\\ &+ u_{2i} - u_{1i}.\end{aligned} \tag{3}$$

The controllers u_{1i} and u_{2i} ($i = 1, 2, 3, \ldots, n$) are chosen in such a way that the coupled system has multistable synchronisation.

Now, according to the proposed scheme, an extreme multistable synchronised systems can be designed by choosing u_{1i} and u_{2i} ($i = 1, 2, 3, \ldots, n$) in such a way that all the corresponding state variables of the two coupled systems keep a constant difference. Therefore, the choice of the controllers will be such that

$$\dot{e}_i = 0, \quad i = 1, 2, 3, \ldots, n. \tag{4}$$

Therefore, $e_1, e_2, e_3, \ldots, e_n$ become constants of motion and henceforth $y_i = x_i + c_i$ ($i = 1, 2, 3, \ldots, n$).

Now, the dynamics of coupled systems (2) is equivalent to the following modified system:

$$\begin{aligned}\dot{x}_i =& f_1(x_1, x_2, \ldots, x_n)\\ &+ u_{1i}(x_1, \ldots, x_n; x_1 + c_1, \ldots, x_n + c_n),\end{aligned} \tag{5}$$

where c_i ($i = 1, 2, 3, \ldots, n$) are the initial condition-dependent constants. The coupled systems (2) show extreme multistable synchronisation behaviour if dynamics of system (5) changes qualitatively with the variation of c_i ($i = 1, 2, 3, \ldots, n$). As the uncoupled system is chaotic, there is a possibility of the existence of extreme multistable synchronisation in the coupled systems.

2.1.1 *Illustration with Rössler system.* In this section, the proposed technique for designing extreme multistable synchronised system will be illustrated with the help of coupled identical chaotic Rössler [28] systems. The famous Rössler system is the following:

$$\begin{aligned}\dot{x} &= -y - z,\\ \dot{y} &= x + ay,\\ \dot{z} &= b - cz + xz,\end{aligned} \tag{6}$$

where a, b and c are positive parameters.

Two identical chaotic Rössler systems are coupled in the following way:

$$\dot{x}_1 = -x_2 - x_3 + u_{11},$$

$$\dot{x}_2 = x_1 + ax_2 + u_{12},$$
$$\dot{x}_3 = b - cx_3 + x_1x_3 + u_{13},$$
$$\dot{y}_1 = -y_2 - y_3 + u_{21},$$
$$\dot{y}_2 = y_1 + ay_2 + u_{22},$$
$$\dot{y}_3 = b - cy_3 + y_1y_3 + u_{23}. \quad (7)$$

The controllers u_{1i} and u_{2i} ($i = 1, 2, 3$), are chosen in such a way that the above system has extreme multistable synchronisation. The dynamics of the synchronisation errors $e_i = y_i - x_i$ ($i = 1, 2, 3$) will be governed by

$$\dot{e}_1 = -e_2 - e_3 + u_{21} - u_{11},$$
$$\dot{e}_2 = e_1 + ae_2 + u_{22} - u_{12},$$
$$\dot{e}_3 = -ce_3 + y_1y_3 - x_1x_3 + u_{23} - u_{13}. \quad (8)$$

Now, the controllers u_{1i} and u_{2i} ($i = 1, 2, 3$) are selected as

$$\begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ a(y_2 - x_2) \\ x_3(y_1 - x_1) \end{pmatrix},$$
$$\begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 + y_2 + y_3 \\ x_1 - y_1 \\ (c - y_1)(y_3 - x_3) \end{pmatrix}. \quad (9)$$

With this choice, the time evolution for the error system will be

$$\dot{e}_i = 0, \quad i = 1, 2, 3. \quad (10)$$

Therefore, e_i = constant = c_i ($i = 1, 2, 3$). Hence, $y_i = x_i + c_i$, where c_i ($i = 1, 2, 3$) are constants depending on the initial conditions of the states of the coupled systems.

The error dynamics (10) possesses a fixed point, (e_1^*, e_2^*, e_3^*), where e_1^*, e_2^* and e_3^* can take any real values. The synchronisation error system is globally stable if we can construct a Lyapunov function.

Let us choose a Lyapunov function (L) corresponding to the above error dynamics as $L = (e_1 - e_1^*)^2 + (e_2 - e_2^*)^2 + (e_3 - e_3^*)^2$. Then, clearly $L(e_1^*, e_2^*, e_3^*) = 0$ and $L(e_1, e_2, e_3) \geq 0$. Also, $\dot{L} = 2(e_1 - e_1^*)\dot{e}_1 + 2(e_2 - e_2^*)\dot{e}_2 + 2(e_3 - e_3^*)\dot{e}_3 = 0$. This establishes that the fixed point, (e_1^*, e_2^*, e_3^*), is stable [29]. However, it is not asymptotically stable. Hence, small perturbation in the synchronisation manifold will not grow and the synchronisation manifold is stable.

Therefore, the dynamics of system (7) is equivalent to the following three-dimensional system:

$$\dot{x}_1 = -x_2 - x_3,$$
$$\dot{x}_2 = x_1 + a(x_2 + c_2),$$
$$\dot{x}_3 = b + x_3(x_1 + c_1 - c). \quad (11)$$

System (7) has extreme multistable synchronisation behaviour if the dynamical behaviour of system (11) changes qualitatively with the variation of the value of c_1 and c_2. Note that for the above choice of the controller, the system dynamics is independent of c_3, i.e. the synchronised dynamics is independent of the choice of initial conditions for x_3 and y_3. One can choose suitable controllers to obtain the dependence of all three constants of motion in the synchronised dynamics.

2.2 *Discrete system*

In this section, the scheme will be discussed for coupled identical discrete dynamical systems. Consider an m-dimensional smooth map of the following type:

$$\mathbf{x}(n+1) = \mathbf{f}(\mathbf{x}(n)), \quad (12)$$

where the state variable $\mathbf{x}$ is an m-dimensional vector, i.e. $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{f} = (f_1, f_2, \ldots, f_m)^{\mathrm{t}}$ is a vector function with m components. Two identical m-dimensional maps of the above types are coupled in the following way:

$$x_p(n+1) = f_p(x_1(n), x_2(n), \ldots, x_m(n)) + u_p(x_1(n), \ldots, x_m(n); y_1(n), \ldots, y_m(n)),$$
$$y_p(n+1) = f_p(y_1(n), y_2(n), \ldots, y_m(n)) + v_p(x_1(n), \ldots, x_m(n); y_1(n), \ldots, y_m(n)), \quad (13)$$

where u_p and v_p ($p = 1, 2, \ldots, m$) are the controllers which will be chosen later. As our aim is to design an extreme multistable synchronised system, the time evolution rule of the synchronisation error is very important. The synchronisation error between systems (13) is defined as

$$e_p(n) = y_p(n) - x_p(n), \quad p = 1, 2, \ldots, m$$

and the dynamical equations for the time evolution of error are obtained as follows:

$$e_p(n+1) = f_p(y_1(n), \ldots, y_m(n)) - f_p(x_1(n), \ldots, x_m(n)) + v_p - u_p \quad (14)$$

($p = 1, 2, \ldots, m$). The extreme multistable synchronised system can be designed by choosing $u_1, u_2, u_3, \ldots, u_m$ and $v_1, v_2, v_3, \ldots, v_m$ in such a way that all the synchronisation errors become initial condition-dependent constants of motion. Therefore, u_p and v_p ($p = 1, 2, \ldots, m$) are chosen in such a way that

$$e_p(n) = c_p \quad \text{for } p = 1, 2, \ldots, m.$$

Therefore, the errors e_p remain constant in time,

$y_p(n) = x_p(n) + c_p\ (p = 1, 2, \ldots, m)$ as $n \to \infty$.

Hence, c_p's are initial condition-dependent constants of motion. In other words, one can say that all the state variable x_i's of one coupled system will be synchronised in the generalised sense with the corresponding state variables of the other coupled system. For the above choice, the dynamics of the coupled systems (13) is equivalent to the following system:

$$x_p(n+1) = f_1(x_1(n), \ldots, x_m(n)) + u_p(x_1(n), \ldots, x_m(n); x_1(n) + c_1, \ldots, x_m(n) + c_m), \quad (15)$$

$p = 1, 2, \ldots, m$ and $c_1, c_2, \ldots, c_m$ are initial condition-dependent constants. System (15) shows extreme multistable synchronisation if infinite number of qualitatively different types of stable synchronised attractors exists with variation of c_p's $(p = 1, 2, \ldots, m)$ only. In other words, the system has infinite number of basins of attraction for infinite number of different types of initial conditions.

2.2.1 *Illustration with Hénon map.* In this section, the proposed method will be illustrated with the help of coupled two-dimensional Hénon maps. The following two-dimensional Hénon map [30] has been considered:

$$x(n+1) = 1 - ax^2(n) + y(n),$$
$$y(n+1) = bx(n), \quad (16)$$

where a and b are parameters. It is well known that for $a = 1.4$ and $b = 0.3$, the map shows chaotic behaviour. Two Hénon maps described by the state variables x_1, y_1 and x_2, y_2 are coupled in a highly nonlinear manner using the controllers u_{ij} $(i, j = 1, 2)$ in the following way:

$$x_1(n+1) = 1 - ax_1^2(n) + y_1(n) + u_{11},$$
$$y_1(n+1) = bx_1(n) + u_{12},$$
$$x_2(n+1) = 1 - ax_2^2(n) + y_2(n) + u_{21},$$
$$y_2(n+1) = bx_2(n) + u_{22}. \quad (17)$$

The error dynamics of the coupled Hénon maps obeys the following evolution equation:

$$e_1(n+1) = -a(x_1(n) + x_2(n))e_1(n) + e_2(n) + u_{21} - u_{11},$$
$$e_2(n+1) = be_1(n) + u_{22} - u_{12}. \quad (18)$$

Now, our task is to choose u_{ij} $(i, j = 1, 2)$ in such a way that $e_p(n) \to c_p$, i.e. $x_2(n) \to x_1(n) + c_1$ and $y_2(n) \to y_1(n) + c_2$ (c_p, $p = 1, 2$, are initial condition-dependent constants of the coupled systems). Therefore, knowing the evolution of the states x_1 and y_1 one can easily determine the evolution of the states x_2 and y_2 and vice versa. This simplification helps us to reduce the dimension of the coupled system.

The controllers are chosen as

$$\begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -b & 0 & b & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(n) \\ y_1(n) \\ x_2(n) \\ y_2(n) \end{pmatrix} + \begin{pmatrix} -a(x_2^2(n) - x_1^2(n)) \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (19)$$

The above choice transforms eq. (18) to

$$e_p(n) = c_p \quad \text{for } p = 1, 2. \quad (20)$$

The error dynamics (20) possesses a fixed point as $(e_1^*(n), e_2^*(n))$, where $e_1^*(n)$ and $e_2^*(n)$ can take any real values. The error dynamics can be rewritten as $e(n+1) = Ae(n)$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us construct a function V as $V\colon G \to \mathbb{R}$, $G \subset \mathbb{R}^m$ and define the variation of V as $\Delta V(n) = V(e(n+1)) - V(e(n))$. Then V is a Lyapunov function on G if

1. V is continuous on G and
2. $\Delta V \le 0$ whenever both $e(n)$ and $e(n+1)$ are in G [31].

Now, we define $V\colon \mathbb{R}^2 \to \mathbb{R}$ by $V(e) = e^{\mathrm{T}}(n)Be(n)$, where $e(n) = (e_1(n), e_2(n))^{\mathrm{T}}$ and B is a positive definite matrix. Clearly, V is continuous on $\mathbb{R}^2$ and $\Delta V = 0$, as $e_1(n+1) = e_1(n)$ and $e_2(n+1) = e_2(n)$. Then, V is a Lyapunov function and hence, the error dynamics is globally stable.

The above choice of controllers immediately tells us that the dynamics of coupled system (17) can be determined completely by the following reduced system of evolution equations:

$$x_2(n+1) = 1 - ax_2^2(n) + y_2(n) + c_1,$$
$$y_2(n+1) = bx_2(n) + c_2. \quad (21)$$

The coupled Hénon maps have extreme multistable synchronised states if system (21) has infinitely many stable attractors with the variation of c_1 and c_2.

3. Theoretical results

Now, system (21) will be analysed analytically. Assume that λ_1 and λ_2 are two roots of the characteristic equation of the Jacobian matrix $J|_{(x^*, y^*)}$. It is well known that a fixed point, (x^*, y^*), is called (i) a sink or locally asymptotically stable if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, (ii) a source or locally unstable if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, (iii) a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 2$ and $|\lambda_2| > 1$) and (iv) non-hyperbolic if either one of eigenvalues is of unit modulus i.e. $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

The fixed points of system (21) will be obtained from the following equations:

$$x^* = 1 - ax^{*2} + y^* + c_1,$$
$$y^* = bx^* + c_2.$$

The two fixed points of system (21) are

$$P_1\left(\frac{b-1+\beta}{2a}, \frac{b(b-1+\beta)}{2a} + c_2\right),$$

and

$$P_2\left(\frac{b-1-\beta}{2a}, \frac{b(b-1-\beta)}{2a} + c_2\right),$$

where

$$\beta = \sqrt{(b-1)^2 + 4a(1+c_1+c_2)}.$$

Therefore, the condition for the existence of fixed point becomes $(b-1)^2 + 4a(1+c_1+c_2) \geq 0$. The Jacobian matrix of (21) at the fixed point (x^*, y^*) can be written as

$$J = \begin{pmatrix} -2ax^* & 1 \\ b & 0 \end{pmatrix}.$$

The characteristic equation of the Jacobian matrix J of system (21) evaluated at the fixed point P_1 can be written as $\lambda^2 + B\lambda + C = 0$, where $B = 2ax$ and $C = -b$. Let $H(\lambda) = \lambda^2 + B\lambda + C$. Then $H(1) = 1 + 2ax - b$ (> 0) and $H(-1) = 1 - 2ax - b$. We know the fixed point P_1 is a sink (locally asymptotically stable) when $(c_1 + c_2) < (3(b-1)^2/4a) - 1$ and $b > -1$; it is a saddle when $(c_1 + c_2) > (3(b-1)^2/4a) - 1$; it is a source (locally unstable) when $(c_1 + c_2) < (3(b-1)^2/4a) - 1$ and $b < -1$, it is non-hyperbolic when $(c_1+c_2) = (3(b-1)^2/4a)-1, c_1 \neq -1, (b/a)-1$ and the system may undergo flip bifurcation (period-doubling bifurcation) in this case. The Hopf bifurcation can occur at P_1 when $c_1 + c_2 < (3/a) - 1$ and $b = -1$.

3.1 *Flip bifurcation*

To study the flip bifurcation, the following theorem will be useful [32–34]:

Theorem 1. *Let $f_\mu\colon \mathbb{R} \to \mathbb{R}$ be a one-parameter family of mappings such that f_{μ_0} has a fixed point x_0 with eigenvalue -1. Assume*
(F1)

$$\left(\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial \mu}\right) = \frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x} - 1\right)\frac{\partial^2 f}{\partial x \partial \mu} \neq 0 \quad at\ (x_0, \mu_0),$$

(F2)

$$a = \left(\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 f}{\partial x^3}\right)\right) \neq 0 \quad at\ (x_0, \mu_0).$$

Then there is a smooth curve of fixed points of f_μ passing through (x_0, μ_0), the stability of which changes at (x_0, μ_0). There is also a smooth curve γ passing through (x_0, μ_0) so that $\gamma - (x_0, \mu_0)$ is a union of hyperbolic period-2 orbits. The curve γ has quadratic tangency with the line $\mathbb{R} \times \mu_0$ at (x_0, μ_0).

Proof. Let

$$S = \left\{(c_1, c_2) : c_1 + c_2 = \frac{3(b-1)^2}{4a} - 1, \quad c_1 + c_2 \neq -1, \quad \frac{b}{a} - 1, a \neq 0, |b| < 1\right\}.$$

We first discuss the flip bifurcation of system (21) at

$$P_1\left(\frac{b-1+\beta}{2a}, \frac{b(b-1+\beta)}{2a} + c_2\right)$$

when parameters vary in a small neighbourhood of S. Taking the arbitrary parameters $(c_1, c_2) \in S$, then system (21) has a unique interior fixed point P_1 and the corresponding eigenvalues are $\lambda_1 = -1$ and $|\lambda_2| \neq 1$. Choosing δ as a bifurcation parameter, we consider a perturbation of system (21) as follows:

$$x(n+1) = 1 - ax(n)^2 + y(n) + c_1 + \delta,$$
$$y(n+1) = bx(n) + c_2, \tag{22}$$

where $|\delta| \ll 1$ which is a small perturbation parameter.

Choose $c_1 = (3(b-1)^2/4a) - c_2 - 1$, then $\beta = 2(1-b)$. In order to transfer the fixed point

$$P_1\left(\frac{1-b}{2a}, \frac{b(1-b)}{2a} + c_2\right)$$

to the origin (0,0),

$$\zeta(n) = x(n) - \frac{1-b}{2a}, \quad \eta(n) = y(n) - \frac{b(1-b)}{2a} - c_2,$$

then we have

$$\begin{pmatrix} \zeta(n+1) \\ \eta(n+1) \end{pmatrix} = \begin{pmatrix} b-1 & 1 \\ b & 0 \end{pmatrix}\begin{pmatrix} \zeta(n) \\ \eta(n) \end{pmatrix} + \begin{pmatrix} -a\zeta(n)^2 + \delta + \dfrac{1-b}{2a} \\ \dfrac{b(1-b)}{2a} + c_2 \end{pmatrix} \tag{23}$$

and construct an invertible matrix

$$T = \begin{pmatrix} 1 & 1 \\ -b & 1 \end{pmatrix}. \tag{24}$$

Now, make a translation

$$\begin{pmatrix}\zeta(n)\\ \eta(n)\end{pmatrix} = T\begin{pmatrix}u(n)\\ v(n)\end{pmatrix} \tag{25}$$

to system (23), then we get

$$\begin{pmatrix}u(n+1)\\ v(n+1)\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}u(n)\\ v(n)\end{pmatrix} + \begin{pmatrix}f_1(u(n), v(n), \delta)\\ f_2(u(n), v(n), \delta)\end{pmatrix}, \tag{26}$$

where

$$\begin{aligned} f_1(u(n), v(n), \delta) &= a_1 u(n)^2 + a_2 u(n)v(n) \\ &\quad + a_3 v(n)^2 + a_4\delta, \\ f_2(u(n), v(n), \delta) &= b_1 u(n)^2 + b_2 u(n)v(n) \\ &\quad + b_3 v(n)^2 + b_4\delta, \end{aligned} \tag{27}$$

$$\begin{aligned} a_1 &= -\frac{a}{1+b}, \quad a_2 = -\frac{2a}{1+b}, \\ a_3 &= -\frac{a}{1+b}, \quad a_4 = \frac{1}{1+b}, \\ b_1 &= -\frac{ab}{1+b}, \quad b_2 = -\frac{2ab}{1+b}, \\ b_3 &= -\frac{ab}{1+b}, \quad b_4 = \frac{b}{1+b}. \end{aligned} \tag{28}$$

Next, we determine the centre manifold $W^c(0, 0)$ of (26) at the fixed point (0, 0) in a small neighbourhood of $\delta = 0$. Based on the centre manifold theory [35–38], we know that there exists a centre manifold $W^c(0, 0)$, which can be approximately represented as follows:

$$\begin{aligned} W^c(0, 0) &= \{(u(n), v(n)) \in R^2 \ : v(n) = W(u(n), \delta) \\ &= \hat{c}_0\delta + \hat{c}_1 u(n)^2 + \hat{c}_2 u(n)\delta + \hat{c}_3\delta^2 \\ &\quad + O((|u(n)| + |\delta|)^3)\}, \end{aligned}$$

where $O((|u(n)| + |\delta|)^3)$ is a function with order at least three of its variables. Now, $f_1 = f_1(u(n), W(u(n), \delta), \delta)$, we have

$$\begin{aligned} v(n+1) &= W(u(n+1), \delta) = W(-u(n) + f_1, \delta) \\ &= \hat{c}_0\delta + \hat{c}_1 u(n)^2 - (2\hat{c}_1 a_4 + \hat{c}_2)u(n)\delta \\ &\quad + (\hat{c}_1 a_4^2 + \hat{c}_2 a_4 + \hat{c}_3)\delta^2 + O(3). \end{aligned} \tag{29}$$

Also,

$$\begin{aligned} v(n+1) &= bv(n) + f_2 \\ &= (b\hat{c}_0 + b_4)\delta + (b\hat{c}_1 + b_1)u(n)^2 \\ &\quad + (b\hat{c}_2 + b_2\hat{c}_0)u(n)\delta + (b\hat{c}_3 + b_3\hat{c}_0^2)\delta^2 \\ &\quad + O(3). \end{aligned} \tag{30}$$

By equating the coefficients of (29) and (30), we have

$$\hat{c}_0 = \frac{b}{1-b^2}, \quad \hat{c}_1 = -\frac{ab}{1-b^2},$$

$$\hat{c}_2 = \frac{2ab}{(1+b)(1-b^2)}, \quad \hat{c}_3 = \frac{ab(b^2+b-1)}{(1-b^2)^3}. \tag{31}$$

Thus, the centre manifold can be approximated as

$$\begin{aligned} v(n) &= \hat{c}_0\delta + \hat{c}_1 u^2(n) + \hat{c}_2 u(n)\delta \\ &\quad + \hat{c}_3\delta^2 + O(3). \end{aligned} \tag{32}$$

Substituting (32) into the first equation of (26), allows us to describe the dynamics on the centre manifold by the following equation:

$$\begin{aligned} u_{n+1} &= -u_n + f_1(u_n, v_n, \delta) \\ &= -u_n + a_1 u_n^2 + a_2 u_n\delta + a_2\hat{c}_1 u_n^3 \\ &\quad + a_2\hat{c}_2 u_n^2\delta + a_2\hat{c}_3 u_n\delta^2 + a_3\hat{c}_0^2\delta^2 \\ &\quad + 2a_3\hat{c}_0\hat{c}_1 u_n^2\delta + 2a_3\hat{c}_0\hat{c}_2 u_n\delta^2 + 2a_3\hat{c}_0\hat{c}_3\delta^3 \\ &\quad + a_4\delta + O(4). \end{aligned} \tag{33}$$

Denote the right-hand side of (33) by F, then the conditions of flip bifurcation are the following two discriminatory quantities: $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ (i.e. conditions (F1) and (F2) of Theorem 1 are equivalent to $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$), where

$$\begin{aligned} \alpha_1 &= \left(\frac{\partial^2 F}{\partial u_n \partial\delta} + \frac{1}{2}\frac{\partial F}{\partial\delta}\frac{\partial^2 F}{\partial u_n^2}\right)\Bigg|_{(0,0)} \neq 0, \\ \alpha_2 &= \left(\frac{1}{6}\frac{\partial^3 F}{\partial u_n^3} + \left(\frac{1}{2}\frac{\partial^2 F}{\partial u_n^2}\right)^2\right)\Bigg|_{(0,0)} \neq 0. \end{aligned} \tag{34}$$

A simple calculation shows that

$$\begin{aligned} \alpha_1 &= a_2 + a_1 a_4 = -\frac{3a + 2ab}{(1+b)^2}, \\ \alpha_2 &= a_1^2 + a_2\hat{c}_1 = \frac{a^2}{1-b^2}. \end{aligned} \tag{35}$$

Based on the above analysis and Theorem 1, we have the following results.

If $(c_1, c_2) \in S$; $\alpha_1 > 0$ and $\alpha_2 > 0$ in (35), then system (21) undergoes a flip bifurcation at the fixed point,

$$P_1\left(\frac{b-1+\beta}{2a}, \frac{b(b-1+\beta)}{2a} + c_2\right).$$

Also, $\alpha_2 > 0$ implies that the flip bifurcation is supercritical, i.e. a period-2 orbit of system (21) appears and it is stable.

In the case of the fixed point,

$$P_2\left(\frac{b-1-\beta}{2a}, \quad \frac{b(b-1-\beta)}{2a} + c_2\right),$$

we have the eigenvalues of Jacobian matrix J as $-\frac{1}{2}(b-1-\beta) \pm \frac{1}{2}\sqrt{(b-1-\beta)^2 + 4b}$ and $\operatorname{tr} J - 1 < -\det J$

and $-\text{tr}\,J - 1 > \det J$. Hence, fixed point P_2 is not a stable fixed point of (21).

Therefore, the existence of period doubling bifurcation in system (21) has been shown analytically with the variation of the parameter c_1 as well as c_2. As c_1 and c_2 are initial condition-dependent parameters, it is clear that with the variation of initial conditions, period doubling may occur in the system. Hence, we have shown analytically that the newly proposed technique is suitable for designing multistable systems. □

4. Extreme multistable synchronisation in logistic map

We have already discussed that extreme multistable synchronisation is possible in coupled one-dimensional discrete dynamical systems. Here the existence of extreme multistable synchronisation is shown in coupled logistic maps.

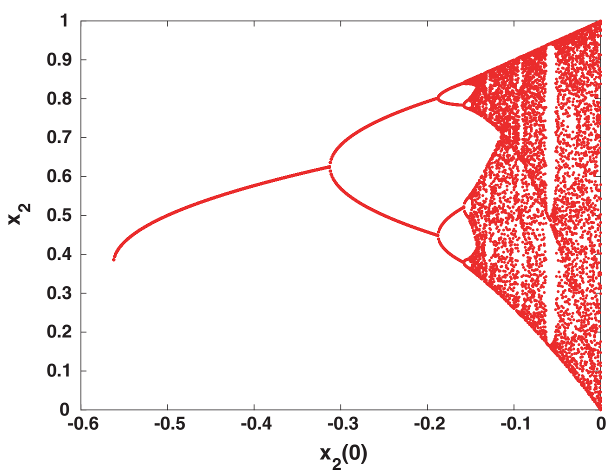

Figure 1. Bifurcation diagram x_2 of the logistic map (39) with respect to parameters $x_1(0) = 0$ and $r = 4$.

We consider the following one-dimensional logistic map [39]:

$$x(n+1) = rx(n)(1 - x(n)), \tag{36}$$

where r is the growth rate. It is well known that for $r = 4$ the map shows a chaotic behaviour. We coupled two logistic maps described by the state variables x_1 and x_2 as follows:

$$x_1(n+1) = rx_1(n)(1 - x_1(n)) + u_1,$$

$$x_2(n+1) = rx_2(n)(1 - x_2(n)) + u_2. \tag{37}$$

The error dynamics of the coupled system is given by the following equation:

$$e(n+1) = re(n) - rx_2^2(n) + rx_1^2(n) + u_2 - u_1. \tag{38}$$

We choose u_i $(i = 1, 2)$ in such a way that $e(n+1) \to e(n)$, i.e. $e(n) \to c$ (c is the initial dependent constant).

Let

$$u_1 = r(x_2(n) - x_1(n)) - rx_2^2(n) + rx_1^2(n),$$

$$u_2 = x_2(n) - x_1(n).$$

By the above choice of the controllers (37) becomes

$$x_2(n+1) = rx_2(n)(1 - x_2(n)) + c. \tag{39}$$

The x_1 state can be obtained by adding a constant with the x_2 state, i.e. the states x_1 and x_2 are synchronised. Numerical simulation results for different initial conditions are shown in figure 1. It is clear from the figure that the coupled system has synchronised fixed points, synchronised period-2 orbits and it becomes chaotic through the synchronised period doubling route. In other

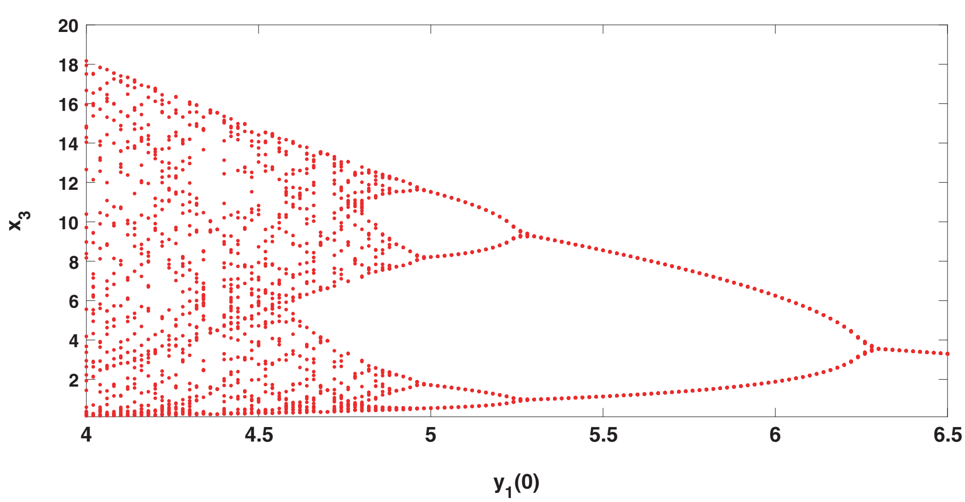

Figure 2. Bifurcation diagram of system (11) with respect to the parameter $y_1(0)$ for $a = b = 0.2$, $c = 9.0$, $c_2 = 0.40$ and $x_1(0) = 0$.

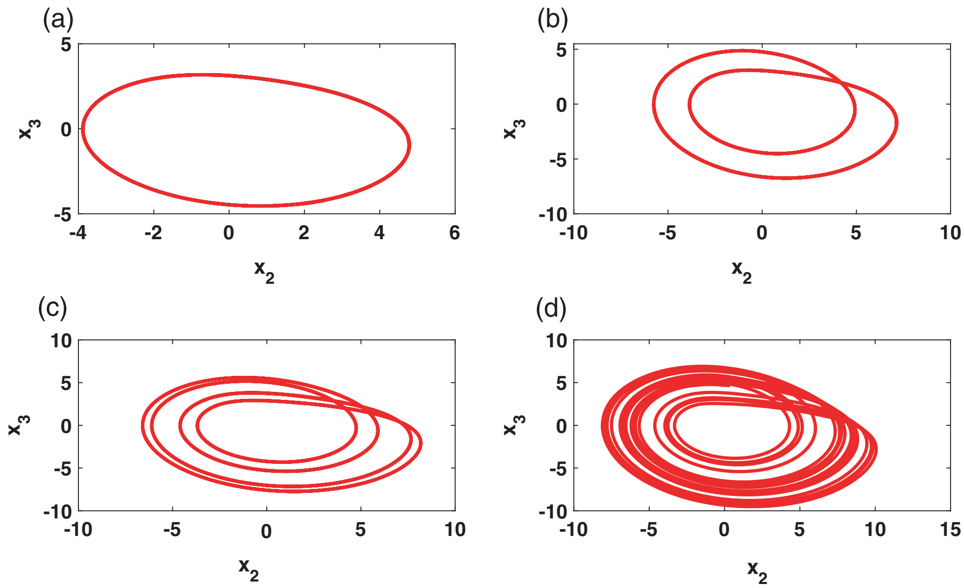

Figure 3. Phase diagrams of system (11) plotted in the yz plane for different values of (**a**) $c_1 = 6.5$ (limit cycle), (**b**) $c_1 = 5.5$ (period-2), (**c**) $c_1 = 5.0$ (period-4) and (**d**) $c_1 = 4.0$ (chaos) by fixing the parameters $a = b = 0.2$, $c = 9.0$ and $c_2 = 0.40$.

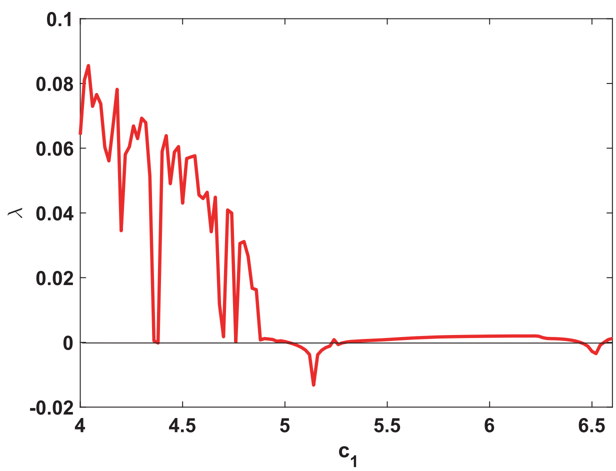

Figure 4. Variation of maximum Lyapunov exponent (λ) of system (11) with respect to $y_1(0)$ for $a = b = 0.2,\ c = 9.0$, $c_2 = 0.40$ and $x_1(0) = 0$.

words, the coupled one-dimensional chaotic maps have extreme multistable synchronisation behaviour.

5. Numerical results

The effectiveness of the proposed scheme is shown in this section. Numerical simulations are done by using MATLAB *R*2016*b*.

5.1 *Continuous system*

According to the theory discussed in §2.1, $c_1 = y_1(0) - x_1(0)$. Now, choosing $x_1(0) = 0$, we have $c_1 = y_1(0)$. Bifurcation diagram of system (11) with respect to initial condition $y_1(0)$ for $a = b = 0.2,\ c = 9.0$ and taking the parameter $c_2 = 0.40$ is shown in figure 2. Extreme multistable synchronisation is obvious from the figure. The phase diagrams of system (11) are plotted in the yz plane for different values of c_1 ($c_1 = 4.0,\ 5.0,\ 5.5$ and 6.5) in figure 3. It is clear from figure 3 that system (11) has qualitatively different dynamical behaviour with the variation of initial condition $y_1(0)$.

The variation of maximum Lyapunov exponent (λ) of the same system with respect to c_1 is shown in figure 4, which again established the extreme multistable synchronisation behaviour of the coupled systems. The existence of qualitatively different types of synchronised attractors as well as synchronised chaos in the coupled systems with the variation of initial conditions is demonstrated via numerical results.

5.2 *Discrete system*

In system (21), both c_1 and c_2 are initial condition-dependent parameters. According to the proposed scheme, $c_1 = x_2(0) - x_1(0)$ and $c_2 = y_2(0) - y_1(0)$. If we choose, $x_1(0) = 0$ and $y_1(0) = 0$, then $c_1 = x_2(0)$ and $c_2 = y_2(0)$, respectively. In figure 5, we have presented the bifurcation diagram with respect to $x_2(0)$ for fixed values of a, b and $y_2(0)$. Figures 5a–5d are drawn, respectively, for (a) $a = 0.1$, $b = 0.3$, $c_2 = 0$; (b) $a = 0.1$, $b = 0.3$, $c_2 = 0.6$; (c) $a = 0.9$, $b = 0.3$, $c_2 = 0.5$ and (d) $a = 1.2$, $b = 0.1$, $c_2 = 0.6$. The variation of the maximum Lyapunov exponent with respect to $x_2(0)$ is shown in figure 6. We can conclude that extreme

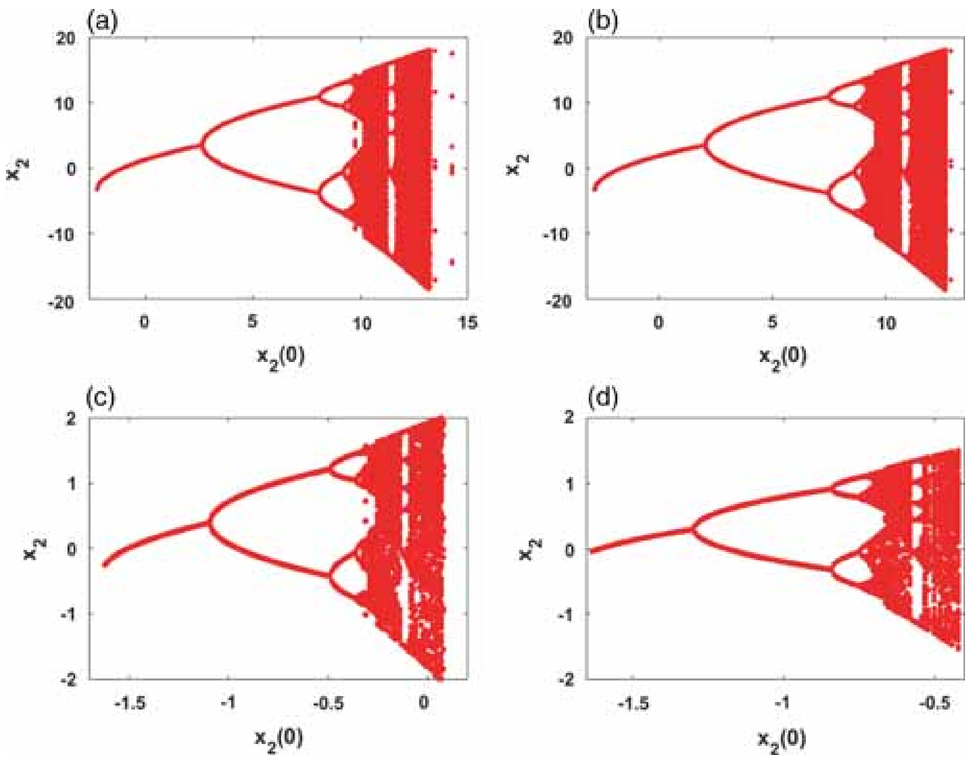

Figure 5. Bifurcation diagrams of x_2 of system (21) with respect to initial condition $x_2(0)$ (related to the control parameter c_1): (**a**) for $a = 0.1$, $b = 0.3$, $c_2 = 0$, $x_1(0) = 0$; (**b**) for $a = 0.1$, $b = 0.3$, $c_2 = 0.6$, $x_1(0) = 0$; (**c**) for $a = 0.9$, $b = 0.3$, $c_2 = 0.5$, $x_1(0) = 0$ and (**d**) for $a = 1.2$, $b = 0.1$, $c_2 = 0.6$, $x_1(0) = 0$.

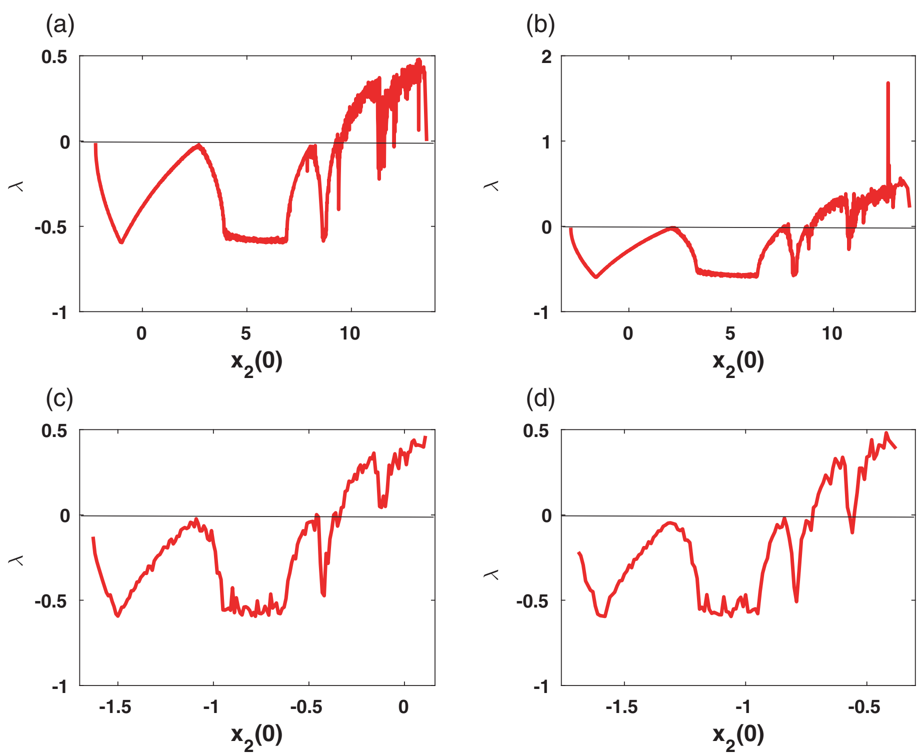

Figure 6. Variation of the maximum Lyapunov exponent (λ) of system (21) with respect to initial condition $x_2(0)$ (control parameter c_1): (**a**) $a = 0.1$, $b = 0.3$, $c_2 = 0$, $x_1(0) = 0$; (**b**) $a = 0.1$, $b = 0.3$, $c_2 = 0.6$, $x_1(0) = 0$; (**c**) $a = 0.9$, $b = 0.3$, $c_2 = 0.5$, $x_1(0) = 0$ and (**d**) $a = 1.2$, $b = 0.1$, $c_2 = 0.6$, $x_1(0) = 0$.

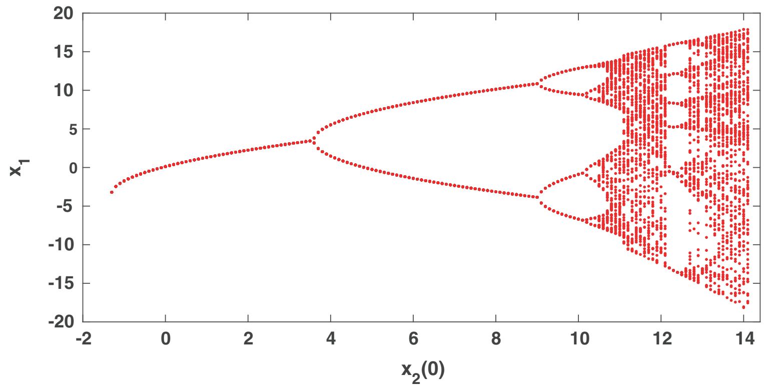

Figure 7. Bifurcation diagrams of x_1 of the coupled systems (17) and (19) with respect to initial condition $x_2(0)$ (related to the control parameter c_1) for $a = 0.1$, $b = 0.3$, $c_2 = 0.6$, $x_1(0) = 0$.

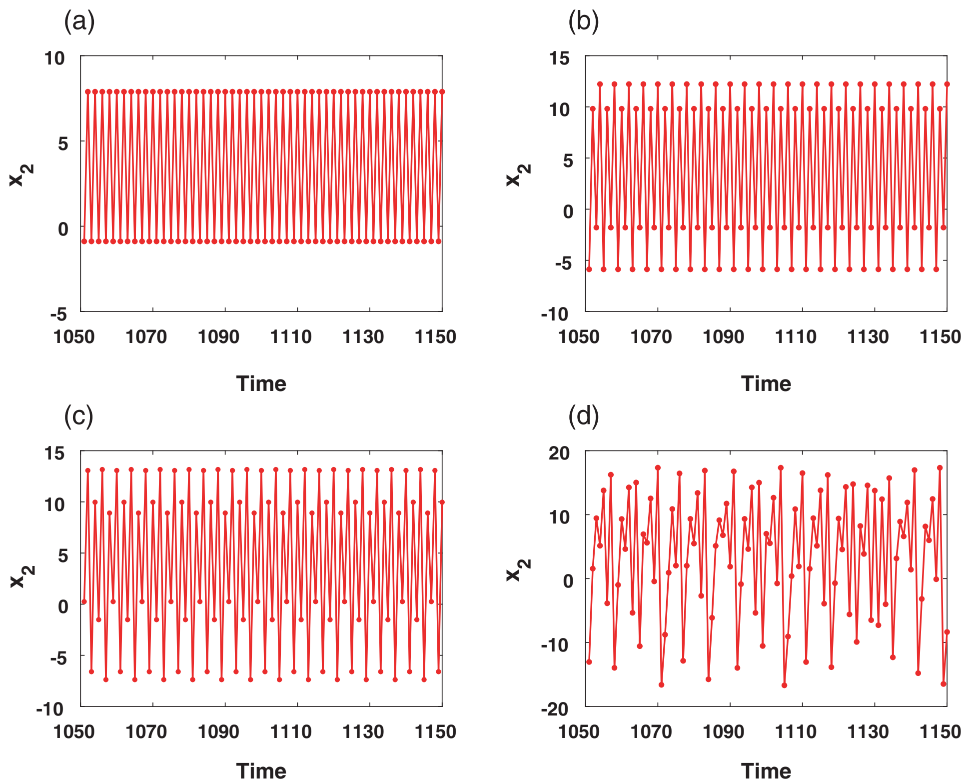

Figure 8. Time evolution of x_2 of (21) keeping $x_1(0) = 0$ fixed: (**a**) period-two orbit for $x_2(0) = 4$, (**b**) period-four orbit for $x_2(0) = 8$, (**c**) period-eight orbit for $x_2(0) = 8.8$ and (**d**) chaotic for $x_2(0) = 12$ ($a = 0.1$, $b = 0.3$, $c_2 = 0.6$).

multistability can be generated successfully via the proposed coupling scheme. Therefore, from our numerical results, it is clear that extreme multistable synchronisation can be generated via the proposed scheme.

We have plotted a bifurcation diagram with respect to initial condition $x_2(0)$ in figure 7 for coupled system eqs (17) and (19) together. The coupled systems clearly show the flip bifurcation with respect to initial conditions. Moreover, it also shows period doubling cascades.

It is important to note that for $a = 0.1, b = 0.3\ (<1)$ and $c_2 = 0.6$, the fixed point

$$P_1\left(\frac{-0.7+\sqrt{1.13+0.4c_1}}{0.2},\ \frac{-0.09+0.3\sqrt{1.13+0.4c_1}}{0.2}\right)$$

satisfies the stability condition $-2.825 < c_1 < 2.075$, and P_1 is asymptotically stable. Moreover for $c_1 = 2.075$,

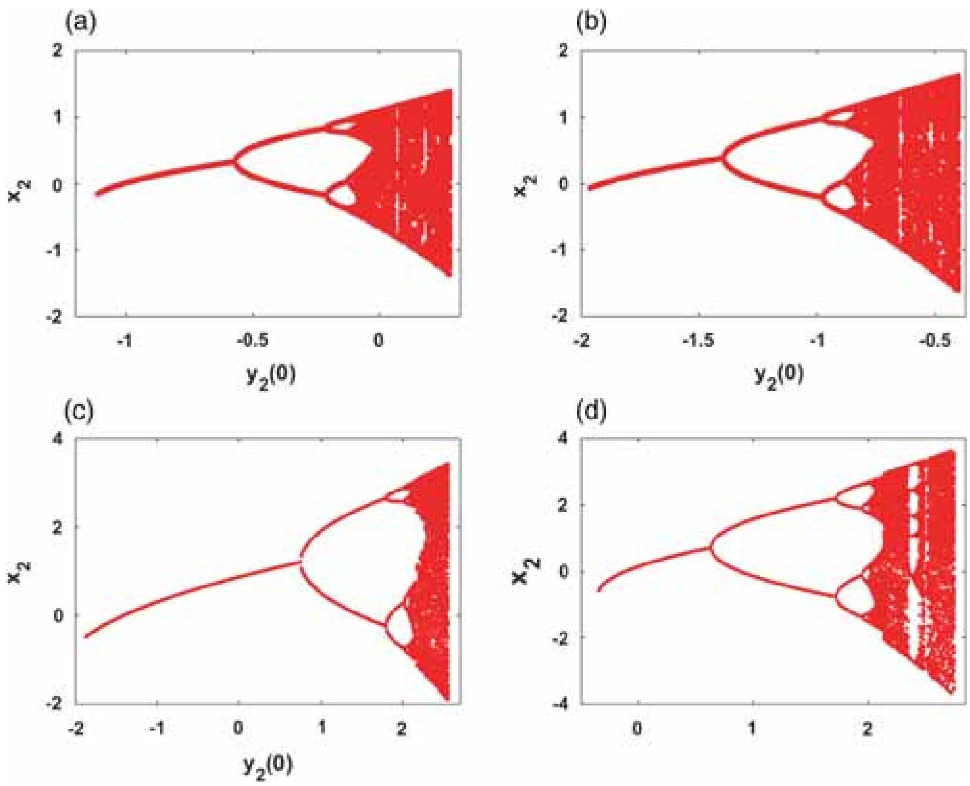

Figure 9. Bifurcation diagram of x_2 of (21) of with respect to initial condition $y_2(0)$ (control parameter c_2) for $y_1(0) = 0$: (**a**) $a = 1.4$, $b = 0.1$, $c_1 = 0$; (**b**) $a = 1.2$, $b = 0.1$, $c_1 = 0.9$; (**c**) $a = 0.5$, $b = -0.2$, $c_1 = 0.4$ and (**d**) $a = 0.5$, $b = 0.3$, $c_1 = -0.9$.

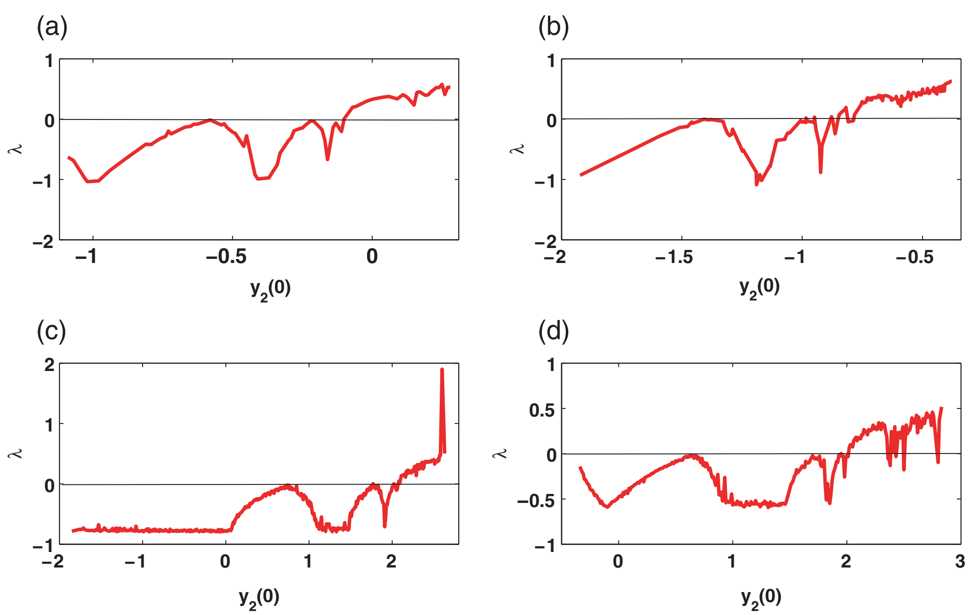

Figure 10. Variation of the maximum Lyapunov exponent (λ) of system (21) with respect to initial condition $y_2(0)$ (control parameter c_2): (**a**) $a = 1.4$, $b = 0.1$, $c_1 = 0$; (**b**) $a = 1.2$, $b = 0.1$, $c_1 = 0.9$; (**c**) $a = 0.5$, $b = -0.2$, $c_1 = 0.4$ and (**d**) $a = 0.5$, $b = 0.3$, $c_1 = -0.9$.

$\alpha_1 = -0.213$ ($\neq 0$) and $\alpha_2 = 0.01$ (>0) and the fixed point P_1 of system (21) undergoes supercritical flip bifurcation as shown in figure 5b. The variation of maximum Lyapunov exponent with respect to $x_2(0)$ is presented in figure 6b. These are in very good agreement with the theoretical predictions. It is observed that there is a cascade of period doubling, and in the period doubling route synchronised chaos appears in the system for $c_1 \geq 8.98$.

The time evolution diagram of x_2 in (21) is presented in figure 8. Existence of synchronised period-two orbit for $x_2(0) = 4$, synchronised period-four orbit for $x_2(0) = 8$, synchronised period-eight orbit for $x_2(0) = 8.8$ and synchronised chaotic orbit for $x_2(0) = 12$ ($a = 0.1$, $b = 0.3$, $c_2 = 0.6$) is observed.

Keeping c_1, a and b fixed, we have plotted bifurcation diagrams with respect to initial condition $y_2(0)$ in figure 9. The initial condition $y_2(0)$ is related to the control parameter c_2 by the relation $c_2 = y_2(0) - y_1(0)$. Setting the parameters (a) $a = 1.4$, $b = 0.1$, $c_1 = 0$; (b) $a = 1.2$, $b = 0.1$, $c_1 = 0.9$; (c) $a = 0.5$, $b = -0.2$, $c_1 = 0.4$; and (d) $a = 0.5$, $b = 0.3$, $c_1 = -0.9$ figure 9a–9d are drawn, respectively. The variation of the maximum Lyapunov exponent of system (21) with respect to $y_2(0)$ is plotted in figure 10. The existence of extreme multistability of the synchronised states is clearly observed from the figures.

6. Conclusion

We proposed a new scheme for designing extreme multistable synchronised systems coupled to two identical dynamical systems. The basic concept behind the proposed scheme is the existence of chaos in the coupled system in the presence of initial condition-dependent constants of motion. Coupling two continuous n ($n \geq 3$)-dimensional chaotic dynamical systems, we have the $2n$-dimensional phase space of the coupled system. Existence of n number of constants of motion in the coupled system reduces the phase-space dimension to n. Because chaos is possible in three- or higher-dimensional autonomous phase space, there is a possibility of obtaining extreme multistable synchronisation. Existence of a wide range of qualitatively different dynamical behaviour with the variation of initial condition-dependent constants of motion in the reduced dynamical system guarantees the existence of extreme multistable synchronisation in the coupled systems. Notice that extreme multistable synchronisation cannot be obtained in this scheme if the original continuous system has phase-space dimension less than or equal to two because chaos is impossible in two- or lower-dimensional autonomous continuous dynamical systems. On the other hand, as chaotic dynamics is possible in one-dimensional discrete dynamical system, extreme multistable synchronisation is possible even in coupled one-dimensional maps in the presence of one initial condition-dependent constant of motion.

In the proposed scheme, we first claim that extreme multistability in synchronised state is possible if all the state variables of the two systems synchronise in the generalised sense, i.e. the difference between the corresponding states of the coupled systems becomes an initial condition-dependent constant. According to Hens *et al* [17], the coexistence of infinitely many attractors in an m-dimensional coupled system will be possible if $m - 1$ of the variables of the two systems are completely synchronised and one of them obeys a constant difference between them. Latter on Pal *et al* [23] generalise it and proposed that the coexistence of infinitely many attractors in an m-dimensional coupled system will be possible if i of the variables of the two systems are completely synchronised and j of them obey a constant difference between them, where $i + j = m$ ($1 \leq i, j \leq m - 1$). In this paper, we introduce the most general precondition for emergence of extreme multistability as: an extreme multistable system can be designed if all m states of one system will synchronise in generalised sense with m states of the other system of the two coupled systems. Moreover, this paper also proposed a technique for designing extreme multistable synchronised systems which was not proposed earlier by any others. In the case of coupled Hénon maps, by using the centre manifold theorem and the bifurcation theory, it has been proved that flip bifurcation exists with the variation of initial conditions. In all cases numerical bifurcation diagrams and variation of largest Lyapunov exponents with respect to the initial condition-dependent parameters are presented to show the effectiveness of the proposed scheme.

This scheme may be useful to find some hidden properties of extreme multistable synchronised systems. This scheme may also be useful to design extreme multistable synchronised systems coupling two non-identical dynamical systems. In future, one can generalise the scheme for designing extreme multistability of synchronised states in coupled different dimensional dynamical systems. Extreme multistability observed in natural systems may be explained with the help of extreme multistable synchronisation behaviour of the network of dynamical systems. The proposed scheme introduces a straightforward way of constructing smooth dynamical systems with extreme multistable nature. Moreover, this work pointed out the control problem of extreme multistable synchronised systems to obtain the desired synchronised state in the case of biological, chemical or engineering systems. Human mind can be thought of as an extreme multistable system because mind can have infinite number of stable states and under small perturbation, mind can change from one stable state to another. Therefore, extreme multistable systems may be very useful for modelling human mind as a dynamical system.

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