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Spherically symmetric wormholes of embedding class one

PETER K F KUHFITTIG

Department of Mathematics, Milwaukee School of Engineering, Milwaukee, WI 53202-3109, USA
E-mail: kuhfitti@msoe.edu

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Abstract. This paper generalises an earlier result by the author based on the well-established embedding theorems that connect the classical theory of relativity to higher-dimensional space–times. In particular, an n-dimensional Riemannian space is said to be of class m if $m + n$ is the lowest dimension of the flat space in which the given space can be embedded. To study traversable wormholes, we concentrate on spacetimes that can be reduced to embedding class one by a suitable transformation. It is subsequently shown that the extra degrees of freedom from the embedding theory provide the basis for a complete wormhole solution in the sense of obtaining both the redshift and shape functions.

Keywords. Traversable wormholes; higher-dimensional Riemannian space; embedding class one.

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1. Introduction

Wormholes are handles or tunnels in space–time that are able to connect widely separated regions of our Universe and may even connect entirely different Universes [1]. Such wormholes can be described by the static and spherically symmetric line element

$$ds^2 = -\,e^{\nu(r)}dt^2 + \frac{dr^2}{1-b(r)/r} + r^2(d\theta^2+\sin^2\theta\, d\phi^2), \quad (1)$$

using units in which $c = G = 1$. Here $\nu = \nu(r)$ is called the 'redshift function', which must be finite everywhere to avoid an event horizon. The function $b = b(r)$ is called the 'shape function'. The spherical surface $r = r_0$ is the 'throat' of the wormhole. Here $b(r)$ must satisfy the following conditions: $b(r_0) = r_0$, $b(r) < r$ for $r > r_0$ and $b'(r_0) < 1$, called the 'flare-out condition'. We also wish to assume that $b'(r) > 0$ due to the field equation $8\pi\rho(r) = b'(r)/r^2$, where ρ is the energy density, normally considered to be positive. The flare-out condition can only be satisfied by violating the null energy condition (NEC), discussed in §5. For a Morris–Thorne wormhole, this violation requires the use of 'exotic matter'.

The discussion in [1] is based on the following strategy: specify the geometric conditions required for a traversable wormhole and then either manufacture or do a search for matter or fields that can produce the desired energy–momentum tensor. The main goal of this paper is to reverse this strategy by showing that the conditions discussed are sufficient for producing a complete solution, i.e. for producing both the redshift and shape functions. The approach in this paper differs significantly from that in [2], which discusses charged wormholes admitting a one-parameter group of conformal motions, together with a new model to explain the flat galactic rotation curves without the need for dark matter.

2. The embedding

Unlike [2], the conditions discussed in this paper are derived directly from the assumption that the space–time is of embedding class one. Here, we need to recall that an n-dimensional Riemannian space is said to be of embedding class m if $m + n$ is the lowest dimension of the flat space in which the given space can be embedded [3–8]. It is well known that the exterior Schwarzschild solution is a Riemannian space of embedding class two. Following [3], we start with the static and spherically symmetric line element

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (2)$$

(For physical reasons, it is generally assumed that $\nu(r)$ is finite and that $\lim_{r\to\infty}\nu(r) = 0$.) It is shown in

[3] that this metric of class two can be reduced to a metric of class one and can therefore be embedded in a five-dimensional flat space–time. The following transformation can accomplish this reduction: $z^1 = r\sin\theta\cos\phi$, $z^2 = r\sin\theta\sin\phi$, $z^3 = r\cos\theta$, $z^4 = \sqrt{K}\,\mathrm{e}^{\nu/2}\cosh(t/\sqrt{K})$ and $z^5 = \sqrt{K}\,\mathrm{e}^{\nu/2}\sinh(t/\sqrt{K})$. The result is [3]

$$\mathrm{d}s^2 = \mathrm{e}^{\nu}\mathrm{d}t^2 - \left(1 + \frac{K\mathrm{e}^{\nu}}{4}\,\nu'^2\right)\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{3}$$

Metric (3) is therefore equivalent to metric (2) if

$$\mathrm{e}^{\lambda} = 1 + \frac{K\mathrm{e}^{\nu}}{4}\,\nu'^2, \tag{4}$$

where $K > 0$ is a free parameter. The condition is equivalent to the following condition due to Karmarkar [9]:

$$R_{1414} = \frac{R_{1212}R_{3434} - R_{1224}R_{1334}}{R_{2323}}, \quad R_{2323} \neq 0. \tag{5}$$

(See [10] for further details.) Hence, while eq. (5) provides justification for the above embedding process, eq. (3) yields a useful mathematical model, helped by the free parameter K. Moreover, this model is consistent with the induced-matter theory in [11], which is discussed further in §5.

3. The solution

To produce the desired wormhole solution, we prefer the opposite signature in line element (2) in order to be consistent with line element (1):

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu(r)}\mathrm{d}t^2 + \mathrm{e}^{\lambda(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{6}$$

Most importantly, no additional assumptions will be made regarding $\nu = \nu(r)$.

The shape function $b = b(r)$ has to incorporate eq. (4) on account of the embedding. An entire class of such shape functions can be readily obtained by inspection:

$$b(r) = r\left(1 - \frac{1}{1 + \frac{1}{4}K\mathrm{e}^{\nu(r)}[\nu'(r)]^2}\right) + \frac{r^n/r_0^{n-1}}{1 + \frac{1}{4}K\mathrm{e}^{\nu(r_0)}[\nu'(r_0)]^2}. \tag{7}$$

Observe that $b(r_0) = r_0$ for all n. Our main task is to show that these shape functions satisfy all the other requirements of shape functions, which is the topic of the next section.

4. The condition $0 < b'(r_0) < 1$

As noted in §1, we need to examine the condition $0 < b'(r_0) < 1$. Hence, we start with

$$\begin{aligned} b'(r_0) = 1 &- \frac{1}{1 + \frac{1}{4}K\mathrm{e}^{\nu(r_0)}[\nu'(r_0)]^2} \\ &+ \frac{n}{1 + \frac{1}{4}K\mathrm{e}^{\nu(r_0)}[\nu'(r_0)]^2} \\ &+ r_0 1\left(1 + \frac{1}{4}K\mathrm{e}^{\nu(r_0)}[\nu'(r_0)]^2\right)^{-2} \\ &\times \frac{1}{4}K\mathrm{e}^{\nu(r_0)}\left(2\nu'(r_0)\nu''(r_0) + [\nu'(r_0]^3\right). \end{aligned} \tag{8}$$

To simplify the analysis, let us introduce the following notations:

$$A = \tfrac{1}{4}\mathrm{e}^{\nu(r_0)}[\nu'(r_0)]^2 \tag{9}$$

and

$$\Omega = \mathrm{e}^{\nu(r_0)}(2\nu'(r_0)\nu''(r_0) + [\nu'(r_0]^3). \tag{10}$$

In view of eq. (8), the condition $0 < b'(r_0) < 1$ now yields

$$\frac{\left(\frac{1-n}{1+AK} - 1\right)(1 + AK)^2}{\frac{1}{4}r_0K} < \Omega < \frac{(1-n)(1+AK)}{\frac{1}{4}r_0K}. \tag{11}$$

In the trivial case $\nu'(r_0) = 0$, the condition $0 < b'(r_0) < 1$ is satisfied provided that $0 < n < 1$. Accordingly, we need to concentrate on the non-trivial case $\nu'(r_0) \neq 0$. As a result, A is positive but Ω can be positive or negative. So the right-hand side of inequality (11) is equivalent to the flare-out condition $b'(r) < 1$ at or near the throat, while the left-hand side is equivalent to $b'(r_0) > 0$. We shall consider the two cases separately.

4.1 *The condition $b'(r_0) < 1$*

To analyse the flare-out condition, we need to consider the two cases, $\Omega > 0$ and $\Omega < 0$. To this end, let us rewrite the right-hand side of inequality (11) as follows:

$$K\left(\frac{1}{4}r_0\Omega - A(1-n)\right) < 1 - n. \tag{12}$$

If $\Omega > 0$, then n must be less than 1 to keep K positive. It also becomes apparent that r_0 is another free parameter. Hence, we can choose r_0 large enough so that

$$\frac{1}{4}r_0\Omega > A(1-n).$$

As a result,

$$K < \frac{1-n}{\frac{1}{4}r_0\Omega - A(1-n)}, \quad n < 1. \tag{13}$$

In inequality (12), if $\Omega < 0$, then we must have $n > 1$ to keep K positive. This time we need to choose r_0 sufficiently large so that

$$\frac{1}{4}r_0|\Omega| > -A(1-n).$$

The result is

$$K > \frac{1-n}{\frac{1}{4}r_0\Omega - A(1-n)}, \quad n > 1. \tag{14}$$

It should be noted that conditions (13) and (14) for the free parameter K can always be met by increasing the throat size of the wormhole. Observe also that $n \neq 1$.

4.2 *The condition $b'(r_0) > 0$*

The left-hand side of inequality (11) is more difficult to analyse since, after simplifying, we get the quadratic inequality

$$A^2K^2 + K\left(\frac{1}{4}r_0\Omega + A + nA\right) + n > 0. \tag{15}$$

Once again we need to consider the two cases $\Omega > 0$, $n < 1$ and $\Omega < 0, n > 1$.

$\Omega > 0, n < 1$:

If $\Omega > 0$ and $0 \leq n < 1$, inequality (15) is automatically satisfied and we have $b'(r_0) > 0$.

If $n < 0$, we first need to solve the quadratic inequality to obtain

$$K < \frac{-(\frac{1}{4}r_0\Omega+A+nA)-\sqrt{(\frac{1}{4}r_0\Omega+A+nA)^2-4A^2n}}{2A^2} \tag{16}$$

or

$$K > \frac{-(\frac{1}{4}r_0\Omega+A+nA)+\sqrt{(\frac{1}{4}r_0\Omega+A+nA)^2-4A^2n}}{2A^2}. \tag{17}$$

Algebraically, the solution is valid for both $\Omega > 0$ and $\Omega < 0$. Because of the 'or', only one of the inequalities is actually needed. (As K has to be positive, the first inequality is unphysical anyway.) For the second inequality, $K > 0$ because $n < 0$. We conclude that for $\Omega > 0$, $n < 0$, the parameter K must satisfy the following inequality:

$$\frac{-(\frac{1}{4}r_0\Omega + A + nA) + \sqrt{(\frac{1}{4}r_0\Omega + A + nA)^2 - 4A^2n}}{2A^2} < K < \frac{1-n}{\frac{1}{4}r_0\Omega - A(1-n)}, \quad n < 0, \tag{18}$$

referring back to inequality (13). Hence if $\Omega > 0$ and $n < 0$, then K must lie between the two positive values. We therefore have a solution for the case $\Omega > 0$, $n < 1$.

$\Omega < 0, n > 1$:

For the case $\Omega < 0$, $n > 1$, the real difficulty is that solutions (16) and (17) may not be real. To avoid this problem, let us choose the free parameter r_0 sufficiently large to start with, i.e. choose r_0 so that $\frac{1}{4}r_0\Omega = -bA$ for some sufficiently large positive constant b to obtain

$$(-bA + A + nA)^2 - 4nA^2 > 0, \tag{19}$$

thereby resulting in a real solution. Consequently, inequality (16) yields

$$K < \frac{-(-b+1+n) - \sqrt{(-b+1+n)^2 - 4n}}{2A} \tag{20}$$

while inequality (14) gives

$$K > \frac{1-n}{\frac{1}{4}r_0\Omega - A(1-n)} = \frac{2(1-n)/(-b-1+n)}{2A}. \tag{21}$$

(Inequality (17) is not needed.)

The significance of the conditions on K can best be seen graphically. Figure 1 shows that for any fixed n,

$$f_1(b) = -(-b+1+n) - \sqrt{(-b+1+n)^2 - 4n} > f_2(b) = \frac{2(1-n)}{-b-1+n}, \tag{22}$$

referring to inequalities (20) and (21). Hence, once again K must lie between two positive values. We, therefore, have a solution for the case $\Omega < 0$, $n > 1$, as well.

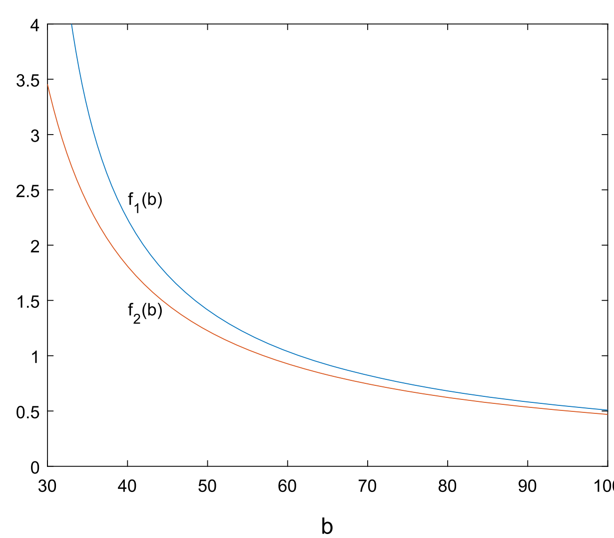

Figure 1. Plots showing $f_1(b)$ and $f_2(b)$.

5. Other conditions

Having shown that the flare-out condition $b'(r_0) < 1$ has been met, let us return to the violation of the NEC, which states that for the energy–momentum tensor $T_{\alpha\beta}$,

$$T_{\alpha\beta}\mu^\alpha\mu^\beta \geq 0$$

for all null vectors. Given the radial outgoing null vector $(1, 1, 0, 0)$, we have $\rho(r_0) + p_r(r_0) < 0$ whenever the condition is violated. By Morris and Thorne [1], this violation is equivalent to the condition

$$\frac{b'(r_0) - b(r_0)/r_0}{2[b(r_0)]^2} < 0, \tag{23}$$

which holds whenever $b'(r) < 1$ at or near the throat. As noted in §1, for a Morris–Thorne wormhole, the violation of the NEC requires the use of 'exotic matter', because ordinary matter normally satisfies the NEC. We have seen, however, that the shape functions and subsequent flare-out conditions were obtained from the embedding theory, which may be viewed as a part of the induced-matter theory [11] in the following sense: according to Wesson [12], the field equations for the five-dimensional flat embedding space yield the Einstein field equations in four dimensions 'containing matter'. The induced-matter theory, therefore, implies that the matter in our Universe actually comes from the geometry and this may very well include the exotic matter. Hence, when the exotic matter cannot be avoided, it may be less problematical in the present context.

Our final observation concerns asymptotic flatness. Because $\nu(r) \to 0$ as $r \to \infty$, we also have $\lim_{r\to\infty}\nu'(r) = 0$. Hence if $n < 1$, we see from eq. (7) that $b(r)/r \to 0$ (in addition to $e^{\nu(r)} \to 1$), resulting in an asymptotically flat space–time.

Unfortunately, this conclusion does not hold for $n>1$. Then the wormhole space–time has to be cut off at some $r = a$ and joined to an external Schwarzschild space–time

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{24}$$

in the usual way. From $e^{\nu(a)} = 1 - 2M/a$, we have $2M = a(1 - e^{\nu(a)})$. But $2M = b(a)$, and so the cut-off at $r = a$ is implicitly determined by the equation $b(a) = (1 - e^{\nu(a)})$, provided, of course, that such a solution exists.

6. Conclusions

An n-dimensional Riemannian space is said to be of embedding class m if $m + n$ is the lowest dimension of the flat space in which the given space can be embedded. Following [3], we assume a spherically symmetric metric of embedding class two that can be reduced to class one by a suitable transformation.

These ideas were applied for obtaining a complete wormhole solution without the usual engineering considerations, i.e. without being required to find or to manufacture matter or fields that produce the desired energy–momentum tensor. The free parameters K and r_0 provided the extra degrees of freedom to obtain both the redshift and shape functions from the embedding theory and may even account for the exotic matter.

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