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Numerical simulation for time-fractional nonlinear coupled dynamical model of romantic and interpersonal relationships

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Abstract. The objective of this paper is to study the nonlinear coupled dynamical fractional model of romantic and interpersonal relationships using fractional variation iteration method (FVIM) and fractional homotopy perturbation transform method (FHPTM). These procedures inspect the dynamics of love affairs among couples. Sufficient conditions for their convergence and error estimates are established. Obtained results are compared with the existing and recently developed methods. It is interesting to observe that these methods also work for those fractional models that do not have an exact solution. Results for different fractional values of time derivative are discussed with the help of figures and tables. Figures are drawn using Maple package. Test examples are provided to illustrate the accuracy and competency of the proposed schemes. Results divulge those schemes that are attractive, accurate, easy to use and highly effective.

Keywords. Fractional variation iteration method; fractional homotopy perturbation transform method; Caputo fractional derivative; He's polynomials.

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1. Introduction

Early references to derivatives of fractional order were made in the 17th century. In the past few decades, fractional-order calculus has emerged as a potential tool in various domains of science and engineering such as neurophysiology [1], fluid dynamic traffic, potential theory, control theory, viscoelasticity, electromagnetic theory, bioengineering, electric technology, plasma physics, mathematical economy, etc. Real-world processes, which we have to deal with, are generally of fractional order. Heat diffusion into a semi-infinite solid where heat flow equals half-derivative of temperature is an example of fractional-order system.

Einstein's mass–energy equation is derived under the assumption of absolute smooth space–time. Actually, space–time is intrinsically discontinuous if it tends to a quantum scale. A Hilbert cube can easily model actual fractal space–time. We can show discontinuity of space–time if we consider a TV screen that is smooth at all observable scales. However, when scale tends to be a very small one, the surface becomes unsmooth and consists of many arrayed pixels. Hence, time is discontinuous when it is tremendously small. A film gives 24 slips per second that gives a continuous movement, but for a case of 10 slips per second, the movement becomes discontinuous. When space–time is discontinuous, fractal theory is embraced to define several phenomena [2]. Molecular diffusion in water is similar to stochastic Brownian motion in view of continuum mechanics, but diffusion follows fractal Fick laws if we detect motion on a molecular scale. Water flow becomes discontinuous and fractal calculus is required to describe the molecule motion that becomes totally unpredictable in continuum mechanics frame. Heat-proof property of cocoon cannot be divulged by advanced calculus. If cocoon wall is supposed to be a continuous medium, then we cannot explain why the temperature change on its inner surface is very slow, irrespective of environmental temperature.

Moreover, time becomes discontinuous in microphysics, i.e. fractal kinetics takes place on a very small time-scale. In a smooth nanofibre membrane, if we study the effects of the diameter of nanofibres on air permeability, then we have to use a nanoscale, and under such case, a nanofibre membrane turns discontinuous and fractal calculus can effectively be used [3].

Differential equations that govern systems with memory are fractional differential equations (FDEs). Arbitrariness in their order introduces more degrees of freedom in design and analysis, resulting in more accurate modelling, better robustness in control and greater flexibility in signal processing. Electrochemical phenomenon like double-layer charge distribution or diffusion process can be better explained with fractional-order system. As a result, modelling of lithium ion battery, fuel cells and super capacitors are carried out with FDEs. Characterisation of ceramic bodies, fractal structures, viscoelastic materials and decay rate of fruits and meat and study of corrosion in metal surface are also promising areas of its applications. Fractional-order system is also a popular choice to study real-time events such as earthquake propagation, volcanic phenomenon, design of thermo-kinetics and modelling of human lungs and skin. Even characteristics of economic market fluctuations adopt fractional calculus-based system modelling. Hence, fractional-order analysis has reached from inert physical network to living network of biology, ecology, physiology and sociology, reminding us Leibnitz's prediction in his letter to L'Hopital in 1695 that fractional differential operator is 'an apparent paradox from which one day useful consequences will be drawn'.

Due to daily interactions, human relationships become disordered. Some inputs are required for a steady relationship, for resolving the differences between them and for not keeping ill feelings for ever [4]. Interpersonal relationships are formed in the context of social, cultural and other influences. In society, they appear in family, companionship, acquaintance, marriage, parent–child, neighbourhood, clubs, work, dating, etc. The most fascinating of all is the romantic relationship which is fundamental in communication and human social life. Interpersonal communication is a central tool for romantic relationships that refer to mutual continuous communication among two or more persons. Researchers studied practices that influence the progress of romantic relationships and considered variables that inspire people to initiate relationships. Research in this context has concentrated both on the behaviour of individual partners and in the pattern of behaviour enacted by romantic pairs. Similarly, when individuals involve in romance, their opinions about the partner and relationship generally affect their relational outcomes. The amount of love one feels towards one's partner is directly proportional to the longevity of that relationship. Romantic relationships are more common among adolescents. Strogatz [5] studied the model of love affair between Romeo and Juliet depicted by Shakespeare by applying a coupled system of differential equations.

Suppose that at any time t, we could measure Romeo's love or hate for Juliet, $\xi\,(t)$, and Juliet's love or hate for Romeo, $\chi(t)$. Positive values of these functions indicate love and negative values indicate hate. The easiest assumption would be that the change in Romeo's love for Juliet is a fraction of his current love added with a fraction of her current love. Similarly, Juliet's love for Romeo will change by a fraction of her current love for Romeo and a fraction of Romeo's love for her. This assumption leads us to the model equation

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \alpha\xi(t) + \beta\chi(t); \qquad \frac{\mathrm{d}\chi}{\mathrm{d}t} = \gamma\xi(t) + \vartheta\chi(t),$$

where α, β, γ and ϑ are constants. Researchers want to know why married couples divorce, and also, some are happy, whereas others are not, with each other. Gottman *et al* [6] worked on dynamical discrete models to define interaction between them. Since design of experiments in these areas is cumbersome and inhibited by ethical reflections, models of Mathematics may play an important part in learning dynamics of relations and behavioural features. A few mathematical models exist for seizing romantic relationship's dynamics, but they are restricted to differential equations of integer order. Cherif and Barley [7] investigated dynamics of romantic and interpersonal relationships by using mathematical models of ordinary, stochastic differential equations to give vision into the behaviour of love. They analysed a deterministic model and then nonlinear stochastic models capturing stochastic rates and ecological factors such as cultural, historical and community conditions that affect proximal experiences and shape patterns of relationship. Their results showed that deterministic models tend to approach locally stable emotional behaviours. The stochastic differential equation extension gives insight into the dynamics of romantic relationships that were not captured by deterministic models, which assumes that love is scalar and individuals respond predictably to their feelings and that of others without external influences such as ecological factors.

A mathematical model of integer order studied in [7] is

$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = -\alpha_1 X_1 + \beta_1 X_2(1 - \epsilon X_2^2) + A_1,$$
$$\frac{\mathrm{d}X_2}{\mathrm{d}t} = -\alpha_2 X_2 + \beta_2 X_1\left(1 - \epsilon X_1^2\right) + A_2.$$

Cherif and Barley [7] gave a new future direction of developing methods to analyse systems that exhibit 'stability boundary crossing' and 'jump between locally stable equilibria' dynamics. However, the biggest advantage of using fractional models of differential equations in physical models is their non-local property. Fractional-order derivative is non-local, whereas integer-order derivative is local in nature. It shows that the upcoming state of physical system is also dependent on all its historical states in addition to its present state. Hence, the fractional models are more realistic. In FDEs, general response expression contains a parameter describing the order of fractional derivative that can be varied to gain various responses.

In the present study, we ponder over a nonlinear system of coupled time-FDEs given as

$$D_t^{\alpha} u = -a_1 u + b_1 v(1 - \epsilon v^2) + c_1,$$
$$D_t^{\alpha} v = -a_2 v + b_1 u(1 - \epsilon u^2) + c_2, \quad 0 < \alpha \leq 1$$

with initial settings $u(0) = 0 = v(0)$, u and v are continuously differentiable state variables, $a_i \geq 0$ and a_i, b_i, c_i $(1 \leq i \leq 2)$ are constants.

The fractional nonlinear coupled dynamical model of romantic and interpersonal relationships was first discussed by Ozalp and Koca [8], in which they obtained a stability condition for equilibrium points. Khader and Alqahtani [9] presented an approximate solution for a nonlinear fractional coupled system of dynamical marriage model by applying Bernstein collocation method and compared the results with those obtained from Runge–Kutta IV order method. They considered fractional derivative in Riemann–Liouville sense and properties of Bernstein polynomials were used to reduce fractional coupled model to a system of nonlinear algebraic equations that were solved by Newton's iterative method. Khader *et al* [10] also applied Legendre spectral collocation method to solve the same model and confirmed natural behaviour of the proposed system. Singh *et al* [11] applied q-homotopy analysis Sumudu transform method (q-HASTM) and Adomian decomposition method (ADM) to solve a fractional model for marriages and compared the results. This model has not yet been studied by fractional variation iteration method (FVIM) and fractional homotopy perturbation transform method (FHPTM). Time-fractional coupled equations describe the motion of particle with memory in time. Space fractional derivatives arise when variations are heavy tailed and describe the particle motion that accounts for variation in flow field over the entire system. Moreover, fraction in time derivative suggests modulation or weighting of system memory. It is apparent that relationships are influenced by memory. This fact marks fractional modelling suitable for such systems. Hence, the study of time-fractional coupled differential equations is very important.

Most nonlinear FDEs do not possess exact solutions, and so some numerical techniques are required for their approximate numerical solution. Reliability of solution schemes is also a very important aspect compared to modelling dimensions of equations [12]. FVIM [13] directly attacks the nonlinear FDEs without a need to find certain polynomials for nonlinear terms and gives result in an infinite series that rapidly converges to analytical solution. This method does not require linearisation, discretisation, perturbations or any restrictive assumptions. It lessens mathematical computations significantly. FVIM has thoroughness in mathematical derivation of Lagrange's multiplier by variational theory for fractional calculus. It leads to solution converging to the exact one. Recently, fractional complex transform [14] is developed to build a simpler variational iteration algorithm for fractional calculus. A complete review on applications of FVIM is available in [15].

Usual analytical methods need more memory in computer as well as time for computation. Hence, to overcome these limitations, they require to be amalgamated with transform operators to work on nonlinear equations. FHPTM shows how Laplace transform may be applied to approximate the solution of nonlinear FDEs by handling homotopy perturbation method (HPM). The perturbation technique has many limitations, e.g., the approximate solution contains a succession of small parameters that have difficulties as most nonlinear problems possess no such parameter. Recently, He [16] suggested the construction of homotopy equation with an auxiliary term that vanishes completely when the embedding parameter $p = 0, 1$ and also it neither affects the initial solution ($p = 0$) nor the real solution ($p = 1$). He also proposed HPM with two expanding parameters that are especially effective for a nonlinear equation with two nonlinear terms. FHPTM [17] is a neat amalgamation of HPM, standard transform of Laplace and He's polynomials. The advantage of FHPTM is its potential for assimilating strong computational methodologies for probing nonlinear coupled FDEs. We have used Caputo fractional derivative because its main advantage is that with these derivatives, the initial conditions for FDEs undertake similar form as for integer-order differential equations. Moreover, Caputo fractional derivative is used for continuously differentiable functions. We have considered u and v as differentiable functions throughout the paper.

The aim of this paper is to obtain numerical solution of nonlinear time-fractional model of coupled differential equations by FVIM and FHPTM and to compare results with those from the existing techniques. This study is structured in the following manner. Section 1 is introductory. In §2, we present a brief review of the preliminary description of Caputo's fractional derivative and some other results. In §3, the nonlinear fractional model of romantic and interpersonal relationships in marriages is described. In §4, the basic plan of the proposed FVIM is provided by taking the problem under consideration. Convergence of FVIM is discussed along with its implementation on a given model. In §5, basic plan of FHPTM is given. Convergence of FHPTM is also discussed along with its implementation on the given model. Section 6 deals with discussion on the obtained numerical results and their significance. In §7, we recapitulate our outcomes and draw inferences.

2. Preliminaries

DEFINITION 2.1

Consider a real function $h(\chi)$, $\chi > 0$.

a. It is called in space $C_\zeta, \zeta \in R$ if $\ni$ a real no. $b(>\zeta)$, s.t. $h(\chi) = \chi^b h_1(\chi)$, $h_1 \in C[0, \infty)$. It is clear that $C_\zeta \subset C_\gamma$, if $\gamma \le \zeta$.
b. It is called in space C_ζ^m, $m \in \mathbb{N}\cup\{0\}$ if $h^{(m)} \in C_\zeta$.

DEFINITION 2.2

Caputo fractional derivative of $h, h \in C_{-1}^m, m \in \mathbb{N}\cup\{0\}$ is

$$D_t^\beta h(t) = \begin{cases} I^{m-\beta} h^{(m)}(t), & m-1 < \beta < m, \quad m \in \mathbb{N}, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} h(t), \beta = m, & \end{cases}$$

a. $I_t^\zeta h(x,t) = \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} h(x,s)\mathrm{d}s$; $\zeta, t > 0$.
b. $D_\tau^\nu V(x,\tau) = I_\tau^{m-\nu} \frac{\partial^m V(x,\tau)}{\partial t^m}$, $m-1 < \nu \le m$.
c. $D_t^\zeta I_t^\zeta h(t) = h(t)$, $m-1 < \zeta \le m$, $m \in \mathbb{N}$.
d. $I_t^\zeta D_t^\zeta h(t) = h(t) - \sum_{k=1}^{m-1} h^k \left(0^+\right) \frac{t^k}{k!}$, $m-1 < \zeta \le m$, $m \in \mathbb{N}$.
e. $I^\upsilon t^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(\upsilon+\zeta+1)} t^{\upsilon+\zeta}$.

DEFINITION 2.3

Laplace transform of Caputo fractional derivative is

$$L\left[D^\alpha g(t)\right] = p^\alpha L[g(t)] - \sum_{k=0}^{n-1} p^{\alpha-k-1} g^{(k)}\left(0^+\right),$$

$n-1 < \alpha \le n$.

3. Model description

To model the behavioural features of the romantic dynamics, we propose a nonlinear fractional deterministic dynamical model with two state variables of the form

$$\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} = -a_1 u + b_1 v(1-\epsilon v^2) + c_1,$$
$$\frac{\mathrm{d}^\alpha v}{\mathrm{d}t^\alpha} = -a_2 v + b_2 u\left(1-\epsilon u^2\right) + c_2,$$

for $(u, v) \in R \times R$ and initial conditions $u(0) = 0 = v(0)$. Moreover, $0 < \alpha \le 1$. State variables u and v stay as measures of love of both individuals for their companions. Positive and negative measures signify feelings. $a_i \ge 0$ is non-negative; a_i, b_i and c_i $(1 \le i \le 2)$ are oblivion, reaction and attraction constants, respectively. Positivity condition is relaxed for b_i and c_i. In this model, it is assumed that there is an exponentially fast decay in feelings if partners are absent. The part of oxytocin in behavioural features, attachment dynamics and cultural conditions is ignored. Parameters a_i, b_i and c_i signify romantic style of both individuals. a_i indicates degree to which one is encouraged by one's personal feeling. It may be used as a level of dependency as well as anxiety on other's endorsement in relationships. Parameter b_i represents level to which one is encouraged by one's partner and/or expects him/her to be helpful. It measures propensity to avoid or seek intimacy in a romantic relationship. The term $-a_i u_i$ expresses that one's love measure declines exponentially in the absence of one's partner; $1/a_i$ shows time needed for love to decrease, and ϵ in return function is compensatory constant. For $\epsilon = 0$, the model reduces to those suggested by Strogatz [5] and Gottman *et al* [6]. In Strogatzian model, equilibrium point $(\bar{u}, \bar{v})$ is satisfied by the equations:

$$\bar{u} = \frac{a_2 c_1 + b_1 c_2}{a_1 a_2 - b_1 b_2}, \quad \bar{v} = \frac{a_1 c_2 + b_2 c_1}{a_1 a_2 - b_1 b_2}.$$

Cherif and Barley [7] proved the condition for asymptotic stability of equilibrium $(\bar{u}, \bar{v})$ that for stable system, the product of the ratio of reactiveness and oblivious coefficients must be < 1.

4. Basic plan of FVIM for nonlinear time-fractional coupled differential equations

Consider the model described by

$$\left.\begin{array}{l} D_t^\alpha u = -a_1 u + b_1 v(1-\epsilon v^2) + c_1 \\ D_t^\alpha v = -a_2 v + b_2 u(1-\epsilon u^2) + c_2, \ (0 < a \le 1) \end{array}\right\}, \tag{1}$$

with initial conditions $u(0) = 0 = v(0)$. $a_i \ge 0, a_i, b_i, c_i (1 \le i \le 2)$ are constants.

Correction functionals are formed for eq. (1) as

$$\left.\begin{aligned} u_{n+1}(t) &= u_n + \int_0^t \lambda(D_t^\alpha u_n + a_1\tilde{u}_n \\ &\quad -b_1\tilde{v}_n(1-\epsilon\tilde{v}_n^2) - c_1)\,(\mathrm{d}\tau)^\alpha \\ v_{n+1}(t) &= v_n + \int_0^t \lambda(D_t^\alpha v_n + a_2\tilde{v}_n \\ &\quad -b_2\tilde{u}_n(1-\epsilon\tilde{u}_n^2) - c_2)\,(\mathrm{d}\tau)^\alpha \end{aligned}\right\}, \tag{2}$$

where λ is the Lagrange's multiplier.

By variational theory, λ must satisfy

$$\frac{\mathrm{d}^\alpha\lambda}{\mathrm{d}\tau^\alpha}\bigg|_{\tau=t} = 0 \quad \text{and} \quad 1+\lambda|_{\tau=t} = 0.$$

We easily get $\lambda = -1$. Then, using it in eq. (2), we get

$$\left.\begin{aligned} u_{n+1}(t) &= u_n - \textstyle\int_0^t (D_t^\alpha u_n + a_1 u_n \\ &\quad -b_1 v_n(1-\epsilon v_n^2) - c_1)\,(\mathrm{d}\tau)^\alpha \\ v_{n+1}(t) &= v_n - \textstyle\int_0^t (D_t^\alpha v_n + a_2 v_n \\ &\quad -b_2 u_n(1-\epsilon u_n^2) - c_2)\,(\mathrm{d}\tau)^\alpha \end{aligned}\right\}. \tag{3}$$

Consecutive approximations $u_n(t), v_n(t), n \geq 0$ can be built henceforth. $\tilde{u}_n$ and $\tilde{v}_n$ are restricted variations, i.e. $\delta\tilde{u}_n = 0$ and $\delta\tilde{v}_n = 0$. Finally, we obtain sequences $u_{n+1}(t), v_{n+1}(t), n \geq 0$ of the solution. Consequently, exact solution is obtained as

$$\left.\begin{aligned} u(t) &= \lim_{n\to\infty} u_n(t) \\ v(t) &= \lim_{n\to\infty} v_n(t) \end{aligned}\right\}. \tag{4}$$

4.1 *Algorithm of FVIM*

Step 1: Find $u_0 = u(0)$ given by initial approximation, set $n = 0$;
Step 2: Use computed values of u_n to obtain u_{n+1} from eq. (3);
Step 3: Define $u_n := u_{n+1}$;
Step 4: If max $|u_{n+1} - u_n| <$ Tol stop, otherwise continue;
Step 5: Set $u_{n+1} := u_n$;
Step 6: Set $n = n + 1$, return to Step 2.

4.2 *Convergence analysis of FVIM*

Now, our emphasis is on the convergence of the proposed FVIM applied to eq. (1) in § 4. Sufficient conditions for the convergence of FVIM and its error estimate [18] are provided.

We define the operators S_1 and S_2 as

$$\left.\begin{aligned} S_1 &= \int_0^t (-1)(D_t^\alpha u_n + a_1 u_n \\ &\quad -b_1 v_n(1-\epsilon v_n^2) - c_1)\,(\mathrm{d}\tau)^\alpha \\ S_2 &= \int_0^t (-1)(D_t^\alpha v_n + a_2 v_n \\ &\quad -b_2 u_n(1-\epsilon u_n^2) - c_2)\,(\mathrm{d}\tau)^\alpha \end{aligned}\right\}. \tag{5}$$

Also, we define the components $v_{k(i)}, k = 0, 1, 2, \ldots, i = 1, 2$ as

$$\left.\begin{aligned} u(t) &= \lim_{n\to\infty} u_n(t) = \sum_{k=0}^{\infty} v_{k(1)} \\ v(t) &= \lim_{n\to\infty} v_n(t) = \sum_{k=0}^{\infty} v_{k(2)} \end{aligned}\right\}. \tag{6}$$

These solutions in series converge very rapidly.

Theorem 1 [19]. *Let S_1 and S_2, defined in eq. (5), be operators from Banach space BS to BS. The series solutions defined in eq. (6) converge if $0 < q < 1$ exists such that*

$$\|S_i[v_{0(i)} + v_{1(i)} + v_{2(i)} + \cdots + v_{k+1(i)}]\| \leq q\|S_i[v_{0(i)} + v_{1(i)} + v_{2(i)} + \cdots + v_{k(i)}]\|, i = 1, 2$$

(i.e. $\|v_{k+1}\| \leq q\|v_k\|$), $\forall k \epsilon \mathbb{N} \cup \{0\}$.

Theorem 1 is an exceptional case of Banach fixed point theorem used in [20] as sufficient condition to discuss convergence of FVIM for various differential equations.

Theorem 2 [19]. *If solution defined in eq. (6) converges, then it is an exact solution of problem* (1).

Theorem 3 [19]. *Suppose series solution (6) converges to solutions $u(t), v(t)$ of problem (1). If truncated series $\sum_{k=0}^{m} v_{k(1)}$ is used as an approximation to solution $u(t)$ of problem (1), then maximum error $E_m(t)$ is assessed as*

$$E_m(t) \leq \frac{1}{1-q} q^{m+1}\|v_0\|.$$

If $\forall i \in \mathbb{N} \cup \{0\}$, then we define the parameters

$$\chi_i = \begin{cases} \frac{\|v_{i+1}\|}{\|v_i\|}, & \|v_i\| \neq 0, \\ 0, & \|v_i\| = 0, \end{cases}$$

then series solution $\sum_{k=0}^{\infty} v_{k(i)}, i = 1, 2$ of problem (1) converges to exact solution (6) when $0 \leq \chi_i < 1, \forall i \in \mathbb{N} \cup \{0\}$. Moreover, as specified in Theorem 3, the maximum absolute truncation error is estimated as

$$\left\| u(t) - \sum_{k=0}^{\infty} v_{k(1)} \right\| \leq \frac{1}{1-\chi} \chi^{m+1} \|v_0\|,$$

where $\chi = \max\{\chi_i, i = 0, 1, 2, \ldots, m\}$.

Remark [19]. If the first finite χ_i's, $i = 1, 2, \ldots, m$, are not less than 1 and $\chi_i \leq 1$ for $i > m$, then obviously, the series solution $\sum_{k=0}^{\infty} v_{k(i)}, i = 1, 2$, of problem (1) converges to an exact solution. It means that first finite terms do not affect the convergence of series solution. Here, the convergence of FVIM depends on χ_i for $i > m$.

4.3 *Numerical implementation of FVIM*

Using conditions, we may initialise with $u_0(t) = 0 = v_0(t)$ and applying FVIM to eq. (1), we get

$$\left.\begin{aligned} u_1(t) &= u_0 - \int_0^t \left(\frac{\mathrm{d}^\alpha u_0}{\mathrm{d}t^\alpha} + a_1 u_0 \right. \\ &\qquad \left. - b_1 v_0(1 - \epsilon v_0^2) - c_1 \right) (\mathrm{d}\tau)^\alpha = \frac{c_1 t^\alpha}{\Gamma(1+\alpha)} \\ v_1(t) &= v_0 - \int_0^t \left(\frac{\mathrm{d}^\alpha v_0}{\mathrm{d}t^\alpha} + a_2 v_0 \right. \\ &\qquad \left. - b_2 u_0(1 - \epsilon u_0^2) - c_2 \right) (\mathrm{d}\tau)^\alpha = \frac{c_2 t^\alpha}{\Gamma(1+\alpha)} \end{aligned}\right\}, \tag{7}$$

$$\left.\begin{aligned} u_2(t) &= u_1 - \int_0^t \left(\frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} + a_1 u_1 \right. \\ &\qquad \left. - b_1 v_1\big(1 - \epsilon v_1^2\big) - c_1 \right) (\mathrm{d}\tau)^\alpha \\ &= \frac{c_1 t^\alpha}{\Gamma(1+\alpha)} - \frac{a_1 c_1 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{b_1 c_2 t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &\quad - \frac{b_1 c_2^3 t^{4\alpha} \epsilon \Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3\, \Gamma(1+4\alpha)} \\ v_2(t) &= v_1 - \int_0^t \left(\frac{\mathrm{d}^\alpha v_1}{\mathrm{d}t^\alpha} + a_2 v_1 \right. \\ &\qquad \left. - b_2 u_1\big(1 - \epsilon u_1^2\big) - c_2 \right) (\mathrm{d}\tau)^\alpha \\ &= \frac{c_2 t^\alpha}{\Gamma(1+\alpha)} - \frac{a_2 c_2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{b_2 c_1 t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &\quad - \frac{b_2 c_1^3 t^{4\alpha} \epsilon \Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3\, \Gamma(1+4\alpha)} \end{aligned}\right\}, \tag{8}$$

$$\begin{aligned} u_3(t) &= \frac{c_1 t^\alpha}{\Gamma(1+\alpha)} - \frac{a_1 c_1 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{b_1 c_2 t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &- \frac{b_1 c_2^3 t^{4\alpha} \epsilon \Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+4\alpha)} + \frac{a_1^2 c_1 t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &+ \frac{b_1 b_2 c_1 t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{a_1 b_1 c_2 t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{a_2 b_1 c_2 t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &- \frac{b_1 b_2 c_1^3 t^{5\alpha} \epsilon \Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+5\alpha)} + \frac{a_1 b_1 c_2^3 t^{5\alpha} \epsilon \Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+5\alpha)} \\ &- \frac{3 b_1 b_2 c_1 c_2^2 t^{5\alpha} \epsilon \Gamma(1+4\alpha)}{\Gamma(1+\alpha)^2 \Gamma(1+2\alpha) \Gamma(1+5\alpha)} \\ &+ \frac{3 a_2 b_1 c_1 c_2^3 t^{5\alpha} \epsilon \Gamma(1+4\alpha)}{\Gamma(1+\alpha)^2 \Gamma(1+2\alpha) \Gamma(1+5\alpha)} \\ &- \frac{3 b_1 b_2^2 c_1^2 c_2 t^{6\alpha} \epsilon \Gamma(1+5\alpha)}{\Gamma(1+2\alpha)^2 \Gamma(1+\alpha) \Gamma(1+6\alpha)} \\ &+ \frac{6 a_2 b_1 b_2 c_1 c_2^2 t^{6\alpha} \epsilon \Gamma(1+5\alpha)}{\Gamma(1+2\alpha)^2 \Gamma(1+\alpha) \Gamma(1+6\alpha)} \\ &- \frac{3 a_2^2 b_1 c_2^3 t^{6\alpha} \epsilon \Gamma(1+5\alpha)}{\Gamma(1+2\alpha)^2 \Gamma(1+\alpha) \Gamma(1+6\alpha)} \\ &- \frac{b_1 b_2^3 c_1^3 t^{7\alpha} \epsilon \Gamma(1+6\alpha)}{\Gamma(1+2\alpha)^3 \Gamma(1+7\alpha)} \\ &+ \frac{3 a_2 b_1 b_2^3 c_1^2 c_2 t^{7\alpha} \epsilon \Gamma(1+6\alpha)}{\Gamma(1+2\alpha)^3 \Gamma(1+7\alpha)} \\ &- \frac{3 a_2^2 b_1 b_2 c_1 c_2^2 t^{7\alpha} \epsilon \Gamma(1+6\alpha)}{\Gamma(1+2\alpha)^3 \Gamma(1+7\alpha)} \\ &+ \frac{a_2^3 b_1 c_2^3 t^{7\alpha} \epsilon \Gamma(1+6\alpha)}{\Gamma(1+2\alpha)^3 \Gamma(1+7\alpha)} \\ &+ \frac{3 b_1 b_2 c_1^3 c_2^2 t^{7\alpha} \epsilon^2 \Gamma(1+3\alpha) \Gamma(1+6\alpha)}{\Gamma(1+\alpha)^5 \Gamma(1+4\alpha) \Gamma(1+7\alpha)} \\ &+ \frac{6 b_1 c_2 b_2^2 c_1^4 t^{8\alpha} \epsilon^2 \Gamma(1+3\alpha) \Gamma(1+7\alpha)}{\Gamma(1+\alpha)^4 \Gamma(1+2\alpha) \Gamma(1+4\alpha) \Gamma(1+8\alpha)} \\ &- \frac{6 a_2 b_1 b_2 c_1^3 c_2^2 c_1^4 t^{8\alpha} \epsilon^2 \Gamma(1+3\alpha) \Gamma(1+7\alpha)}{\Gamma(1+\alpha)^4 \Gamma(1+2\alpha) \Gamma(1+4\alpha) \Gamma(1+8\alpha)} \\ &+ \frac{3 b_1 b_2^3 c_1^5 t^{9\alpha} \epsilon^2 \Gamma(1+3\alpha) \Gamma(1+8\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+2\alpha)^2 \Gamma(1+4\alpha) \Gamma(1+9\alpha)} \\ &+ \frac{6 a_2 b_1 b_2^2 c_1^4 c_2 t^{9\alpha} \epsilon^2 \Gamma(1+3\alpha) \Gamma(1+8\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+2\alpha)^2 \Gamma(1+4\alpha) \Gamma(1+9\alpha)} \\ &+ \frac{3 a_2^2 b_1 b_2 c_1^3 c_2^2 t^{9\alpha} \epsilon^2 \Gamma(1+3\alpha) \Gamma(1+8\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+2\alpha)^2 \Gamma(1+4\alpha) \Gamma(1+9\alpha)} \\ &- \frac{3 b_1 b_2^2 c_1^6 c_2 t^{10\alpha} \epsilon^3 \Gamma(1+3\alpha)^2 \Gamma(1+9\alpha)}{\Gamma(1+\alpha)^7 \Gamma(1+4\alpha)^2 \Gamma(1+10\alpha)} \\ &- \frac{3 b_1 b_2^3 c_1^7 t^{11\alpha} \epsilon^3 \Gamma(1+3\alpha)^2 \Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6 \Gamma(1+4\alpha)^2 \Gamma(1+2\alpha) \Gamma(1+11\alpha)} \\ &+ \frac{3 a_2 b_1 b_2^2 c_1^6 c_2 t^{11\alpha} \epsilon^3 \Gamma(1+3\alpha)^2 \Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6 \Gamma(1+4\alpha)^2 \Gamma(1+2\alpha) \Gamma(1+11\alpha)} \\ &+ \frac{b_1 b_2^3 c_1^9 t^{13\alpha} \epsilon^4 \Gamma(1+3\alpha)^3 \Gamma(1+12\alpha)}{\Gamma(1+\alpha)^9 \Gamma(1+4\alpha)^3 \Gamma(1+13\alpha)}, \end{aligned}$$

$$\begin{aligned} v_3(t) = -t^\alpha \Bigg[& \frac{3 b_2 c_1^2 (a_1 c_1 - b_1 c_2) t^{4\alpha} \epsilon \Gamma(1+4\alpha)}{\Gamma(1+\alpha)^2 \Gamma(1+2\alpha) \Gamma(1+5\alpha)} \\ & + \frac{3 b_2 c_1 (a_1 c_1 - b_1 c_2)^2 t^{5\alpha} \epsilon \Gamma(1+5\alpha)}{\Gamma(1+\alpha) \Gamma(1+2\alpha)^2 \Gamma(1+6\alpha)} \\ & - \frac{c_2}{\Gamma(1+\alpha)} + t^\alpha \Bigg(\frac{-b_2 c_1 + a_2 c_2}{\Gamma(1+2\alpha)} \\ & + \frac{\big(a_1 b_2 c_1 + a_2 b_2 c_1 - a_2^2 c_2 - b_1 b_2 c_2\big) t^\alpha}{\Gamma(1+3\alpha)} \Bigg) \end{aligned}$$

$$
\begin{aligned}
&-\frac{b_2(a_1c_1-b_1c_2)^3t^{5\alpha}\epsilon\Gamma(1+6\alpha)}{\Gamma(1+2\alpha)^3\Gamma(1+7\alpha)}\\
&-\frac{3b_1b_2c_1c_2^3t^{6\alpha}\epsilon^2\Gamma(1+3\alpha)\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^5\Gamma(1+4\alpha)\Gamma(1+7\alpha)}\\
&+\frac{6b_1b_2c_1c_2^3(a_1c_1-b_1c_2)t^{7\alpha}\epsilon^2\Gamma(1+3\alpha)\Gamma(1+7\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+8\alpha)}\\
&+\frac{b_2t^{3\alpha}\epsilon\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3}\left(\frac{(-a_2c_1^3+b_1c_2^3)t^{\alpha}}{\Gamma(1+5\alpha)}\right.\\
&-\frac{3b_1c_2^3(a_1c_1-b_1c_2)^2t^{5\alpha}\epsilon\Gamma(1+8\alpha)}{\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)\Gamma(1+9\alpha)}\\
&\left.+\frac{c_1^3}{\Gamma(1+4\alpha)}\right)\\
&+\frac{3b_2b_1^2c_2^6c_1t^{9\alpha}\epsilon^3\Gamma(1+3\alpha)^2\Gamma(1+9\alpha)}{\Gamma(1+\alpha)^7\Gamma(1+4\alpha)^2\Gamma(1+10\alpha)}\\
&+\frac{3b_2b_1^2c_2^6(a_1c_1-b_1c_2)t^{10\alpha}\epsilon^3\Gamma(1+3\alpha)^2\Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6\Gamma(1+4\alpha)^2\Gamma(1+2\alpha)\Gamma(1+11\alpha)}\\
&\left.-\frac{b_2b_1^3c_2^9t^{12\alpha}\epsilon^4\Gamma(1+3\alpha)^3\Gamma(1+12\alpha)}{\Gamma(1+\alpha)^9\Gamma(1+4\alpha)^3\Gamma(1+13\alpha)}\right].
\end{aligned}
$$

Proceeding in this way, rest of the components may be obtained using Mathematica package.

At last, we get solution as

$$
\left.\begin{aligned}
u(t)&=\lim_{n\to\infty}u_n(t)\\
v(t)&=\lim_{n\to\infty}v_n(t)
\end{aligned}\right\}. \tag{9}
$$

In view of eqs (5) and (6), iteration formula for problem (1) can be built as

$$
\begin{aligned}
v_{0(1)}&=0,\quad v_{0(2)}=0,\\
v_{1(1)}&=\frac{c_1t^{\alpha}}{\Gamma(1+\alpha)},\quad v_{1(2)}=\frac{c_2t^{\alpha}}{\Gamma(1+\alpha)},\\
v_{2(1)}&=\frac{-a_1c_1t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{b_1c_2t^{2\alpha}}{\Gamma(1+2\alpha)}\\
&\quad-\frac{b_1c_2^3\epsilon t^{4\alpha}\Gamma(1+3\alpha)}{(\Gamma(1+\alpha))^3\Gamma(1+4\alpha)},\\
v_{2(2)}&=\frac{-a_2c_2t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{b_2c_1t^{2\alpha}}{\Gamma(1+2\alpha)}\\
&\quad-\frac{b_2c_1^3\epsilon t^{4\alpha}\Gamma(1+3\alpha)}{(\Gamma(1+\alpha))^3\Gamma(1+4\alpha)},
\end{aligned}
$$

and so on.

By computing χ_i's for this problem, we have

$$
\begin{aligned}
\chi_i&=\frac{\|v_{i+1(1)}\|}{\|v_{i(1)}\|}\\
&=\left\|t^{\alpha}\frac{\Gamma(1+i\alpha)}{\Gamma(1+(i+1)\alpha)}\left[\left(-a_1+\frac{b_1c_2}{c_1}\right)-\left(\frac{b_1c_2^3\epsilon}{c_1}\right)\right.\right.\\
&\quad\left.\left.\frac{t^{2\alpha}\Gamma(1+(i+1)\alpha)\Gamma(1+(i+2)\alpha)}{(\Gamma(1+i\alpha))^3\Gamma(1+(i+3)\alpha)}\right]\right\|<1,\\
\chi_i'&=\frac{\|v_{i+1(2)}\|}{\|v_{i(2)}\|}\\
&=\left\|t^{\alpha}\frac{\Gamma(1+i\alpha)}{\Gamma(1+(i+1)\alpha)}\left[\left(-a_2+\frac{b_2c_1}{c_2}\right)-\left(\frac{b_2c_1^3\epsilon}{c_2}\right)\right.\right.\\
&\quad\left.\left.\frac{t^{2\alpha}\Gamma(1+(i+1)\alpha)\Gamma(1+(i+2)\alpha)}{(\Gamma(1+i\alpha))^3\Gamma(1+(i+3)\alpha)}\right]\right\|<1,
\end{aligned}
$$

when for example, $i>1$ and $0<\alpha\le 1$. Constants a_i, b_i, c_i, $i=1,2$ are taken as 0.05, 0.04, 0.2 and 0.07, 0.06, 0.3, respectively, and $\epsilon=0.01$. This confirms that variational approach for problem (1) gives positive and bounded solution that ultimately converges to exact solution. Problem (1) is considered when $0<t\le 1$ to discuss condition of convergence. Obviously, we can get the length of the interval and examine condition of convergence after neglecting the first few terms of the series solution.

5. Basic plan of FHPTM for nonlinear time-fractional coupled differential equations

To explain the process of FHPTM, we ponder over a coupled fractional nonlinear system of ordinary differential equations (ODEs):

$$
\left.\begin{aligned}
D_t^{\alpha}u(t)+R_1(u,v)+Q_1(u,v)&=g_1(t)\\
D_t^{\alpha}v(t)+R_2(u,v)+Q_2(u,v)&=g_2(t)
\end{aligned}\right\}, \tag{10}
$$

with initial values

$$
u(0)=0=v(0). \tag{11}
$$

Here, D_t^{α} is the Caputo's fractional derivative of arbitrary order α, R_1, R_2 and Q_1, Q_2 are linear and nonlinear operators, respectively, g_1, g_2 are source terms. Moreover, $0<\alpha\le 1$.

FHPTM comprises taking transform of Laplace on eq. (10) and using differentiation property,

$$
\left.\begin{aligned}
L[u(t)]&=p^{-\alpha}L[g_1(t)]\\
&\qquad-p^{-\alpha}L[R_1(u,v)+Q_1(u,v)]\\
L[v(t)]&=p^{-\alpha}L[g_2(t)]\\
&\qquad-p^{-\alpha}L[R_2(u,v)+Q_2(u,v)]
\end{aligned}\right\}. \tag{12}
$$

Taking inverse transform, we get

$$\left.\begin{aligned} u(t) &= G_1(t) - L^{-1}\big[p^{-\alpha}L\{R_1(u,v) \\ &\quad +Q_1(u,v)\}\big] \\ v(t) &= G_2(t) - L^{-1}\big[p^{-\alpha}L\{R_2(u,v) \\ &\quad +Q_2(u,v)\}\big] \end{aligned}\right\}. \tag{13}$$

Here, $G_1(t)$ and $G_2(t)$ come from the source term and the prescribed initial values.

Applying HPM, it is assumed that the results may be articulated as a power series:

$$\left.\begin{aligned} u(t) &= \sum_{n=0}^{\infty} p^n u_n(t) \\ v(t) &= \sum_{n=0}^{\infty} p^n v_n(t) \end{aligned}\right\}. \tag{14}$$

Here, $p \in [0, 1]$ is a homotopy parameter.

Nonlinear terms are expressed as

$$\left.\begin{aligned} Nu(t) &= \sum_{n=0}^{\infty} p^n H_n(u) \\ Nv(t) &= \sum_{n=0}^{\infty} p^n H_n'(v) \end{aligned}\right\}. \tag{15}$$

Here, H_n and H_n' denote He's polynomials of $u_0, u_1, u_2, \ldots, u_n$ and $v_0, v_1, v_2, \ldots, v_n$ given by

$$\left.\begin{aligned} &H_n(u_0, u_1, u_2, \ldots) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0} \\ &\quad n = 0, 1, 2, 3, \ldots \\ &H_n'(v_0, v_1, v_2, \ldots) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right]_{p=0} \\ &\quad n = 0, 1, 2, 3, \ldots \end{aligned}\right\}. \tag{16}$$

Using eqs (15) and (16) in eq. (13) and applying HPM, we get

$$\left.\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(t) &= G_1(t) - pL^{-1}\big[p^{-\alpha}L\{R_1(u,v) \\ &\quad +Q_1(u,v)\}\big], \\ \sum_{n=0}^{\infty} p^n v_n(t) &= G_2(t) - pL^{-1}\big[p^{-\alpha}L\{R_2(u,v) \\ &\quad +Q_2(u,v)\}\big] \end{aligned}\right\}. \tag{17}$$

This is a pairing of HPM and transform of Laplace using He's polynomials.

Equating coefficients of identical powers of p on each side, we get

$$\left.\begin{aligned} &p^0 : u_0(t) = G_1(t), \quad v_0(t) = G_2(t) \\ &p^n : u_n(t) = -L^{-1}\big[p^{-\alpha}L\{R_1(u_{n-1}, v_{n-1}) \\ &\qquad +H_{n-1}(u,v)\}\big], \quad n > 0, n \in \mathbb{N} \\ &v_n(t) = -L^{-1}\big[p^{-\alpha}L\{R_2(u_{n-1}, v_{n-1}) \\ &\qquad +H_{n-1}'(u,v)\}\big], \quad n > 0, n \in \mathbb{N} \end{aligned}\right\}. \tag{18}$$

Continuing in this way, the enduring components can completely be achieved also. Therefore, the series solution is fully calculated. At last, the analytical solution is approximated by the series

$$\left.\begin{aligned} u(t) &= \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n(t) \\ v(t) &= \lim_{p\to 1}\sum_{n=0}^{\infty} p^n v_n(t) \end{aligned}\right\}. \tag{19}$$

The above solutions in series converge rapidly.

5.1 *Convergence analysis of FHPTM*

We focus on the convergence of the proposed FHPTM applied to eq. (1) in §4. Sufficient conditions for its convergence are presented.

Series (19) is convergent for most of the cases. Still, ensuing suggestions were specified by He's to treasure convergence rate on the nonlinear operator.

(1) Second derivatives of Nu and Nv with respect to u and v, respectively, must be small as parameter can be relatively large, i.e. $p \to 1$.
(2) Norm of $L^{-1}(\partial N/\partial u)$ and $L^{-1}(\partial N/\partial v)$ must be less than 1 so that the series converges.

Theorem 4 [21]. *Let X and Y be Banach spaces and $T : X \to Y$, be a contraction nonlinear mapping, i.e. $\forall \xi, \tilde{\xi} \in X \,\|T(\xi) - T(\tilde{\xi})\| \le \chi \|\xi - \tilde{\xi}\|, 0 < \chi < 1$, which by Banach fixed point theorem, having fixed point u, i.e. $T(u) = u$.*

Sequence created by HPM is regarded as $V_n = T(V_{n-1})$, $V_{n-1} = \sum_{i=0}^{n-1} u_i$, $n = 1, 2, 3, \ldots$ and suppose $V_0 = v_0 = u_0 \in B_r(u)$ where $B_r(u) = \{u^* \in X \mid \|u^* - u\| < r\}$, then

(i) $\|V_n - u\| \le \chi^n \|v_0 - u\|$,
(ii) $V_n \in B_r(u)$ and
(iii) $\lim_{n\to\infty} V_n = u$.

5.2 *Implementation of FHPTM*

We show the applicability and efficiency of FHPTM for examining an inhomogeneous nonlinear time-fractional coupled dynamical system of ODEs given as

$$\left.\begin{aligned} D_t^{\alpha} u &= -a_1 u + b_1 v(1-\epsilon v^2) + c_1 \\ D_t^{\alpha} v &= -a_2 v + b_2 u(1-\epsilon u^2) + c_2 \end{aligned}\right\}, \tag{20}$$

with conditions $u(0) = 0 = v(0)$ and $0 < \alpha \le 1$. a_i, b_i, c_i $(1 \le i \le 2)$ and ϵ are constants.

To solve eq. (20), we apply transform of Laplace on each side and using initial conditions as

$$\left.\begin{aligned} u(p) &= \frac{c_1}{p^{\alpha+1}} + p^{-\alpha} L\big[-a_1 u + b_1 v\big(1-\epsilon v^2\big)\big] \\ v(p) &= \frac{c_2}{p^{\alpha+1}} + p^{-\alpha} L\big[-a_2 v + b_2 u\big(1-\epsilon u^2\big)\big] \end{aligned}\right\}. \tag{21}$$

Taking the inverse Laplace transform, we get

$$\left.\begin{aligned} u(t) &= \frac{c_1 t^{\alpha}}{\Gamma(1+\alpha)} \\ &\quad + L^{-1}\big[p^{-\alpha} L\big\{-a_1 u + b_1 v\big(1-\epsilon v^2\big)\big\}\big] \\ v(t) &= \frac{c_2 t^{\alpha}}{\Gamma(1+\alpha)} \\ &\quad + L^{-1}\big[p^{-\alpha} L\big\{-a_2 v + b_2 u\big(1-\epsilon u^2\big)\big\}\big] \end{aligned}\right\}. \tag{22}$$

Applying HPM on eq. (22), we have

$$\left.\begin{aligned} \sum p^n u_n(t) &= \frac{c_1 t^{\alpha}}{\Gamma(1+\alpha)} + pL^{-1}\big[p^{-\alpha} L\big\{-a_1 u \\ &\quad + b_1 v\big(1-\epsilon v^2\big)\big\}\big] \\ \sum p^n v_n(t) &= \frac{c_2 t^{\alpha}}{\Gamma(1+\alpha)} + pL^{-1}\big[p^{-\alpha} L\big\{-a_2 v \\ &\quad + b_2 u\big(1-\epsilon u^2\big)\big\}\big] \end{aligned}\right\}. \tag{23}$$

Equating coefficients of like powers of p on each side of eq. (23), we have

$$p^0\colon u_0(t) = \frac{c_1 t^{\alpha}}{\Gamma(1+\alpha)}, \quad v_0(t) = \frac{c_2 t^{\alpha}}{\Gamma(1+\alpha)},$$

$$p^1\colon u_1(t) = \frac{-a_1 c_1 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{b_1 c_2 t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{b_1 c_2^3 t^{4\alpha} \epsilon \Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+4\alpha)},$$

$$v_1(t) = \frac{-a_2 c_2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{b_2 c_1 t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{b_2 c_1^3 t^{4\alpha} \epsilon \Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+4\alpha)},$$

$$\begin{aligned} p^2\colon u_2(t) &= \frac{(a_1^2 c_1 + b_1 b_2 c_1 - a_1 b_1 c_2 - a_2 b_1 c_2) t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &\quad - \frac{b_1 (b_2 c_1^3 - a_1 c_2^3) t^{5\alpha} \epsilon \Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+5\alpha)} \\ &\quad - \frac{b_1 (b_2 c_1 - a_2 c_2)^3 t^{7\alpha} \epsilon \Gamma(1+6\alpha)}{\Gamma(1+2\alpha)^3 \Gamma(1+7\alpha)} \\ &\quad + \frac{3 b_1 b_2 c_1^3 (b_2 c_1 - a_2 c_2)^2 t^{9\alpha} \epsilon^2 \Gamma(1+3\alpha) \Gamma(1+8\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+2\alpha)^2 \Gamma(1+4\alpha) \Gamma(1+9\alpha)} \\ &\quad + \frac{3 b_1 b_2^2 c_1^6 (b_2 c_1 - a_2 c_2) t^{11\alpha} \epsilon^3 \Gamma(1+3\alpha)^2 \Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6 \Gamma(1+2\alpha) \Gamma(1+4\alpha)^2 \Gamma(1+11\alpha)} \\ &\quad + \frac{b_1 b_2^3 c_1^9 t^{13\alpha} \epsilon^4 \Gamma(1+3\alpha)^3 \Gamma(1+12\alpha)}{\Gamma(1+\alpha)^9 \Gamma(1+4\alpha)^3 \Gamma(1+11\alpha)}, \end{aligned}$$

$$\begin{aligned} v_2(t) &= \frac{(-a_1 b_2 c_1 - a_2 b_2 c_1 + a_2^2 c_2 + b_1 b_2 c_2) t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &\quad + \frac{b_2 (a_2 c_1^3 - b_1 c_2^3) t^{5\alpha} \epsilon \Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+5\alpha)} \\ &\quad + \frac{b_2 (a_1 c_1 - b_1 c_2)^3 t^{7\alpha} \epsilon \Gamma(1+6\alpha)}{\Gamma(1+2\alpha)^3 \Gamma(1+7\alpha)} \\ &\quad + \frac{3 b_1 b_2 c_2^3 (a_1 c_1 - b_1 c_2)^2 t^{9\alpha} \epsilon^2 \Gamma(1+3\alpha) \Gamma(1+8\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+2\alpha)^2 \Gamma(1+4\alpha) \Gamma(1+9\alpha)} \\ &\quad + \frac{3 b_2 b_1^2 c_2^6 (a_1 c_1 - b_1 c_2) t^{11\alpha} \epsilon^3 \Gamma(1+3\alpha)^2 \Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6 \Gamma(1+2\alpha) \Gamma(1+4\alpha)^2 \Gamma(1+11\alpha)} \\ &\quad + \frac{b_2 b_1^3 c_2^9 t^{13\alpha} \epsilon^4 \Gamma(1+3\alpha)^3 \Gamma(1+12\alpha)}{\Gamma(1+\alpha)^9 \Gamma(1+4\alpha)^3 \Gamma(1+13\alpha)}. \end{aligned}$$

Next components of the solution can be found using Mathematica package. Therefore, the solution is

$$\left.\begin{aligned} u(t) &= \lim_{p\to 1} \sum_{n=0}^{\infty} p^n u_n(t) \\ v(t) &= \lim_{p\to 1} \sum_{n=0}^{\infty} p^n v_n(t) \end{aligned}\right\}. \tag{24}$$

6. Numerical results and discussion

Numerical simulations are carried out for both state variables $u(t)$ and $v(t)$ at distinct fractional Brownian motions given by $\alpha = 0.75, 0.5, 0.25$ and the standard motion $\alpha = 1$. The numerical approximate solutions of dynamical nonlinear time-fractional model of relationships are obtained by applying FVIM and FHPTM using the parameters $a_1 = 0.05, b_1 = 0.04, c_1 = 0.2, a_2 = 0.07, b_2 = 0.06, c_2 = 0.3$ and $\varepsilon = 0.01$. These numerical results are compared those obtained from q-HASTM and ADM as shown in tables 1 and 2. From these tables, it can be perceived that solution by FVIM and FHPTM at distinct grid points is in a very good agreement. Tables 3 and 4 show that the absolute error in $u(t)$ and $v(t)$ between consecutive iterations is negligible and reduces to zero as the number of iterations increases for diverse values of fractional order α. Hence, we get an inference that FVIM and FHPTM

Table 1. Comparison among FHPTM, FVIM, q-HASTM and ADM for u at different values of t when $\alpha = 1$, $a_1 = 0.05$, $b_1 = 0.04$, $c_1 = 0.2$, $a_2 = 0.07$, $b_2 = 0.06$, $c_2 = 0.3$ and $\epsilon = 0.01$.

t	FHPTM solution	FVIM solution	q-HASTM solution [9]	ADM solution [9]
0	0	0	0	0
2	0.5829269235	0.5829798397	0.5829274000	0.5829289274
4	1.1348238851	1.1356623036	1.134838400	1.134891434
6	1.6590147543	1.6632168681	1.659119400	1.659550026
8	2.1569971387	2.1701479503	2.157414400	2.159330141
10	2.6284249746	2.6602389625	2.629625000	2.635727526
12	3.0711102653	3.1365549432	3.073910400	3.089599578
14	3.4810466296	3.6015225079	3.486687400	3.521399539
16	3.8524586546	4.0571000495	3.862630400	3.931326626
18	4.1778828109	4.5050409566	4.194671400	4.319398257
20	4.4482882407	4.9472377208	4.474000000	4.685515415

Table 2. Comparison among FHPTM, FVIM, q-HASTM and ADM for v at different values of t when $\alpha = 1$, $a_1 = 0.05$, $b_1 = 0.04$, $c_1 = 0.2$, $a_2 = 0.07$, $b_2 = 0.06$, $c_2 = 0.3$ and $\epsilon = 0.01$.

t	HPTM solution	FVIM solution	q-HASTM solution [9]	ADM solution [9]
0	0	0	0	0
2	0.4033793415	0.4033470590	0.4033788000	0.4033806333
4	0.8109843482	0.8105148646	0.8109674666	0.8110188698
6	1.2189279147	1.2167810965	1.218802800	1.219148908
8	1.6232483995	1.6171661535	1.622732800	1.624049986
10	2.0199588477	2.0067697404	2.018416666	2.022138493
12	2.4050950676	2.3810760260	2.401324800	2.410163488
14	2.7747665001	2.7362278879	2.766738800	2.785587314
16	3.1252142165	3.0692672718	3.109751466	3.147219587
18	3.4528810743	3.3783356037	3.425266800	3.496167079
20	3.7545004984	3.6628256642	3.708000000	3.837175265

Table 3. Comparison of absolute error in $u(t)$ between consecutive approximations when the exact solution is not known.

t	$\alpha = 1$ (FVIM)		$\alpha = 1$ (FHPTM)		$\alpha = 0.50$ (FVIM)		$\alpha = 0.50$ (FHPTM)	
	$\lvert u_2 - u_1\rvert$	$\lvert u_3 - u_2\rvert$	$\lvert u_2 - u_1\rvert$	$\lvert u_3 - u_2\rvert$	$\lvert u_2 - u_1\rvert$	$\lvert u_3 - u_2\rvert$	$\lvert u_2 - u_1\rvert$	$\lvert u_3 - u_2\rvert$
0	0	0	0	0	0	0	0	0
0.2	3.99×10^{-5}	6.13×10^{-7}	4.06×10^{-5}	6.16×10^{-7}	3.99×10^{-4}	3.09×10^{-5}	4.30×10^{-4}	3.20×10^{-5}
0.4	1.59×10^{-4}	4.90×10^{-6}	1.64×10^{-4}	4.96×10^{-6}	7.98×10^{-4}	8.74×10^{-5}	8.85×10^{-4}	9.17×10^{-5}
0.6	3.59×10^{-4}	1.65×10^{-5}	3.76×10^{-4}	1.68×10^{-5}	1.19×10^{-3}	1.60×10^{-4}	1.35×10^{-3}	1.70×10^{-4}
0.8	6.38×10^{-4}	3.92×10^{-5}	6.78×10^{-4}	4.01×10^{-5}	1.59×10^{-3}	2.47×10^{-4}	1.84×10^{-3}	2.64×10^{-4}
1.0	9.97×10^{-4}	7.65×10^{-5}	1.07×10^{-3}	7.88×10^{-5}	1.98×10^{-3}	3.45×10^{-4}	2.33×10^{-3}	3.72×10^{-4}

Table 4. Comparison of absolute error in $v(t)$ between consecutive approximations when the exact solution is not known.

t	$\alpha = 1$ (FVIM)		$\alpha = 1$ (FHPTM)		$\alpha = 0.50$ (FVIM)		$\alpha = 0.50$ (FHPTM)	
	$\lvert v_2 - v_1\rvert$	$\lvert v_3 - v_2\rvert$	$\lvert v_2 - v_1\rvert$	$\lvert v_3 - v_2\rvert$	$\lvert v_2 - v_1\rvert$	$\lvert v_3 - v_2\rvert$	$\lvert v_2 - v_1\rvert$	$\lvert v_3 - v_2\rvert$
0	0	0	0	0	0	0	0	0
0.2	1.80×10^{-4}	9.99×10^{-7}	1.81×10^{-4}	1.00×10^{-6}	1.80×10^{-3}	5.04×10^{-5}	1.85×10^{-3}	5.20×10^{-5}
0.4	7.20×10^{-4}	7.99×10^{-6}	7.28×10^{-4}	8.08×10^{-6}	3.60×10^{-3}	1.42×10^{-4}	3.74×10^{-3}	1.49×10^{-4}
0.6	1.62×10^{-3}	2.69×10^{-5}	1.64×10^{-3}	2.74×10^{-5}	5.40×10^{-3}	2.62×10^{-4}	5.66×10^{-3}	2.76×10^{-4}
0.8	2.88×10^{-3}	6.39×10^{-5}	2.94×10^{-3}	6.53×10^{-5}	7.20×10^{-3}	4.03×10^{-4}	7.60×10^{-3}	4.29×10^{-4}
1.0	4.50×10^{-3}	1.24×10^{-4}	4.62×10^{-3}	1.28×10^{-4}	9.00×10^{-3}	5.63×10^{-4}	9.56×10^{-3}	6.04×10^{-4}

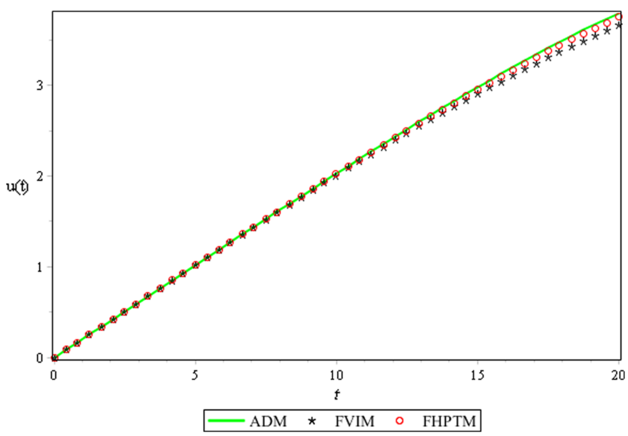

Figure 1. Comparison of approximate solutions $u(t)$ at different values of t for $\alpha = 1$.

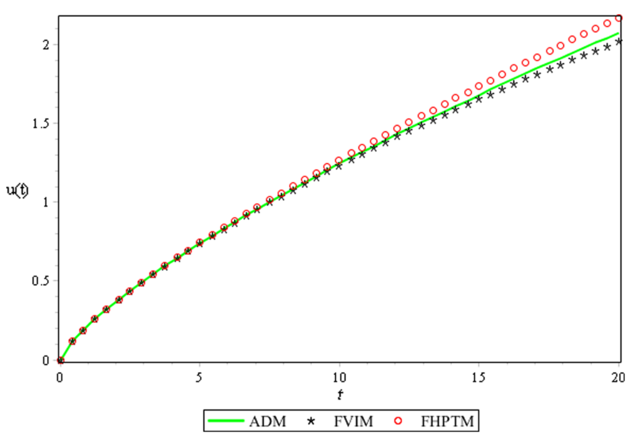

Figure 2. Comparison of approximate solutions $u(t)$ at different values of t for $\alpha = 0.75$.

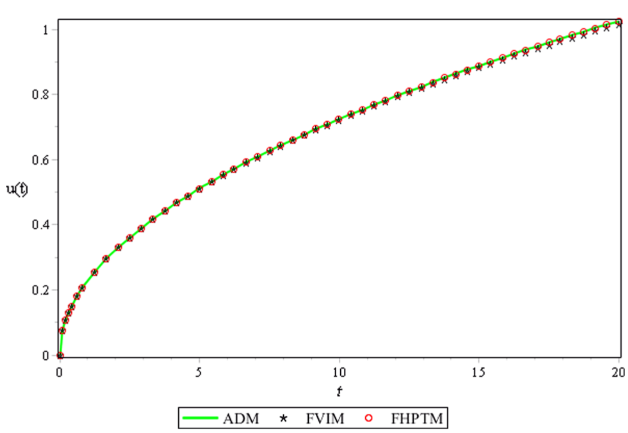

Figure 3. Comparison of approximate solutions $u(t)$ at different values of t for $\alpha = 0.50$.

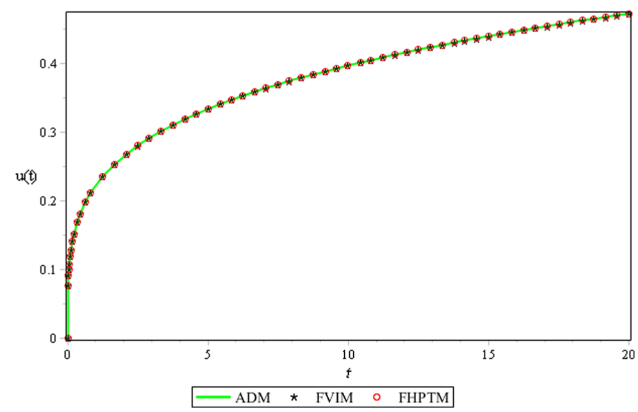

Figure 4. Comparison of approximate solutions $u(t)$ at different values of t for $\alpha = 0.25$.

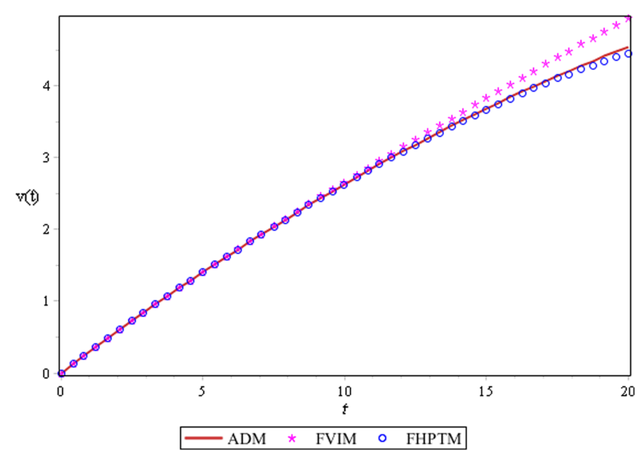

Figure 5. Comparison of approximate solutions $v(t)$ at different values of t for $\alpha = 1$.

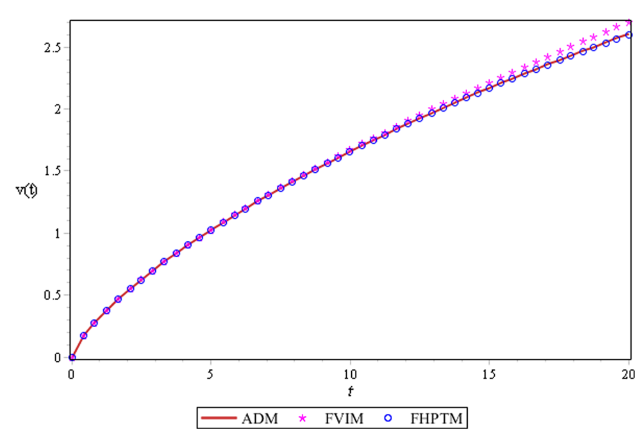

Figure 6. Comparison of approximate solutions $v(t)$ at different values of t for $\alpha = 0.75$.

also work for those fractional models that do not possess exact solution. Figures 1–8 depict the approximate behaviour of $u(t)$ and $v(t)$ for different values of fractional order α. Values of $u(t)$ and $v(t)$ have been taken up to third approximation at different values of t using FVIM, FHPTM and ADM. Figures 1–4 reveal that by increasing α in fractional dynamical model, the love of the first person increases for his/her partner. Figures 5–8 illustrate that by decreasing α in the same model, love of the second person decreases for his/her

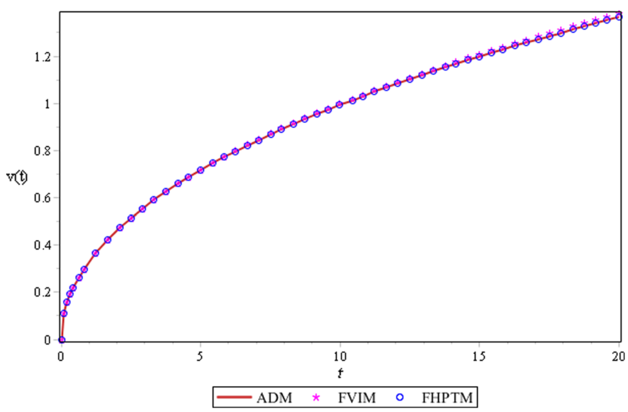

Figure 7. Comparison of approximate solutions $v(t)$ at different values of t for $\alpha = 0.50$.

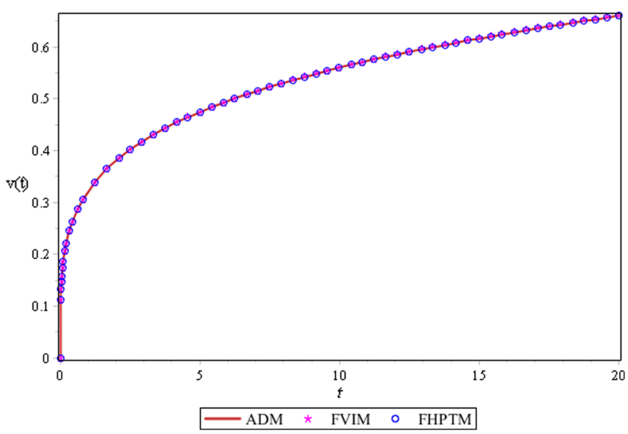

Figure 8. Comparison of approximate solutions $v(t)$ at different values of t for $\alpha = 0.25$.

partner. It is observed that results obtained by the proposed schemes converge faster in comparison to other existing numerical schemes.

7. Conclusions

In this paper, FVIM is successfully applied to obtain a rapidly convergent approximate numerical solution of coupled nonlinear dynamical fractional model of romantic and interpersonal relationships for marriages. Results obtained are compared with those from FHPTM and ADM. It is seen that FVIM is capable of reducing the size of calculations and is easy to use for both small and large parameters in nonlinear fractional problems. Numerical results indicate that FVIM works very well, even if lower-order approximations are used. However, the accuracy can be improved by using higher-order approximations in the solution. It should be noted that FVIM is used directly without using linearisation, perturbation, Adomian polynomials or other restrictive assumptions. It is shown by convergence analysis that the obtained numerical solutions are positive and bounded. Sufficient conditions for the convergence of the methods are established. It is interesting to note that the proposed method also works efficiently when exact solution for integer-order model is unknown. Therefore, we conclude that the presented scheme is highly reliable and powerful in getting numerical approximate solutions for distinct classes of nonlinear fractional models of the system of differential equations.

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