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General electrodynamics of non-abelian vector bosons of $SU(2)$

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Abstract. Generalised Dirac–Maxwell equations (GDM) are extended to describe non-abelian vector bosons by forming $SU(2)$ multiplet. Noether's conserved current is investigated by forming suitable Lagrangian for the theory. General electrodynamics (GED) equations are obtained as Euler–Lagrange equations. Higgs mechanism leads to eigenvalue problem with masses of the bosons as eigenvalues. The sources of the fields have only improper conservation. Analogous to abelian vector bosons, non-abelian vector bosons also are seen to have nuclear structure with massive nucleus. There occur two types of $SU(2)$ sheets, each of three non-abelian vector bosons: one group contains one bradyon and two tachyon vector bosons, whereas the other group contains one tachyon and two bradyon vector bosons. Physical Z and W bosons are formed from the eigenvectors of $U(1)$ and $SU(2)$. The Z and W bosons do not have the same coupling strengths in $SU(2)$.

Keywords. Non-abelian bosodynamics; Noether's theorem; nuclear structure; eigenvalue problem; bradyon vector bosons; tachyon vector bosons.

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1. Introduction

Analogous to the developed general electrodynamics (GED) theory [1], we develop in this paper the GED theory for non-abelian vector bosons of $SU(2)$ along the lines of Noether's theorem [2,3]. In this section, we formulate the concept of $SU(2)$ multiplet, discuss its gauge transformation and form suitable Lagrangian density. In §2 Noether's theorem is investigated. It is found that non-abelian vector bosons also have the same Noether's current as for abelian vector bosons. They, however, have improper conservation. Section 3 is fully devoted to the determination of masses of the vector bosons through Higgs mechanism [4–6] which leads to a mass–eigenvalue problem. Two groups of $SU(2)$ are seen, each containing three vector bosons. One group contains one bradyon vector and two tachyon vectors, whereas the other group contains two bradyons and one tachyon. Physical Z and W bosons are obtained along the lines of the standard model [7,8]. They are seen to have different coupling strengths.

1.1 *SU(2) multiplet and gauge variation*

Starting with the working vector bosons $W_{i\mu}$ (which need not be physical), we describe the $SU(2)$ isovector as the multiplet Z_μ defined as

$$Z_\mu = \tau_i W_{i\mu} = \boldsymbol{\tau} \cdot \mathbf{W}_\mu, \quad i = 1, 2, 3, \tag{1.1}$$

where τ_i are the Pauli matrices and are generators of $SU(2)$.

The vectors transform under $SU(2)$ as [9]

$$Z'_\mu = U Z_\mu U^{-1} - \mathrm{i}\big(\partial_\mu U\big)U^{-1}, \tag{1.2}$$

where U is the $SU(2)$ unitary transformation operator given by

$$U = \exp\left(-\frac{2\mathrm{i}}{g_2}\boldsymbol{\tau} \cdot \boldsymbol{\theta}\right) \tag{1.3}$$

and $\boldsymbol{\theta}$ is the gauge vector. The factor $2/g_2$ is for convenience. Here, g_2 is the $SU(2)$ charge of the Z_μ bosons and it is the same for all $W_{i\mu}$.

For infinitesimal transformations, we get

$$Z'_\mu = Z_\mu - \frac{2}{g_2}\big(\partial_\mu \boldsymbol{\tau} \cdot \boldsymbol{\theta}\big) - \frac{2\mathrm{i}}{g_2}\big(\boldsymbol{\tau} \cdot \boldsymbol{\theta} Z_\mu - Z_\mu \boldsymbol{\tau} \cdot \boldsymbol{\theta}\big). \tag{1.4}$$

Using the properties of Pauli matrices, this gives component-wise transformations as

$$W'_{i\mu} = W_{i\mu} - \frac{2}{g_2}\left(\partial_\mu \theta_i\right) + \frac{2}{g_2}\left(\theta_j W_{k\mu} - \theta_k W_{j\mu}\right). \tag{1.5}$$

The gauge variation of $W_{i\mu}$ is given by

$$\delta W_{i\mu} = W'_{i\mu} - W_{i\mu} = -\frac{2}{g_2}\left(\partial_\mu \theta_i\right) + \frac{2}{g_2}\left(\theta_j W_{k\mu} - \theta_k W_{j\mu}\right). \tag{1.6}$$

The first term on the right-hand side of this equation is parallel to $W_{i\mu}$, whereas the second term is orthogonal to it. Furthermore,

$$Z^2 = Z^\mu Z_\mu = w^2 = w_i^2 + w_j^2 + w_k^2. \tag{1.7}$$

Its gauge variation gives

$$\delta Z^2 = 2Z^\mu \delta Z_\mu = 2W_i^\mu \delta W_{i\mu} = -\frac{4}{g_2}\left(W_i^\mu \partial_\mu \theta_i\right). \tag{1.8}$$

Covariant derivative is defined as

$$D_\lambda = \partial_\lambda - \frac{1}{2}\mathrm{i}g_2 Z_\lambda. \tag{1.9}$$

The formula of field tensor for abelian vector bosons [1] becomes a formula for non-abelian vector bosons as

$$F^{\mu\nu} = d^{\mu\nu\lambda\rho} D_\lambda Z_\rho = G^{\mu\nu} - \frac{1}{2}\mathrm{i}g_2 Z^{\mu\nu}, \tag{1.10}$$

where

$$Z^{\mu\nu} = d^{\mu\nu\lambda\rho} Z_\lambda Z_\rho = Z^2 g^{\mu\nu} - Z^\mu Z^\nu + Z^\nu Z^\mu - \mathrm{i}\varepsilon^{\mu\nu\lambda\rho} Z_\lambda Z_\rho, \tag{1.10a}$$

$$d^{\mu\nu\lambda\rho} = g^{\mu\nu} g^{\lambda\rho} - g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\nu\lambda} - \mathrm{i}\varepsilon^{\mu\nu\lambda\rho}. \tag{1.10b}$$

The tensor $Z^{\mu\nu}$ simplifies to

$$Z^{\mu\nu} = w^2 g^{\mu\nu} - 2\mathrm{i}\varepsilon_{ijk}\tau_k [W_i^\mu W_j^\nu + \mathrm{i}\varepsilon^{\mu\nu\lambda\rho} W_{i\lambda} W_{j\rho}], \tag{1.10c}$$

where $\varepsilon_{123} = 1$.

1.2 *The Lagrangian density*

The Lagrangian density can be written in two ways with coupling charges g_2 and g'_2 as

$$\begin{aligned}\Im(Z_\mu, D_\nu Z_\mu) &= \frac{1}{16}\left(\mathbf{G}^{\mu\nu} - \frac{1}{2}\mathrm{i}g_2 Z^{\mu\nu}\right)\cdot\left(\mathbf{G}_{\mu\nu} + \frac{1}{2}\mathrm{i}g_2 Z_{\mu\nu}\right) \\ &+ \frac{1}{16}\left(\mathbf{G}^{\mu\nu} + \frac{1}{2}\mathrm{i}g_2 Z^{\mu\nu}\right)\cdot\left(\mathbf{G}_{\mu\nu} - \frac{1}{2}\mathrm{i}g_2 Z_{\mu\nu}\right) \\ &- \frac{1}{2}m_2^2 Z^\mu Z_\mu + Z^\mu \Sigma_\mu + \text{c.c.}\end{aligned} \tag{1.11a}$$

and

$$\begin{aligned}\Im'(Z_\mu, D_\nu Z_\mu) &= \frac{1}{16}\left(\mathbf{G}^{\mu\nu} + \frac{1}{2}\mathrm{i}g'_2 Z^{\mu\nu}\right)\cdot\left(\mathbf{G}_{\mu\nu} + \frac{1}{2}\mathrm{i}g'_2 Z_{\mu\nu}\right) \\ &+ \frac{1}{16}\left(\mathbf{G}^{\mu\nu} - \frac{1}{2}\mathrm{i}g'_2 Z^{\mu\nu}\right)\cdot\left(\mathbf{G}_{\mu\nu} - \frac{1}{2}\mathrm{i}g'_2 Z_{\mu\nu}\right) \\ &- \frac{1}{2}m_2'^2 Z^\mu Z_\mu + Z^\mu \Sigma_\mu + \text{c.c.},\end{aligned} \tag{1.11b}$$

where $\Sigma_\mu = \boldsymbol{\tau}\cdot\mathbf{J}_\mu$ is the multiplet source of $SU(2)$ and m^2 is the constant mass parameter. The Lagrangian has reflection symmetry under $Z_\mu \to -Z_\mu$ and $\Sigma_\mu \to -\Sigma_\mu$.

The Lagrangian densities simplify to

$$\Im = \frac{1}{8}\left(\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} + \frac{1}{4}g_2^2 Z^{\mu\nu} Z_{\mu\nu}\right) - \frac{1}{2}m_2^2 w^2 + Z^\mu \Sigma_\mu \tag{1.12a}$$

and

$$\Im' = \frac{1}{8}\left(\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{4}g_2'^2 Z^{\mu\nu} Z_{\mu\nu}\right) - \frac{1}{2}m_2'^2 w^2 + Z^\mu \Sigma_\mu. \tag{1.12b}$$

The features of the non-abelian vector bosons are contained in the term $Z^{\mu\nu}Z_{\mu\nu}$, which simplifies to

$$Z^{\mu\nu} Z_{\mu\nu} = 4\left(Z^\mu Z^\nu Z_\mu Z_\nu + \mathrm{i}\varepsilon^{\mu\nu\lambda\rho} Z_\mu Z_\nu Z_\lambda Z_\rho\right). \tag{1.13a}$$

The term $\varepsilon^{\mu\nu\lambda\rho} Z_\mu Z_\nu Z_\lambda Z_\rho$ vanishes for $SU((2)$ because there are only three non-abelian vector bosons, and so in this term, there occurs at least a pair of identical vector bosons. We then have

$$Z^{\mu\nu} Z_{\mu\nu} = 4 Z^\mu Z^\nu Z_\mu Z_\nu. \tag{1.13b}$$

To evaluate this further, we use the following identities:

$$Z^\mu Z^\nu = \left(W_i^\mu W_i^\nu + \mathrm{i}\varepsilon_{klm} W_k^\mu W_l^\nu \tau_m\right) \tag{1.14a}$$

and

$$Z^\mu Z^\nu + Z^\nu Z^\mu = 2 W_i^\mu W_i^\nu. \tag{1.14b}$$

We then obtain for the term

$$Z^\mu Z^\nu Z_\mu Z_\nu = 2\left(W_i^\mu W_i^\nu\right)\left(W_{j\mu} W_{j\nu}\right) - w^4 + 2\mathrm{i}\left(W_i^\mu W_i^\nu\right)\varepsilon_{klm} W_{k\mu} W_{l\nu} \tau_m. \tag{1.15}$$

However, the last term that involves the factor $\varepsilon_{klm} W_{k\mu} W_{l\nu} \tau_m$ is symmetric in the indices μ and ν, whereas it is antisymmetric in the indices k, l, m and so it vanishes.

Now, as the three W bosons are linearly independent, they can form orthonormal triads:

$$W_i^\alpha W_{j\alpha} = W_i^\alpha W_{i\alpha}\delta_{ij} = w_i^2\delta_{ij} \text{(the index } i \text{ is not summed).} \tag{1.16a}$$

Hence,

$$\mathbf{W}^\alpha \cdot \mathbf{W}_\alpha = w_i^2 + w_j^2 + w_k^2 = w^2 \tag{1.16b}$$

and

$$\left(W_i^\mu W_i^\nu\right)\left(W_{j\mu}W_{j\nu}\right) = \left(W_i^\mu W_{j\mu}\right)\left(W_i^\nu W_{j\nu}\right) = w_i^2\delta_{ij}w_i^2\delta_{ij} = w_i^4 + w_j^4 + w_k^4. \tag{1.16c}$$

Finally, we get the important result

$$Z^\mu Z^\nu Z_\mu Z_\nu = 2(w_i^4 + w_j^4 + w_k^4) - w^4. \tag{1.17}$$

The Lagrangian densities now reduce to

$$\Im = \frac{1}{8}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{2}m_2^2 w^2 - \frac{1}{8}g_2^2[w^4 - 2(w_i^4 + w_j^4 + w_k^4)] + \mathbf{W}^\mu\cdot\mathbf{J}_\mu \tag{1.18a}$$

and

$$\Im = \frac{1}{8}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{2}m_2'^2 w^2 + \frac{1}{8}g_2'^2[w^4 - 2(w_i^4 + w_j^4 + w_k^4)] + \mathbf{W}^\mu\cdot\mathbf{J}_\mu. \tag{1.18b}$$

2. Noether's theorem

With this background, we now frame Noether's theorem. The gauge variation of the Lagrangian density eq. (1.18a) gives

$$\delta\Im = \frac{1}{4}\mathbf{G}^{\mu\nu}\cdot\delta\mathbf{G}_{\mu\nu} - \big[M_2^2\mathbf{W}^\mu\cdot\delta\mathbf{W}_\mu - g_2^2\big(w_i^2 W_i^\mu\delta W_{i\mu} + w_j^2 W_j^\mu\delta W_{j\mu} + w_k^2 W_k^\mu\cdot\delta W_{k\mu}\big)\big] + \big(\mathbf{J}^\mu\cdot\delta\mathbf{W}_\mu\big), \tag{2.1a}$$

where M_2^2 is the mass function of the $SU(2)$ multiplet given by

$$M_2^2 = \left(m_2^2 + \frac{1}{2}g_2^2 w^2\right) \tag{2.2b}$$

and

$$\mathbf{J}^\mu\cdot\delta\mathbf{W}_\mu = -\frac{2}{g_2}\mathbf{J}^\mu\cdot\partial_\mu\boldsymbol{\theta}. \tag{2.2c}$$

This gives the following Euler–Lagrange equations for W_i^μ as equations of GED:

$$\partial_\lambda G_i^{\rho\lambda} + K_i^2 W_i^\rho = J_i^\rho. \tag{2.2d}$$

The gauge variation of $\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu}$ is given by

$$\delta\big(\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu}\big) = 2\mathbf{G}^{\mu\nu}\cdot\delta\mathbf{G}_{\mu\nu} = 2\mathbf{f}_0 g^{\mu\nu}\cdot\delta\mathbf{f}_0 g_{\mu\nu} \tag{2.2e}$$

$$= 8\mathbf{f}_0\cdot\delta\mathbf{f}_0 = -\frac{16}{g_2}\mathbf{f}_0\cdot\partial^2\boldsymbol{\theta}.$$

Defining the mass functions for the component vector bosons W_i^μ as

$$K_i^2 = M_2^2 - g_2^2 w_i^2, \quad K_j^2 = M_2^2 - g_2^2 w_j^2, \quad K_k^2 = M_2^2 - g_2^2 w_k^2, \tag{2.3}$$

eq. (2.1a) becomes

$$\delta\Im = -\frac{2}{g_2}\mathbf{f}_0\cdot\partial^2\boldsymbol{\theta} + \frac{2}{g_1}(K_i^2 W_i^\mu\partial_\mu\theta_i + K_j^2 W_j^\mu\partial_\mu\theta_j + K_k^2 W_k^\mu\cdot\partial_\mu\theta_k) - \frac{2}{g_1}\mathbf{J}^\mu\cdot\partial_\mu\boldsymbol{\theta}. \tag{2.4}$$

We write the terms in the bracket as

$$K_i^2 W_i^\mu\partial_\mu\theta_i = \partial_\mu\big(K_i^2 W_i^\mu\theta_i\big) - \theta_i K_i^2\partial_\mu W_i^\mu - \theta_i W_i^\mu\partial_\mu K_i^2 = \partial_\mu\big(K_i^2 W_i^\mu\theta_i\big) - K_i^2 f_{0i}\theta_i - \theta_i W_i^\mu\partial_\mu K_i^2 \tag{2.5}$$

and

$$\mathbf{J}^\mu\cdot\partial_\mu\boldsymbol{\theta} = \partial_\mu\big(\mathbf{J}^\mu\cdot\boldsymbol{\theta}\big) - \boldsymbol{\theta}\cdot\partial_\mu\mathbf{J}^\mu. \tag{2.6}$$

Then, eq. (2.4) converts to

$$\delta\Im = \frac{2}{g_2}\partial_\mu[(K_i^2 W_i^\mu - J_i^\mu)\theta_i + (K_j^2 W_j^\mu - J_j^\mu)\theta_j + (K_k^2 W_k^\mu - J_k^\mu)\theta_k] - \frac{2}{g_2}[f_{0i}(\partial^2 + K_i^2)\theta_i + f_{0j}(\partial^2 + K_j^2)\theta_j + f_{0k}\big(\partial^2 + K_k^2\big)\theta_k] + \frac{2}{g_2}\big[\theta_i\big(\partial_\mu J_i^\mu - W_i^\mu\partial_\mu K_i^2\big) + \theta_j\big(\partial_\mu J_j^\mu - W_j^\mu\partial_\mu K_j^2\big) + \theta_k\big(\partial_\mu J_k^\mu - W_k^\mu\partial_\mu K_k^2\big)\big]. \tag{2.7}$$

This contains various information. Noether's theorem requires that the gauge variation of the Lagrangian should be a perfect divergence of some current. This demands that the second and the third bracket terms on the right-hand side of this equation should vanish. As f_{0i} and θ_i are arbitrary, the vanishing of these terms gives for each i, j and k:

$$(\partial^2 + K_i^2)\theta_i = 0 \tag{2.8a}$$

and

$$\partial_\mu J_i^\mu - W_i^\mu \partial_\mu K_i^2 = 0. \tag{2.8b}$$

Equation (2.8a) is the homogeneous wave equation satisfied by the gauges θ_i, whereas eq. (2.8b) is concerned with the conservation law. To investigate this, we consider the divergence of the Euler–Lagrange equations for W_i^μ, as given by eq. (2.2d). It gives

$$\left(\partial^2 + K_i^2\right) f_{0i} = \partial_\mu J_i^\mu - W_i^\mu \partial_\mu K_i^2. \tag{2.9a}$$

However, the source J_i of eq. (2.2e) contains Maxwell's divergenceless current j_{0i}:

$$\partial_\lambda f_i^{\rho\lambda} = j_{0i}^\rho. \tag{2.9b}$$

Writing the total current source as the mixture of conserved and non-conserved currents as

$$J_i^\mu = j_{0i}^\mu + S_i^\mu, \tag{2.10}$$

eqs (2.8b) and (2.9a), respectively, become

$$\partial_\mu S_i^\mu - W_i^\mu \partial_\mu K_i^2 = 0 \tag{2.11a}$$

and

$$\left(\partial^2 + K_i^2\right) f_{0i} = 0. \tag{2.11b}$$

Thus, according to eqs (2.8a) and (2.11b), the scalar bosons f_{0i} and the gauges θ_i obey the same wave equation with the same mass function.

Equations (2.8a), (2.8b) and (2.11b) reduce the gauge variation of the Lagrangian density to

$$\delta\Im = -\frac{2}{g_2}\partial_\rho[\theta_i \partial_\lambda G_i^{\rho\lambda}], \quad \text{index } i \text{ is summed.} \tag{2.12a}$$

Exchanging the dummy indices, we write this as

$$\delta\Im = -\frac{2}{g_2}\partial_\lambda[\theta_i \partial_\rho G_i^{\rho\lambda}], \quad \text{index } i \text{ is summed.} \tag{2.12b}$$

Now, the Lagrangian density equation (1.18a) gives the current N_i^λ as

$$N_i^\lambda = \frac{\partial\Im}{\partial\left(\partial_\lambda W_i^\rho\right)} = -\frac{2}{g_2} G_i^{\rho\lambda}\partial_\rho\theta_i. \tag{2.13}$$

Noether's theorem is stated as [1]

$$\delta\Im = \partial_\lambda N_i^\lambda = -\frac{2}{g_2}\partial_\lambda[\theta_i \partial_\rho G_i^{\lambda\rho}], \quad \text{index } i \text{ is summed.} \tag{2.14}$$

Substituting for $\delta\Im$ eq. (2.13) into eq. (2.14), we get

$$-\frac{2}{g_2}\partial_\lambda[G_i^{\rho\lambda}\partial_\rho\theta_i + G_j^{\rho\lambda}\partial_\rho\theta_j + G_k^{\rho\lambda}\partial_\rho\theta_k]$$
$$= -\frac{2}{g_2}\partial_\lambda[\partial_\rho G_i^{\lambda\rho}\theta_i + \partial_\rho G_i^{\lambda\rho}\theta_j + \partial_\rho G_i^{\lambda\rho}\theta_k].$$

This gives the conservation law as

$$\partial_\lambda \Pi^\lambda = 0, \tag{2.15a}$$

where Π^λ is Noether's conserved current given by

$$\begin{aligned}\Pi^\lambda = -\frac{2}{g_2}[&(G_i^{\rho\lambda}\partial_\rho\theta_i - \partial_\rho G_i^{\lambda\rho}\theta_i) \\ &+(G_j^{\rho\lambda}\partial_\rho\theta_j - \partial_\rho G_j^{\lambda\rho}\theta_j) \\ &+(G_k^{\rho\lambda}\partial_\rho\theta_k - \partial_\rho G_k^{\lambda\rho}\theta_k)].\end{aligned} \tag{2.15b}$$

This total current is the sum of the three partial currents Π_i^λ given by

$$\Pi_i^\lambda = -\frac{2}{g_2}(G_i^{\rho\lambda}\partial_\rho\theta_i - \partial_\rho G_i^{\lambda\rho}\theta_i). \tag{2.15c}$$

Its expansion gives

$$\begin{aligned}\Pi_i^\lambda = &-\frac{2}{g_2}\left(f_{0i}\partial^\lambda\theta_i - \partial^\lambda f_{0i}\theta_i\right) \\ &-\frac{2}{g_2}(f_i^{\rho\lambda}\partial_\rho\theta_i - \partial_\rho f_i^{\lambda\rho}\theta_i).\end{aligned} \tag{2.16a}$$

The antisymmetric nature of $f_i^{\lambda\rho}$ reduces this further to

$$\begin{aligned}\Pi_i^\lambda = &-\frac{2}{g_2}\left(f_{0i}\partial^\lambda\theta_i - \partial^\lambda f_{0i}\theta_i\right) \\ &-\frac{2}{g_2}\partial_\rho\left(f_i^{\rho\lambda}\theta_i\right).\end{aligned} \tag{2.16b}$$

These component currents of non-abelian vector bosons are identical with Noether's current for abelian vector bosons.

The second term on the right-hand side of eq. (2.16b) is divergenceless, and so Noether's conservation law (eq. (2.15a)) reduces to

$$\partial_\lambda \Pi_i^\lambda = -\frac{2}{g_2}\partial_\lambda\left(f_{0i}\partial^\lambda\theta_i - \partial^\lambda f_{0i}\theta_i\right) = 0. \tag{2.16c}$$

2.1 *Noether's improper conservation theorem and GED*

The conservation equation (2.8b) indicates that sources for non-abelian vector bosons are not conserved because the mass functions K_i^2 are not constant. It, however, looks like the vanishing of covariant divergence. To see this, we select the sources S_i^μ as non-conserved currents defined as

$$S_i^\mu = -\frac{2\mathrm{i}}{g_2}\partial_\mu K_i^2. \tag{2.17a}$$

Then eq. (2.8b) becomes the vanishing of the covariant divergence of the mass current $\partial_\mu K_i^2$:

$$-\frac{2\mathrm{i}}{g_2} D_\mu \partial^\mu K_i^2 = 0. \tag{2.17b}$$

This is Noether's improper conservation law [3].

Equations (2.9a), (2.10) and (2.17a) give the GED equations for non-abelian vector bosons in two forms as

$$\partial^\rho f_{0i} + K_i^2 W_i^\rho = -\frac{2\mathrm{i}}{g_2}\partial^\rho K_i^2 \tag{2.18a}$$

and

$$\partial_\lambda G_i^{\rho\lambda} + K_i^2 W_i^\rho = j_{0i}^\rho - \frac{2\mathrm{i}}{g_2}\partial^\rho K_i^2 \tag{2.18b}$$

or

$$\partial^2 W_i^\rho + K_i^2 W_i^\rho = j_{0i}^\rho - \frac{2\mathrm{i}}{g_2}\partial^\rho K_i^2. \tag{2.18c}$$

These are, respectively, equivalent to

$$\partial^\rho f_{0i} + \frac{2\mathrm{i}}{g_2} D^\rho K_i^2 = 0 \tag{2.19a}$$

and

$$\partial_\lambda G_i^{\rho\lambda} + \frac{2\mathrm{i}}{g_2} D^\rho K_i^2 = j_{0i}^\rho. \tag{2.19b}$$

2.2 *Nuclear structure of non-abelian vector bosons*

Equation (2.19b) is the totality of eqs (2.9b) and (2.19a) and so these equations provide nuclear structure to massive spin-1 non-abelian vector bosons and collectively describe them in the form of eq. (2.19b). Equation (2.19a) describes the massive nucleus f_{0i}, whereas eq. (2.9b) describes the massless photon cloud $f_i^{\mu\nu}$ surrounding the nucleus. This structure is similar to the nuclear structure of abelian vector bosons.

3. Higgs mechanism and boson masses

To determine the masses of the vector bosons we use the GED equation (2.18c). These are written as

$$\partial^2 W_i^\rho + K_i^2 W_i^\rho + \frac{2\mathrm{i}}{g_2}\partial^\rho K_i^2 = j_{0i}^\rho, \tag{3.1a}$$

where K_i^2 in full form is given by

$$K_i^2 = m_2^2 + \frac{1}{2} g_2^2 (w_j^2 + w_k^2 - w_i^2). \tag{3.1b}$$

Let $w_i = v_i =$ const. be the non-zero vacuum expectation values. Then eq. (3.1a) in vacuum becomes

$$\left[m_2^2 + \frac{1}{2} g_2^2 (v_j^2 + v_k^2 - v_i^2)\right] v_i = 0. \tag{3.2a}$$

Making cyclic rotations of the indices i, j, k, we get equations for the additional two bosons as

$$\left[m_2^2 + \frac{1}{2} g_2^2 (v_k^2 + v_i^2 - v_j^2)\right] v_j = 0 \tag{3.2b}$$

and

$$\left[m_2^2 + \frac{1}{2} g_2^2 (v_i^2 + v_j^2 - v_k^2)\right] v_k = 0. \tag{3.2c}$$

There are two types of solutions for vacuum. For $m^2 > 0$, the vacuum has zero expectation values, i.e., $v_{i,j,k} = 0$. For $m^2 < 0$, on the other hand, these equations give non-zero vacuum expectation values as

$$m_2^2 + \frac{1}{2} g_2^2 (v_j^2 + v_k^2 - v_i^2) = 0, \tag{3.3a}$$

$$m_2^2 + \frac{1}{2} g_2^2 (v_k^2 + v_i^2 - v_j^2) = 0, \tag{3.3b}$$

$$m_2^2 + \frac{1}{2} g_2^2 (v_i^2 + v_j^2 - v_k^2) = 0. \tag{3.3c}$$

These are simultaneous equations. Adding any two of these equations, we get

$$v_i^2 = v_j^2 = v_k^2 = -\frac{2m_2^2}{g_2^2} = b^2. \tag{3.4}$$

All the W bosons have the same vacuum expectation value.

Addition of all the three equations gives

$$3m_2^2 + \frac{1}{2} g_2^2 v^2 = 0, \tag{3.5a}$$

where

$$v^2 = v_i^2 + v_j^2 + v_k^2. \tag{3.5b}$$

The vacuum expectation value is given by

$$v^2 = -\frac{6m_2^2}{g_2^2} = 3b^2 = r^2. \tag{3.5c}$$

We take $v_i^2 > 0$ and so $m_2^2 < 0$. The locus of the vacuum is a sphere of radius $v = \sqrt{-(6m_2^2/g_2^2)} = r$. However, there are only eight points on the sphere at which there occur vacuum. These vacuum points are

$$r(1,1,1),\ r(1,1,-1),\ r(1,-1,1),\ r(-1,1,1),$$
$$r(1,-1,-1),\ r(-1,-1,1),\ r(-1,1,-1)$$

and

$$r(-1,-1,-1). \tag{3.6}$$

These are connected to each other by reflection transformations $v_i \to -v_i$, $v_j \to -v_j$, $v_k \to -v_k$.

3.1 *Spontaneous symmetry breaking and mass eigenvalue problem*

For determining the masses of the vector bosons, we select the point $r(1,1,1)$ on the locus of the vacuum so that $v_i = v_j = v_k = b$ and neglect all the other points.

For the sake of simplifying the calculations, we consider the boson vector W_i^α as real and so expand it around the selected point as

$$W_i^\alpha = v_i^\alpha + h_i^\alpha. \tag{3.7}$$

Here, v_i^α is at the point $a(1,1,1)$ and h_i^α is the Higgs vector boson. This choice just neglects the Goldstone-like massless vector bosons [10]. Now the reflection symmetry of the Lagrangian density is spontaneously broken. From eq. (3.7), we get

$$\begin{aligned} w_i^2 &= v_i^2 + 2v_i h_i \cos\theta_i + h_i^2 \\ &= b^2 + 2bh_i \cos\theta_i + h_i^2 \end{aligned} \tag{3.8a}$$

and

$$\begin{aligned} w^2 = v^2 + 2v_i\big(&h_i \cos\theta_i + h_j \cos\theta_j \\ &+h_k \cos\theta_k) + h^2 \end{aligned} \tag{3.8b}$$

or

$$\begin{aligned} w^2 = 3b^2 + 2b\big(&h_i \cos\theta_i + h_j \cos\theta_j \\ &+h_k \cos\theta_k) + h^2, \end{aligned} \tag{3.8c}$$

where we have defined

$$v_{i\alpha} h_i^\alpha = v_i h_i \cos\theta_i, \tag{3.9}$$

where θ_i is the angle between the vacuum v_i^α and the Higgs vector boson h_i^α. Further, we obtain

$$\begin{aligned} w^2 - 2w_i^2 = b^2 + 2b\big(&h_j \cos\theta_j + h_k \cos\theta_k \\ &-h_i \cos\theta_i) + (h_j^2 + h_k^2 - h_i^2). \end{aligned} \tag{3.10}$$

The GED equation (3.1a) now transforms to

$$\begin{aligned} \partial^2 h_i^\rho + \Big[&m_2^2 + \frac{1}{2} g_2^2 b^2 + g_2^2 b(h_j \cos\theta_j \\ &+h_k \cos\theta_k - h_i \cos\theta_i) \\ &+\frac{1}{2} g_2^2 (h_j^2 + h_k^2 - h_i^2)\Big]\big(v_i^\rho + h_i^\rho\big) = 0. \end{aligned}$$

Using the vacuum values, eq. (3.4), this simplifies to

$$\begin{aligned} \partial^2 h_i^\rho + \big[&g_2^2 b\big(h_j \cos\theta_j + h_k \cos\theta_k - h_i \cos\theta_i\big) \\ &+\frac{1}{2} g_2^2 (h_j^2 + h_k^2 - h_i^2)\big]\big(v_i^\rho + h_i^\rho\big) = 0. \end{aligned} \tag{3.11a}$$

For further evaluation, we organise this equation as

$$\begin{aligned} \partial^2 h_i^\rho &+ g_2^2 b\big(h_j \cos\theta_j + h_k \cos\theta_k - h_i \cos\theta_i\big) v_i^\rho \\ &+g_2^2 b\big(h_j \cos\theta_j + h_k \cos\theta_k - h_i \cos\theta_i\big) h_i^\rho \\ &+\frac{1}{2} g_2^2 (h_j^2 + h_k^2 - h_i^2)\big(v_i^\rho + h_i^\rho\big) = 0. \end{aligned} \tag{3.11b}$$

Neglecting the higher-order interaction terms, which vanish around the vacuum point, we get the linear equation

$$\begin{aligned} \partial^2 h_i^\rho + g_2^2 b\big(&h_j \cos\theta_j + h_k \cos\theta_k \\ &-h_i \cos\theta_i) v_i^\rho = 0. \end{aligned} \tag{3.12}$$

Taking scalar product with $v_{i\rho}$, we get by noting that $v_i^\rho v_{i\rho} = b^2$ and $h_i^\rho v_{i\rho} = h_i b \cos\theta_i$

$$\begin{aligned} \partial^2 h_i \cos\theta_i + g_2^2 b^2 (&h_j \cos\theta_j + h_k \cos\theta_k \\ &-h_i \cos\theta_i) = 0. \end{aligned} \tag{3.13a}$$

Making cyclic rotations of the indices, we get two more equations

$$\begin{aligned} \partial^2 h_j \cos\theta_j + g_2^2 b^2 (&h_k \cos\theta_k + h_i \cos\theta_i \\ &-h_j \cos\theta_j) = 0 \end{aligned} \tag{3.13b}$$

and

$$\begin{aligned} \partial^2 h_k \cos\theta_k + g_2^2 b^2 (&h_i \cos\theta_i + h_j \cos\theta_j \\ &-h_k \cos\theta_k) = 0. \end{aligned} \tag{3.13c}$$

To compose these three simultaneous equations together, we define a three-vector (triad) as

$$\begin{aligned} h^{\mathrm{T}} &= (h_1, h_2, h_3) \\ &= \big(h_i \cos\theta_i, h_j \cos\theta_j, h_k \cos\theta_k\big). \end{aligned} \tag{3.14}$$

Equation (3.13) is then converted to the matrix form as

$$\big(\partial^2 + g_2^2 b^2 A\big) h = 0, \tag{3.15a}$$

where the matrix A (reduced mass matrix) is given by

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}. \tag{3.15b}$$

Its eigenvalues λ are given by the secular equation

$$\begin{vmatrix} -(1+\lambda) & 1 & 1 \\ 1 & -(1+\lambda) & 1 \\ 1 & 1 & -(1+\lambda) \end{vmatrix} = 0. \tag{3.16}$$

This simplifies to

$$(\lambda - 1)(\lambda + 2)(\lambda + 2) = 0. \tag{3.17}$$

The three eigenvalues are a singlet $\lambda_1 = +1$ and doublets $\lambda_2 = \lambda_3 = -2$.

Similarly, we evaluate the normalised eigenvectors of A as

$$x^{\mathrm{T}} = \frac{1}{\sqrt{3}}(1,1,1), \quad y^{\mathrm{T}} = \frac{1}{\sqrt{2}}(1,-1,0),$$

$$z^{\mathrm{T}} = \frac{1}{\sqrt{2}}(1, 0, -1). \tag{3.18}$$

However, the vector x is orthogonal to the vectors y and z but y and z are not mutually orthogonal. They are made orthogonal by Schmidt's method which gives

$$z'^{\mathrm{T}} = \frac{2}{\sqrt{3}}\left[z^{\mathrm{T}} - \left(y^{\mathrm{T}}z\right)y^{\mathrm{T}}\right] = \frac{1}{\sqrt{6}}(1, 1, -2). \tag{3.19}$$

This vector is orthogonal to x as well as to y. The matrix E, which diagonalises A, is composed of the eigenvectors as

$$E = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}. \tag{3.20}$$

The matrix E transforms the vectors h to

$$h' = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}}(h_1 + h_2 + h_3) \\ \frac{1}{\sqrt{2}}(h_1 - h_2) \\ \frac{1}{\sqrt{6}}(h_1 + h_2 - 2h_3) \end{bmatrix}. \tag{3.21}$$

Thus, the vector bosons h'_1, h'_2, h'_3 are linear combinations of the vector bosons h_1, h_2, h_3. Premultiplying eq. (3.15a) by the matrix E and using the above results, we get the Klein–Gordon equations satisfied by vector bosons h'_1, h'_2, h'_3 as

$$\left(\partial^2 + \lambda g_2^2 b^2\right)h' = 0. \tag{3.22}$$

Their masses are given by

$$\mu_i^2 = g_2^2 b^2 \lambda_i, \quad i = 1, 2, 3. \tag{3.23}$$

Out of the three eigenvalues, two of them are equal and this doublet corresponds to h'_2 and h'_3 bosons. The singlet eigenvalue corresponds to h'_1 boson. The mass of the h'_1 boson is given by $\mu_1^2 = g_2^2 b^2$, whereas h'_2 and h'_3 bosons have equal mass given by $\mu_2^2 = -2g_2^2 b^2$.

However, the particles having negative squared mass are faster than light particle tachyons (t) [11–13]. Therefore, h'_2 and h'_3 bosons associated with the bradyons (slower than light particles) h'_1 are tachyons.

3.2 *SU(2) of two sheets*

To obtain the two missing bradyons, we repeat the above procedure with the Lagrangian density $\Im'$ given in eq. (1.18b) as

$$\Im' = \frac{1}{8}G^{\mu\nu}G_{\mu\nu} - \frac{1}{2}m'^2_2 w^2 + \frac{1}{8}g'^2_2[w^4 - 2(w_i^4 + w_j^4 + w_k^4)] + \mathbf{W}^{\mu} \cdot \mathbf{J}_{\mu}. \tag{3.24}$$

This gives the Euler–Lagrange equations as

$$\partial^2 W_i^{\rho} + \left[m'^2_2 - \frac{1}{2}g'^2_2\left(w^2 - 2w_i^2\right)\right]W_i^{\rho} = 0. \tag{3.25a}$$

The non-zero vacuum expectation values are given by

$$m'^2_2 - \frac{1}{2}g'^2_2(v_j^2 + v_k^2 - v_i^2) = 0, \tag{3.25b}$$

$$m'^2_2 - \frac{1}{2}g'^2_2(v_k^2 + v_i^2 - v_j^2) = 0, \tag{3.25c}$$

$$m'^2_2 - \frac{1}{2}g'^2_2(v_i^2 + v_j^2 - v_k^2) = 0. \tag{3.25d}$$

These have solutions

$$v_i^2 = v_j^2 = v_k^2 = \frac{2m'^2_2}{g'^2_2} = b'^2 \tag{3.26a}$$

and

$$v^2 = \frac{6m'^2_2}{g'^2_2} = 3b'^2 = r'^2. \tag{3.26b}$$

As $v^2 > 0$, this implies that $m'^2_2 > 0$. The locus of the vacuum is a sphere of radius $v = \sqrt{(6m'^2_2/g'^2_2)} = r'$. There are again only eight points on the sphere at which there occur vacuum. These vacuum points are

$r'(1, 1, 1)$, $r'(1, 1, -1)$, $r'(1, -1, 1)$,
$r'(-1, 1, 1)$, $r'(1, -1, -1)$, $r'(-1, -1, 1)$,
$r'(-1, 1, -1)$

and

$r'(-1, -1, -1)$.

We transform the vector bosons around the vacuum point $r'(1, 1, 1)$ as

$$W_i^{\alpha} = v_i^{\alpha} + h_i^{\alpha}. \tag{3.27}$$

Following the same procedure as above, we arrive at the following eigenvalue equation:

$$\left(\partial^2 - \lambda' g'^2_2 b'^2\right)h' = 0. \tag{3.28}$$

The eigenvectors h'_1, h'_2, h'_3 have eigenvalues given by

$$\mu'^2_{1(t)} = -g'^2_2 b'^2, \quad \mu^2_{2,3(b)} = 2g'^2_2 b'^2. \tag{3.29}$$

Now, $h'_1(t)$ is a tachyon and $h'_2(b)$ and $h'_3(b)$ are bradyons of equal mass.

The set of the six vector bosons forms two sheets of $SU(2)$: one for $\sigma = 1$ (up) and the other for

$\sigma = -1$ (down). The unified Klein–Gordon equation for the eigenvectors $h'_{\sigma\lambda}$ can then be written as

$$\left[\partial^2 + \sigma\lambda(g_2)^2 b^2\right] h'_{\sigma\lambda} = 0. \tag{3.30}$$

The mass of the vector boson of quantum numbers σ and λ is given by

$$\mu^2(\sigma\lambda) = \sigma\lambda\left(g_2^2\right) b^2. \tag{3.31a}$$

3.3 *The γ photon, Z and the W bosons in $U(1) \otimes SU(2)$*

The masses of the six vector bosons are

$$\mu^2_{1(b)} = g_2^2 b^2, \quad \mu'^2_{1(t)} = -g'^2_2 b'^2,$$
$$\mu^2_{2,3(b)} = 2g'^2_2 b'^2, \quad \mu'^2_{2,3(t)} = -2g_2^2 b^2. \tag{3.31b}$$

Note that $g_2^2 \neq g'^2_2$. The ratio of the masses of h'_2 and h'_1 bradyons comes out to be

$$\frac{\mu_{2,3(b')}}{\mu_{1(b)}} = \frac{\sqrt{2} g'_2 b'}{g_2 b} = \sqrt{2}\left|\frac{m'_2}{m_2}\right|. \tag{3.32}$$

This ratio becomes greater than 1 if $\left|m'_2/m_2\right| = 1$. This is against physical observation (e.g., consider the case of Z and W bosons). Hence, $\left|m'_2/m_2\right|$ should be less than 1.

Alternatively, the singlet h'_1 can combine with the $U(1)$ boson B to form physical Z boson and massless photon as is done in the standard model of $U(1) \otimes SU(2)$ symmetry [7,8]. The mixing of B and h'_1 with Weinberg's mixing angle θ_{w} is written as

$$\gamma = B\cos\theta_{\mathrm{w}} - h'_1 \sin\theta_{\mathrm{w}},$$
$$Z = h'_1 \cos\theta_{\mathrm{w}} + B\sin\theta_{\mathrm{w}}. \tag{3.33a}$$

We take this mixing also to apply to the masses of these particles, and so write

$$\mu_\gamma = \mu_{\mathrm{B}} \cos\theta_{\mathrm{w}} - \mu_1 \sin\theta_{\mathrm{w}},$$
$$\mu_Z = \mu_1 \cos\theta_{\mathrm{w}} + \mu_{\mathrm{B}} \sin\theta_{\mathrm{w}}, \tag{3.33b}$$

where (here Y is $U(1)$ hypercharge) [1]

$$\mu^2_{\mathrm{B}} = g_1^2 Y^2 a^2 = 2\left|m_1^2\right|. \tag{3.33c}$$

We select the angle θ_{w} such that the photon mass becomes exactly zero. We then have from eq. (3.33b)

$$\tan\theta_{\mathrm{w}} = \frac{\mu_{\mathrm{B}}}{\mu_1} = \frac{g_1 Y a}{g_2 b}. \tag{3.34}$$

Substituting this in the mass formula for Z, we get

$$\mu_Z = \sqrt{\left(\mu_1^2 + \mu_2^2\right)} = \sqrt{\left(g_1^2 Y^2 a^2 + g_2^2 b^2\right)}. \tag{3.35}$$

$W^{\pm}$ defined as follows also have the mass as that of h'_2:

$$W^{\pm} = \frac{1}{\sqrt{2}}\left(h'_2 \pm \mathrm{i}\, h'_3 \right). \tag{3.36}$$

3.4 *The W/Z mass ratio*

The ratio of $W^{\pm}$ mass to that of Z mass now becomes

$$\frac{\mu_W}{\mu_Z} = \frac{\sqrt{2} g'_2 b'}{\sqrt{g_1^2 Y^2 a^2 + g_2^2 b^2}} = \frac{\mu'_2}{\sqrt{\mu_1^2 + \mu_2^2}}. \tag{3.37}$$

To simplify the analysis, we assume that all the vector bosons including the abelian ones have the same non-zero vacuum expectation values, i.e., $B = v_i = v_j = v_k = v$ and so $a = b = b'$. The mass ratio then simplifies to

$$\frac{\mu_W}{\mu_Z} = \frac{\sqrt{2} g'_2}{\sqrt{g_1^2 Y^2 + g_2^2}}. \tag{3.38}$$

The standard model [7,8] defines

$$g_1 \cos\theta_{\mathrm{w}} = g_2 \sin\theta_{\mathrm{w}} = e. \tag{3.39}$$

By taking $Y^2 = 1$, the mass ratio then reduces to

$$\frac{\mu_W}{\mu_Z} = \frac{\sqrt{2} g'_2 \cos\theta_{\mathrm{w}} \sin\theta_{\mathrm{w}}}{e}. \tag{3.40}$$

However, the standard model obtains

$$\frac{\mu_W}{\mu_Z} = \cos\theta_{\mathrm{w}}. \tag{3.41}$$

Therefore, in order that our GED theory agrees with the standard model, the coupling g'_2 should be given by

$$g'_2 \sin\theta_{\mathrm{w}} = \frac{e}{\sqrt{2}} \quad \text{or} \quad g'_2 = \frac{g_2}{\sqrt{2}}. \tag{3.42}$$

This agreement, however, shows that the Z and the W bosons do not have the same coupling in the Lagrangian density which is not so in the standard model. According to our present theory, the $SU(2)$ symmetry appears as a two-sheet symmetry ($\sigma = \pm 1$) of coupling constants related by $g'_2 = (g_2/\sqrt{2})$. The down $\sigma = -1$ sheet is weak compared to the up sheet $\sigma = +1$. The Z and the W bosons do not belong to the same sheet of $SU(2)$, and so the coupling of W bosons is weaker than the coupling of Z bosons in the weak interactions. Also, the bradyon boson and the same tachyon boson do not have the same mass. As such there is breaking of up–down (sheet) and bradyon–tachyon symmetries. These symmetry breakings are due to the non-abelian character of the vector bosons.

The experimental values of the bradyon masses $\mu_{W^+} = \mu_{W^-} = 82$ GeV and $\mu_Z = 94$ GeV give $\cos\theta_{\mathrm{w}} = 0.8723$.

4. Conclusions

The GED theory for non-abelian vector bosons developed here is the smooth generalisation of electrodynamics. Apart from the results in agreement with experiments it gives additional details of vector bosons. All the vector bosons have massive nuclear structure and all have the same vacuum expectation values. $SU(2)$ has two-sheet symmetry accommodating three tachyons in addition to the Z and the W bosons. The Z and the W bosons are seen not to have equal couplings in the $SU(2)$ interaction. They have sources having improper conservation as in the theory of gravitation [3].

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