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Long-term dynamics of a q-deformed discrete susceptible–infected–susceptible epidemic model with delay

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Abstract. The main thrust of this paper is to consider a delayed q-deformed discrete susceptible–infected–susceptible (SIS) epidemic model. Parametric conditions on the local stability of the disease-free fixed point and the endemic fixed points are obtained. A codimension-one bifurcation analysis at the fixed points of the model is discussed. The model has a variety of bifurcations such as flip, transcritical, and pitchfork bifurcations. Numerical simulations including trajectories, bifurcation diagrams, maximal Lyapunov exponent, and phase portraits are illustrated to verify the obtained analytical results. It has been noticed that introducing the delay in the absence of deformations recovers the chaotic behaviour of the model. Meanwhile, introducing both deformations and delay suppress the chaotic behaviour of the model. The disease will be eradicated by increasing the value of both deformation and delay strength parameters.

Keywords. Susceptible–infected–susceptible epidemic model; q-deformation; delay; bifurcation; chaos.

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1. Introduction

Recently, q-deformed physical systems have attracted many researchers due to the emergence of the so-called quantum group structures in some physical problems [1–3]. For instance, it is observed in [3] that the constructed one-dimensional map for the Belousov–Zhabotinskii reaction in a stirred chemical reactor has a striking similarity to the q-deformed logistic map for negative deformation value. Thus, q-deformed numbers and functions succeeded in explaining many experimental observations [4–6]. The q-deformation is non-trivial in the sense that the emerging deformed algebra is no longer linear. Applications of the theory of quantum groups are effective in clarifying many complex physical phenomena. Following the applications, quantum group theory has attracted considerable interest of physicists and mathematicians towards the special branch of mathematics dealing with q-deformed systems [7–11]. The q-deformed logistic map considered in [11] exhibits a variety of dynamic behaviours such as fixed points, periodic and surprisingly the coexistence of attractors which rarely happen in one-dimensional nonlinear maps. In [9], a q-deformed Henon map is studied and the authors succeeded in obtaining the desired behaviour by slightly changing the deformation parameter without disturbing the canonical form of the system. In [12], q-deformation is employed to describe the population of the larch budmoth insect. As it is usually assumed that the host density is totaly converted into parasitoid density which results in over-counting of the parasitoid numbers, the authors used q-deformation to avoid the over-counting of this problem by taking $q_{\text{parasitoid}} \neq 1$.

Mathematical models have played significant roles in describing the dynamical evolution of infectious diseases. SIS model is one of the simplest epidemic models with compartment structure in which infectious diseases such as gonorrhea and malaria are transmitted without immunity. In the SIS model, the host population is divided into two epidemiological groups: the susceptible and the infective. Indeed, discrete models have been used to formulate some epidemic models because they possess their unique dynamic characteristics such as period-doubling (flip) bifurcation and chaotic behaviour in low dimension [13–18].

In this paper, we consider the following simple discrete SIS epidemic model with surprising dynamics [19]

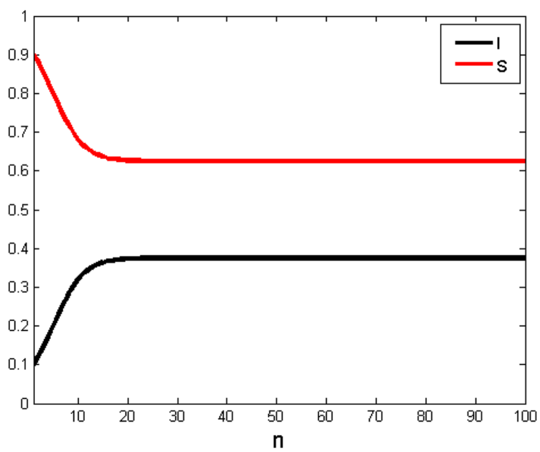

Figure 1. Time series of model (1) with $\alpha = 0.2$, $k = 0.5$, and $\gamma = 4$.

$$S_{n+1} = S_n + kI_n - \alpha\gamma S_n I_n,$$
$$I_{n+1} = I_n - kI_n + \alpha\gamma S_n I_n, \quad n = 1, 2, 3, ..., \tag{1}$$

where α is the transmission rate, γ is the infection rate, and k measures the percentage of population that recovers from a disease each period. The time series of model (1) is illustrated in figure 1 with parameters $\alpha = 0.2$, $k = 0.5$, and $\gamma = 4$ starting with an initial value $I_0 = 0.1$.

As we can see in figure 1, the infected individuals increase quickly, then settle down at the fixed point 0.375, while the susceptible individuals decrease quickly and then settle down at the fixed point $S = (k/\gamma\alpha) = 0.625$.

The total population size N is constant because from eq. (1) we have $N = S + I$. Thus, we can reduce the dimension of model (1) to a one-dimensional model and rewrite it in the following form:

$$I_{n+1} = \alpha\gamma(N - I_n)I_n + (1 - k)I_n. \tag{2}$$

Considering a time delay in eq. (2) will put it into the following form:

$$I_{n+1} = \alpha\gamma(N - I_n)I_n + (1 - k)(1 - \beta)I_n + \beta I_{n-\tau}, \tag{3}$$

where β is the delay strength and τ is the time delay.

As q-deformation of a function is based on introducing an additional parameter q in the function definition such that taking the limit $q \to 1$, the original function is obtained. There are many deformation schemes for the same function, one can see for example [11]. In this paper, we shall adopt the q-deformation given by Tsallis in [20]

$$[x] = \frac{x}{1 + (1 - q)(1 - x)} \quad \text{and} \quad -\infty < q < 2. \tag{4}$$

Now, we introduce the q-deformation definition given in (4) to eq. (3) which in return will be in the following form:

$$I_{n+1} = \alpha\gamma(N-[I_n])[I_n] + (1-k)(1-\beta)[I_n] + \beta[I_{n-\tau}]. \tag{5}$$

In what follows, we discuss model (5) from the dynamic perspective and investigate the effect of deformation and delay on its long-term behaviour. In the following analytical study of model (5), we take $\tau = 1$ and in the numerical simulations section we consider further values of τ. Thus, taking $\tau = 1$, eq. (5) will be transformed into the following model:

$$I_{n+1} = \alpha\gamma\left(\frac{N(1 + (1 - q_I)(1 - I_n))I_n - I_n^2}{(1 + (1 - q_I)(1 - I_n))^2}\right) + \frac{(1 - k)(1 - \beta)I_n}{1 + (1 - q_I)(1 - I_n)} + \frac{\beta L_n}{1 + (1 - q_L)(1 - L_n)}.$$
$$L_{n+1} = I_n. \tag{6}$$

As the disease represented by the SIS model (6) is expected to reshape the infected cells, q-deformation of variables is employed in this work to model the reshaping of those cells. Two different values of deformations q_I and q_L are proposed in the underlined model because cells infected by the disease in the current moment may respond differently from those cells infected by the disease in the previous moment.

The rest of the paper is structured as follows. Section 2 investigates the existence criteria of fixed points and their local stability. In §3, a detailed codimension-one bifurcation at fixed points is performed. All analytical findings are confirmed in §4 with the help of numerical simulations. Finally, we conclude in §5.

2. Fixed points classifications

Model (6) has at most 4 fixed points

- A disease-free fixed point $E_0 = (0, 0)$ which exists for all parameter values,
- Endemic fixed points $E^* = (I^*, L^*)$, where I^* is the positive root of the following equation:

$$R^2TI^3 - (3R^2T + \theta - \alpha\gamma T)I^2 + (3R^2T + 2 + \phi - \alpha\gamma T + \alpha\gamma)I - (R^2T + \theta + \phi + D) = 0, \tag{7}$$

where

$$R = 1 - q_I, \quad T = 1 - q_L,$$
$$\theta = 2RT + R^2 - T\alpha\gamma NR - (1 - k)(1 - \beta)RT - \beta^2R^2,$$
$$\phi = T + 2R - \alpha\gamma NR - (1 - k)(1 - \beta)R - \alpha\gamma TN - (1 - k)(1 - \beta)T - 2\beta R,$$
$$D = 1 - \alpha\gamma N - (1 - k)(1 - \beta) - \beta,$$

and

$$L^* = I^*.$$

Now we can state the following lemma.

Lemma 1. *There is at least one positive root for eq.* (7)

(i) *if* $3R^2T + \theta \le \alpha\gamma T$, *then eq.* (7) *has a unique positive root,*
(ii) *if* $3R^2T + \theta > \alpha\gamma T$ *and* $3R^2T + 2 + \phi + \alpha\gamma > \alpha\gamma T$, *then eq.* (7) *has three positive roots,*
(iii) *if* $3R^2T + \theta > \alpha\gamma T$ *and* $3R^2T + 2 + \phi + \alpha\gamma < \alpha\gamma T$, *then eq.* (7) *has a unique positive root.*

The proof is omitted.

The Jacobian matrix of model (6) calculated at any point (I, L) is

$$J(I,L) = \begin{pmatrix} \dfrac{IR(N\alpha\gamma + (1-k)(1-\beta))}{(1+R(1-I))^2} + \dfrac{N\alpha\gamma + (1-k)(1-\beta)}{1+R(1-I)} - \dfrac{2\alpha\gamma I(1+R)}{(1+R(1-I))^3} & \dfrac{(1+T)\beta}{(1+T(1-L))^2} \\ 1 & 0 \end{pmatrix}.$$

In order to classify fixed points, we need the two following lemmas.

Lemma 2 [21]. *Let* $F(\lambda) = \lambda^2 + P\lambda + Q$ *be the characteristic equation of eigenvalues associated with the Jacobian matrix evaluated at a fixed point* (I, L). *Then* (I, L) *is*

1. *a sink (thus locally asymptotically stable) if* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$;
2. *a source (thus locally unstable) if* $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$;
3. *a saddle if* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *(or* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$*);*
4. *non-hyperbolic if either* $|\lambda_1| = 1$ *or* $|\lambda_2| = 1$.

Lemma 3 [21]. *Let* $F(\lambda) = \lambda^2 + P\lambda + Q$. *Suppose that* $F(1) > 0$ *and* $F(\lambda) = 0$ *has two roots* λ_1 *and* λ_2. *Then*

1. $F(-1) > 0$ *and* $Q < 1$ *if and only if* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$;
2. $F(-1) < 0$ *if and only if* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*);*
3. $F(-1) > 0$ *and* $Q > 1$ *if and only if* $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$;
4. $F(-1) = 0$ *and* $P \neq 0, 2$ *if and only if* $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$;
5. $P^2 - 4Q < 0$ *and* $Q = 1$ *if and only if* λ_1 *and* λ_2 *are complex and* $|\lambda_{1,2}| = 1$.

The corresponding characteristic equation at E^* is given by

$$\xi(\lambda) = \lambda^2 + p^*\lambda + q^* = 0,$$

where

$$p^* = -\left(\frac{IR(N\alpha\gamma + (1-k)(1-\beta))}{(1+R(1-I))^2} + \frac{N\alpha\gamma + (1-k)(1-\beta)}{1+R(1-I)} - \frac{2\alpha\gamma I(1+R)}{(1+R(1-I))^3}\right)$$

and

$$q^* = -\frac{\beta(1+T)}{(1+T(1-L))^2}.$$

The dynamics of the fixed points of model (6) are summarised in the following proposition.

PROPOSITION 1

1. E_0 *is a sink if*
$$\frac{N\alpha\gamma + (1-k)(1-\beta)}{R} > \frac{\beta}{T} - 1$$
and $\beta > T$.
2. E_0 *is a source if*
$$\frac{N\alpha\gamma + (1-k)(1-\beta)}{R} > \frac{\beta}{T} - 1$$
and $\beta < T$.
3. E_0 *is a saddle if*
$$\frac{N\alpha\gamma + (1-k)(1-\beta)}{R} < \frac{\beta}{T} - 1.$$
4. E_0 *is nonhyperbolic if*
$$\gamma_1 = \frac{-(1-k)(1-\beta)}{N\alpha} - \frac{1+R}{N\alpha} + \frac{\beta(1+R)}{N\alpha(1+T)} = \gamma_{01}$$
or
$$\gamma_2 = \frac{-(1-k)(1-\beta)}{N\alpha} + \frac{1+R}{N\alpha} - \frac{\beta(1+R)}{N\alpha(1+T)} = \gamma_{02}.$$

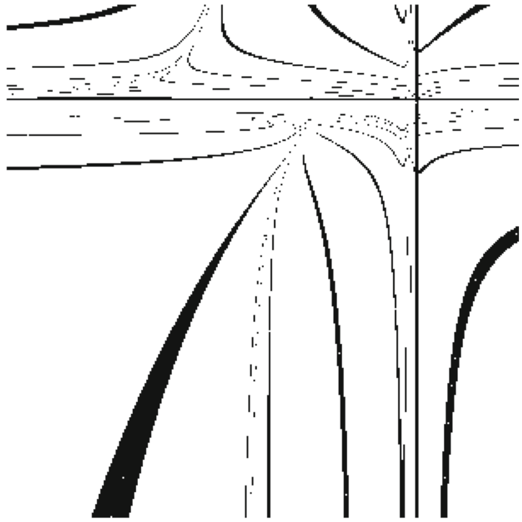

Figure 2. Basin of attraction of fixed points of model (6).

5. *E^* is a sink if* $1 + q^* > p^*$ *and*
$$\beta > -\frac{(1+T(1-L^*))^2}{1+T}.$$
6. *E^* is a source if* $1 + q^* > p^*$ *and*
$$\beta < -\frac{(1+T(1-L^*))^2}{1+T}.$$
7. *E^* is a saddle if* $1 + q^* < p^*$.
8. *E^* is nonhyperbolic if* $q^* = -(1 + p^*)$ *or* $q^* = p^* - 1$.

Figure 2 shows the basin of attraction of fixed points of model (6) for $q_I = -0.95, q_L = -0.8, N = 10, \alpha = 0.2, \beta = 0.2, k = 0.5$, and $\gamma = 0.2$. The two fixed points with these values are $E_0 = (0, 0)$ and $E^* = (3.2783, 3.2783)$ where black is the basin of infinity and white is the basin of attraction.

3. Codimension-one bifurcation

In this section, the occurrence of flip, transcritical, and pitchfork bifurcations in model (1) is investigated at the disease-free fixed point E_0 and the flip bifurcation is discussed at the endemic fixed point E^* where γ is taken as the bifurcation parameter.

3.1 *Bifurcation at* E_0

Model (6) undergoes flip, transcritical, and pitchfork bifurcations at E_0 as discussed below.

3.1.1 *Flip bifurcation (FB).* In this part, we discuss the occurrence of flip bifurcation at E_0. Let

$$\mathrm{FB} = \{(q_I, q_L, \alpha, \beta, k, \gamma_1) : \\ \gamma_1 = \gamma_{01}, \alpha, \beta, k, \gamma_{01} > 0, -\infty < q_I, q_L < 2\}.$$

If γ_{01} is taken from FB, then the characteristic equation associated with the Jacobian matrix of model (6) evaluated at E_0 has two eigenvalues:

$$\lambda_1 = -1$$

and

$$\lambda_2 = \frac{\beta - 2(1+T)}{2(1+T)} - \frac{1}{2}\sqrt{\frac{(\beta - 2(1+T))^2}{4(1+T)^2} + \frac{4\beta}{1+T}}$$

with $|\lambda_2| \neq 1$. We discuss the flip bifurcation of model (6) at E_0 when parameters vary in a small neighbourhood of FB. Model (6) is described by

$$\begin{cases} I \to \alpha\gamma_{01}\left(\dfrac{N(1+R(1-I))I - I^2}{(1+R(1-I))^2}\right) \\ \quad + \dfrac{(1-k)(1-\beta)I}{1+R(1-I)} + \dfrac{\beta L}{1+T(1-L)}, \\ L \to I. \end{cases} \tag{8}$$

Take γ_{01} as the bifurcation parameter and consider a limited perturbation of system (8) as follows:

$$\begin{cases} I \to \alpha(\gamma_{01} + \gamma_{01}^*)\left(\dfrac{N(1+R(1-I))I - I^2}{(1+R(1-I))^2}\right) \\ \quad + \dfrac{(1-k)(1-\beta)I}{1+R(1-I)} + \dfrac{\beta L}{1+T(1-L)}, \\ L \to I, \end{cases} \tag{9}$$

where $|\gamma_{01}^*| \ll 1$. Expanding system (9) in Taylor series we obtain

$$\begin{cases} I \to a_1 I + a_2 L + a_{11} I^2 + a_{13}\gamma_{01}^* I + a_{22} L^2 \\ \quad + a_{111} I^3 + a_{222} L^3 + O((|I| + |L| + |\gamma_{01}^*|)^4), \\ L \to b_1 I, \end{cases} \tag{10}$$

where

$$\begin{cases} a_1 = \dfrac{R-\beta-k-R\beta-Rk+10\alpha\gamma+\beta k+10R\alpha\gamma+R\beta k+1}{(R+1)^2}, \\ a_2 = \dfrac{\beta}{T+1}, \quad a_{11} = \dfrac{\beta-1}{2(R+1)}, \quad a_{13} = \dfrac{10\alpha}{R+1}, \\ a_{22} = \dfrac{1}{2(T+1)}, \quad a_{111} = \dfrac{1}{6(R+1)}, \\ a_{222} = \dfrac{-1}{6(T+1)^2}, \quad b_1 = 1. \end{cases} \tag{11}$$

Now we construct an invertible matrix

$$G = \begin{pmatrix} a_2 & a_2 \\ -1-a_1 & \lambda_2 - a_1 \end{pmatrix},$$

and use the translation

$$\begin{pmatrix} I \\ L \end{pmatrix} = G \begin{pmatrix} \tilde I \\ \tilde L \end{pmatrix}.$$

We obtain

$$\begin{pmatrix} \tilde I \\ \tilde L \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde I \\ \tilde L \end{pmatrix} + \begin{pmatrix} f_1(\tilde I, \tilde L, \gamma_{01}^*) \\ f_2(\tilde I, \tilde L, \gamma_{01}^*) \end{pmatrix}, \tag{12}$$

where

$$f_1(\tilde I, \tilde L, \gamma_{01}^*) = \frac{(\lambda_2-a_1)(a_{11}I^2+a_{22}L^2+a_{111}I^3+a_{222}L^3+a_{13}I\gamma_{01}^*)}{a_2(1+\lambda_2)} + O((|I|+|L|+|\gamma_{01}^*|)^4),$$

$$f_2(\tilde I, \tilde L, \gamma_{01}^*) = \frac{(1+a_1)(a_{11}I^2+a_{22}L^2+a_{111}I^3+a_{222}L^3+a_{13}I\gamma_{01}^*)}{a_2(1+\lambda_2)} + O((|I|+|L|+|\gamma_{01}^*|)^4),$$

and

$$I = a_2(\tilde I + \tilde L), \;\; L = -(1+a_1)\tilde I + (\lambda_2 - a_1)\tilde L,$$

$$I^2 = a_2^2(\tilde I^2 + \tilde I \tilde L + \tilde L^2),$$

$$IL = a_2(-(1+a_1)\tilde I^2 + (\lambda_2 - 2a_1 - 1)\tilde I \tilde L + a_2(\lambda_2 - a_1)\tilde L^2),$$

$$L^2 = (1+a_1)^2 \tilde I^2 + (\lambda_2 - a_1)^2 \tilde L^2 - 2(1+a_1)(\lambda_2-a_1)\tilde I \tilde L,$$

$$L^3 = -(1+a_1)^3 \tilde I^3 + 3(1+a_1)(\lambda_2-a_1)\tilde I^2 \tilde L - 3(1+a_1)(\lambda_2-a_1)\tilde I \tilde L^2 + (\lambda_2-a_1)^3 \tilde L^3.$$

According to the centre manifold theorem [22], there exists a centre manifold $W_c(0,0,0)$ of (12) at E_0 in a small neighbourhood of γ_{01}^* in the following form:

$$W_c(0,0,0) = \{(\tilde I, \tilde L, \gamma_{01}^*) \in R^3,$$

$$\tilde L = h(\tilde I, \gamma_{01}^*), h(0,0) = 0, Dh(0,0) = 0\},$$

for $\tilde I$ and γ_{01}^* sufficiently small. Suppose there exists a centre manifold given by

$$h(\tilde I, \gamma_{01}^*) = c_0 \tilde I^2 + c_1 \tilde I \gamma_{01}^* + c_2 {\gamma_{01}^*}^2 + O((|\tilde I| + |\gamma_{01}^*|)^3). \tag{13}$$

The following relation must be satisfied by the centre manifold:

$$h(-\tilde I + f_1(\tilde I, h(\tilde I, \gamma_{01}^*), \gamma_{01}^*), \gamma_{01}^*) = \lambda_2 h(\tilde I, \gamma_{01}^*) + f_2(\tilde I, h(\tilde I, \gamma_{01}^*), \gamma_{01}^*). \tag{14}$$

Substituting (13) into (14), then equating like powers' coefficients in (14), we obtain

$$c_0 = \frac{(1+a_1)(1+\lambda_2)(a_{11}a_2^2 + a_{22}a_1)}{a_2(1+\lambda_2)^2 + a_2 b_1^2(\lambda_2-a_1)^2 - a_2\lambda_2(1+\lambda_2)^2},$$

$$c_1 = \frac{-a_{13}}{(1+\lambda_2)(1+\lambda_2+a_2^2 b_1)},$$

$$c_2 = 0.$$

Now, system (12) restricted to the centre manifold $W_c(0,0,0)$ is given by

$$F: \tilde I \to A\tilde I + B\tilde I \gamma_{01}^* + C\tilde I^2 + D\tilde I^2 \gamma_{01}^* + E\tilde I^3 + O((|\tilde I| + |\gamma_{01}^*|)^4), \tag{15}$$

where

$$A = -1 + \frac{b_1(\lambda_2 - a_1)}{(\lambda_2+1)}, \quad B = \frac{b_1 a_2 c_1(\lambda_2 - a_1)}{a_2(\lambda_2+1)},$$

$$C = \frac{1}{a_2(\lambda_2+1)}(b_1 a_2 c_0(\lambda_2-a_1) - a_2^3 a_{11} - 2a_2^3 a_{11} c_0 - a_2 a_{22}(1+a_1)^2 - a_2 a_{22} c_0(\lambda_2 - a_1)),$$

$$D = \frac{1}{a_2(\lambda_2+1)}(-2a_2^3 a_{11} c_1 - 2a_2 a_{22} c_1 (1+a_1)(\lambda_2 - a_1)),$$

$$E = \frac{1}{a_2(\lambda_2+1)}(-2a_2a_{22}c_0(1+a_1)(\lambda_2-a_1) \\ -a_2^4a_{111}c_0^3 - a_2^4a_{111} + a_2a_{222}(1+a_1)^3). \qquad (16)$$

Thus, system (15) admits a flip bifurcation if the quantities α_1 and α_2 are not equal to zero, where

$$\alpha_1 = \left(2\frac{\partial^2 F}{\partial\gamma_{01}^*\partial\tilde{I}} + \frac{\partial F}{\partial\gamma_{01}^*}\frac{\partial F}{\partial\tilde{I}}\right)_{(0,0)} = 2B \neq 0,$$
$$\alpha_2 = \left(\frac{1}{2}\left(\frac{\partial^2 F}{\partial\tilde{I}^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 F}{\partial\tilde{I}^3}\right)\right)_{(0,0)} = 2(C^2+E).$$

3.1.2 *Transcritical bifurcation (TS).* Now we investigate the possibility of the occurrence of transcritical bifurcation at E_0. Let

$$\mathrm{TS} = \{(q_I, q_L, \alpha, \beta, k, \gamma_2) : \\ \gamma_2 = \gamma_{02}, \alpha, \beta, k, \gamma_{02} > 0, -\infty < q_I, q_L < 2\},$$

If γ_{02} is taken from TS, then the characteristic equation associated with the Jacobian matrix of model (6) evaluated at E_0 has two eigenvalues:

$$\lambda_1 = 1$$

and

$$\lambda_2 = \frac{2(1+T)-\beta}{2(1+T)} - \frac{1}{2}\sqrt{\frac{(2(1+T)-\beta)^2}{4(1+T)^2} + \frac{4\beta}{1+T}}$$

with $|\lambda_2| \neq 1$. We discuss the flip bifurcation of model (6) at E_0 when parameters vary in a small neighbourhood of TS. Model (6) is described by

$$\begin{cases} I \to \alpha\gamma_{02}\left(\dfrac{N(1+R(1-I))I-I^2}{(1+R(1-I))^2}\right) + \dfrac{(1-k)(1-\beta)I}{1+R(1-I)} \\ \quad + \dfrac{\beta L}{1+T(1-L)}. \\ L \to I. \end{cases} \qquad (17)$$

Take γ_{02} as the bifurcation parameter and consider a limited perturbation of system (17) as follows:

$$\begin{cases} I \to \alpha(\gamma_{02}+\gamma_{02}^*)\left(\dfrac{N(1+R(1-I))I-I^2}{(1+R(1-I))^2}\right) \\ \quad + \dfrac{(1-k)(1-\beta)I}{1+R(1-I)} + \dfrac{\beta L}{1+T(1-L)}, \\ L \to I, \end{cases} \qquad (18)$$

where $|\gamma_{02}^*| \ll 1$. Expanding system (18) in Taylor series we obtain

$$\begin{cases} I \to a_1 I + a_2 L + a_{11}I^2 + a_{13}\gamma_{02}^* I + a_{22}L^2 + a_{111}I^3 \\ \quad + a_{222}L^3 + O((|I|+|L|+|\gamma_{02}^*|)^4), \\ L \to b_1 I, \end{cases} \qquad (19)$$

where $a_1,\ a_2,\ a_{13},\ a_{11},\ a_{111},\ a_{222}$ and b_1 are defined in (11).

Constructing an invertible matrix

$$M = \begin{pmatrix} a_2 & a_2 \\ -1-a_1 & \lambda_2 - a_1 \end{pmatrix},$$

and using the translation

$$\begin{pmatrix} I \\ L \end{pmatrix} = M\begin{pmatrix} \tilde{I} \\ \tilde{L} \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \tilde{I} \\ \tilde{L} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \tilde{I} \\ \tilde{L} \end{pmatrix} + \begin{pmatrix} \Theta_1(\tilde{I},\tilde{L},\gamma_{02}^*) \\ \Theta_2(\tilde{I},\tilde{L},\gamma_{02}^*) \end{pmatrix}, \qquad (20)$$

where

$$\Theta_1(\tilde{I},\tilde{L},\gamma_{01}^*) = \frac{(\lambda_2-a_1)b_1 I - a_2(a_{11}I^2 + a_{22}L^2 + a_{111}I^3 + a_{222}L^3 + a_{13}I\gamma_{02}^*)}{a_2(1+\lambda_2)} + O((|I|+|L|+|\gamma_{02}^*|)^4),$$
$$\Theta_2(\tilde{I},\tilde{L},\gamma_{02}^*) = \frac{(1+a_1)(a_{11}I^2 + a_{22}L^2 + a_{111}I^3 + a_{222}L^3 + a_{13}I\gamma_{02}^*)}{a_2(1+\lambda_2)} + O((|I|+|L|+|\gamma_{02}^*|)^4).$$

There exists a centre manifold $W_c^1(0,0,0)$ of (20) at E_0 in a small neighbourhood of γ_{02}^* which can be represented in the form

$$W_c^1(0,0,0) = \{(\tilde{T},\tilde{L},\gamma_{02}^*) \in R^3, \\ \tilde{y} = \aleph(\tilde{I},\gamma_{02}^*), \aleph(0,0) = 0, D\aleph(0,0) = 0\},$$

for $\tilde{I}$ and γ_{02}^* sufficiently small. Suppose there exists a centre manifold given by

$$\aleph(\tilde{I},\gamma_{01}^*) = m_0\tilde{I}^2 + m_1\tilde{I}\gamma_{02}^* + m_2{\gamma_{01}^*}^2 \\ + O((|\tilde{I}|+|\gamma_{02}^*|)^3). \qquad (21)$$

The following relation must be satisfied by the centre manifold

$$\aleph(\tilde{I} + f_1(\tilde{I}, \aleph(\tilde{I}, \gamma_{02}^*), \gamma_{02}^*), \gamma_{02}^*) = \lambda_2 h(\tilde{I}, \gamma_{02}^*) + f_2(\tilde{I}, \aleph(\tilde{I}, \gamma_{02}^*), \gamma_{02}^*). \tag{22}$$

Substituting (21) into (22), then equating like powers' coefficients in (14), we obtain

$$m_0 = \frac{(1+a_1)(1+\lambda_2)(a_{11}a_2^2 + a_{22}a_1)}{a_2(1+\lambda_2)^2 + a_2 b_1^2(\lambda_2 - a_1)^2 - a_2\lambda_2(1+\lambda_2)^2},$$
$$m_1 = \frac{-a_{13}}{(1+\lambda_2)(1+\lambda_2+a_2^2 b_1)},$$
$$m_2 = 0.$$

$$\gamma_* = \frac{\dfrac{(1-k)(1-\beta)}{-(1+R(1-I^*))} + \dfrac{\beta(1+T)}{(1+T(1-L^*))^2} - \dfrac{I^*R^*(1-k)(1-\beta)}{(1+R(1-I^*))^2} - 1}{\dfrac{-2I^*\alpha(1+R)}{(1+R(1-I^*))^3} + \dfrac{N\alpha}{(1+R(1-I^*))} + \dfrac{NRI^*\alpha}{(1+R(1-I^*))^2}},$$

Now, system (20) restricted to the centre manifold $W_c^1(0,0,0)$ is given by

$$F_1 : \tilde{I} \to A_1\tilde{I} + B\tilde{I}\gamma_{02}^* + C\tilde{I}^2 + D\tilde{I}^2\gamma_{02}^* + E\tilde{I}^3 + O((|\tilde{I}| + |\gamma_{02}^*|)^4), \tag{23}$$

where

$$A_1 = 1 + \frac{b_1(\lambda_2 - a_1)}{(\lambda_2 + 1)}$$

and B, C, D and E are the same as defined in eq. (33). System (23) undergoes a transcritical bifurcation according to the following conditions:

$$F_1(0,0) = 0, \quad \left(\frac{\partial^2 F_1}{\partial\gamma_{02}^*\partial\tilde{I}}\right)_{(0,0)} = B \neq 0,$$
$$\left(\frac{\partial^2 F_1}{\partial\tilde{I}^2}\right)_{(0,0)} = 2C \neq 0. \tag{24}$$

3.1.3 *Pitchfork bifurcation.* Depending on the preceding analysis in §3.1.2, we can easily make sure that the conditions for pitchfork bifurcation at E_0 hold true as follows:

$$\left(\frac{\partial F_1}{\partial\tilde{I}}\right)_{(0,0)} = 1, \quad \left(\frac{\partial^2 F_1}{\partial\gamma_{02}^*\partial\tilde{I}}\right)_{(0,0)} = B \neq 0,$$
$$\left(\frac{\partial^3 F_1}{\partial\tilde{I}^3}\right)_{(0,0)} = 6E \neq 0. \tag{25}$$

3.2 *Bifurcation at E^**

In this part, we perform a codimension-one flip bifurcation analysis at the endemic fixed point E^*. If then the characteristic equation at E^* will have two eigenvalues $\lambda_1 = -1$ and $|\lambda_2| \neq 1$. Let

$$\mathrm{FB}_E^* = \{(q_I, q_L, \alpha, \beta, k, \gamma_{03}) : \gamma_{03} = \gamma^*, \alpha, \beta, k, \gamma^* > 0, -\infty < q_I, q_L < 2\}.$$

A flip bifurcation may happen at E^* when parameters vary in a small neighbourhood of FB_E^*. Model (6) takes the form

$$\begin{cases} I \to \alpha(\gamma + \gamma^*)\left(\dfrac{N(1+R(1-I^*))I - I^{*^2}}{(1+R(1-I^*))^2}\right) \\ \qquad + \dfrac{(1-k)(1-\beta)I^*}{1+R(1-I^*)} + \dfrac{\beta L^*}{1+T(1-L^*)}, \\ L \to I^*, \end{cases} \tag{26}$$

where $|\gamma^*| \ll 1$ is a limited perturbation parameter.

Transform E^* to the origin by introducing $u = I - I^*$, $L = L - L^*$, and we obtain

$$\begin{cases} u \to a_1 u + a_2 v + a_{11}u^2 + a_{22}v^2 + a_{13}u\gamma^* + a_{22}v^2 \\ \qquad + a_{111}u^3 + a_{222}v^3 + O((|u| + |v| + |\gamma^*|)^4), \\ v \to b_1 u, \end{cases} \tag{27}$$

where

$$\begin{cases} a_1 = \dfrac{(1+R)(R-\beta-k-R\beta-Rk+\beta k-I^*R+I^*R\beta+I^*Rk-2I^*\alpha\gamma+N\alpha\gamma+R\beta k-I^*R\beta k+NR\alpha\gamma-I^*NR\alpha\gamma+1)}{(1+R(1-I^*))^3}, \\ a_2 = \dfrac{\beta(T+1)}{1+T(1-L^*)}, \quad a_{13} = \dfrac{(1+R)(-2\alpha I^* - N\alpha RI^* + NR(1+\alpha))}{(1+R(1-I^*))^3}, \\ a_{11} = \dfrac{RI^*(\beta-1)}{2(1+R(1-I^*))^2} - \dfrac{\beta-1}{2(R(I^*-1)-1)}, \quad a_{111} = \dfrac{RI^*}{6(R(I^*-1)-1)^2} - \dfrac{1}{6(R(I^*-1)-1)}, \\ a_{222} = \dfrac{(2L^*-T+L^*T-1)}{6(T-L^*T+1)^3}, \quad b_1 = 1. \end{cases} \tag{28}$$

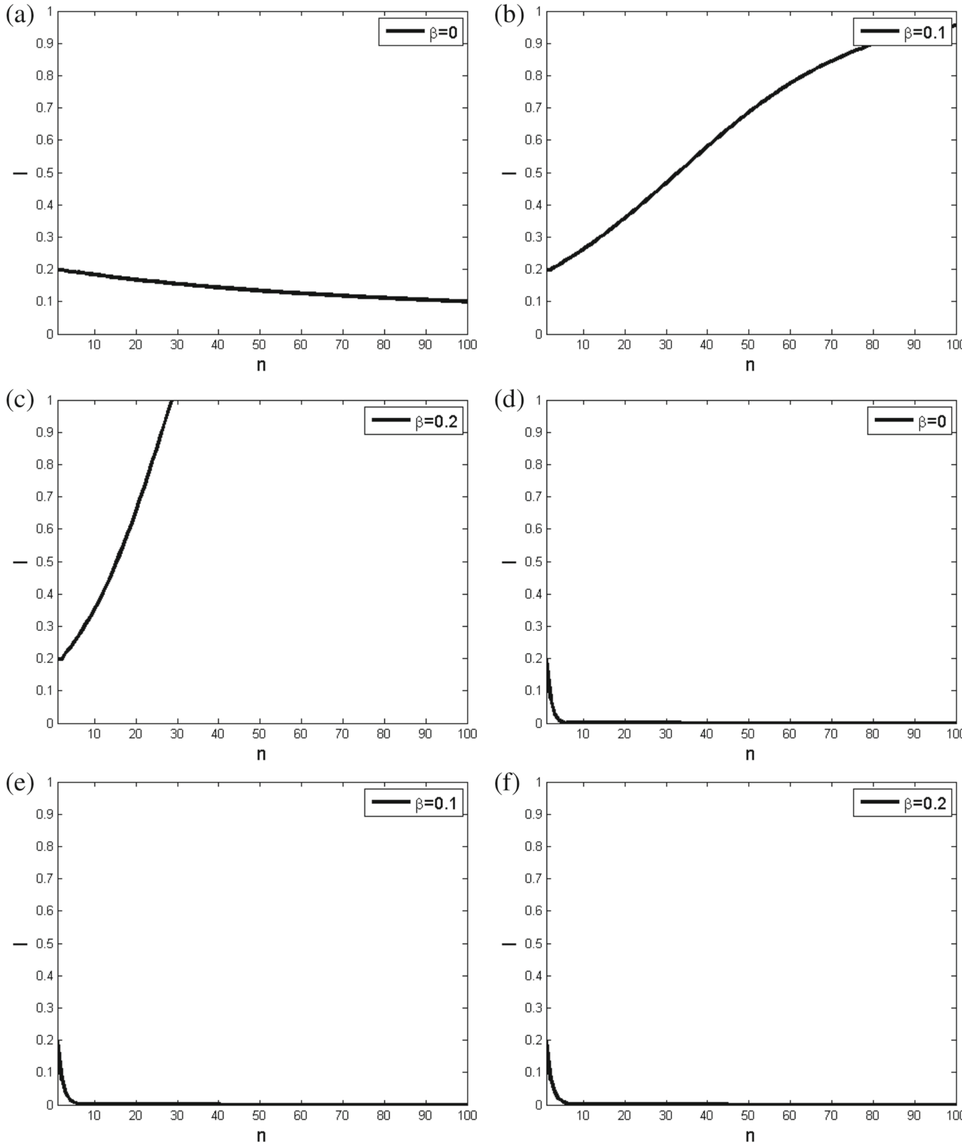

Figure 3. Trajectories of model (6) for $N = 10, \alpha = 0.2, k = 0.5, \gamma = 0.25$. (**a**, **b**, **c**) $q_I = q_L = 1$ and (**d**, **e**, **f**) $q_I = -0.95, q_L = -0.8$.

Constructing an invertible matrix

$$Y = \begin{pmatrix} a_2 & a_2 \\ -1 - a_1 & \lambda_2 - a_1 \end{pmatrix},$$

and using the translation

$$\begin{pmatrix} u \\ v \end{pmatrix} = Y \begin{pmatrix} \tilde{I} \\ \tilde{L} \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \tilde{I} \\ \tilde{L} \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde{I} \\ \tilde{L} \end{pmatrix} + \begin{pmatrix} \Phi_1(\tilde{I}, \tilde{L}, \gamma^*) \\ \Phi_2(\tilde{I}, \tilde{L}, \gamma^*) \end{pmatrix}, \tag{29}$$

where

$$\Phi_1(\tilde{u}, \tilde{v}, \gamma^*) = \frac{(\lambda_2 - a_1)(a_{11}u^2 + a_{22}v^2 + a_{111}u^3 + a_{222}v^3 + a_{13}u\gamma^*)}{a_2(1 + \lambda_2)} + O((|u| + |v| + |\gamma^*|)^4),$$

$$\Phi_2(\tilde{u}, \tilde{v}, \gamma^*) = \frac{(1 + a_1)(a_{11}u^2 + a_{22}v^2 + a_{111}u^3 + a_{222}v^3 + a_{13}u\gamma^*)}{a_2(1 + \lambda_2)} + O((|u| + |v| + |\gamma^*|)^4),$$

and

$$u = a_2(\tilde{I} + \tilde{L}), \quad v = -(1 + a_1)\tilde{I} + (\lambda_2 - a_1)\tilde{L}.$$

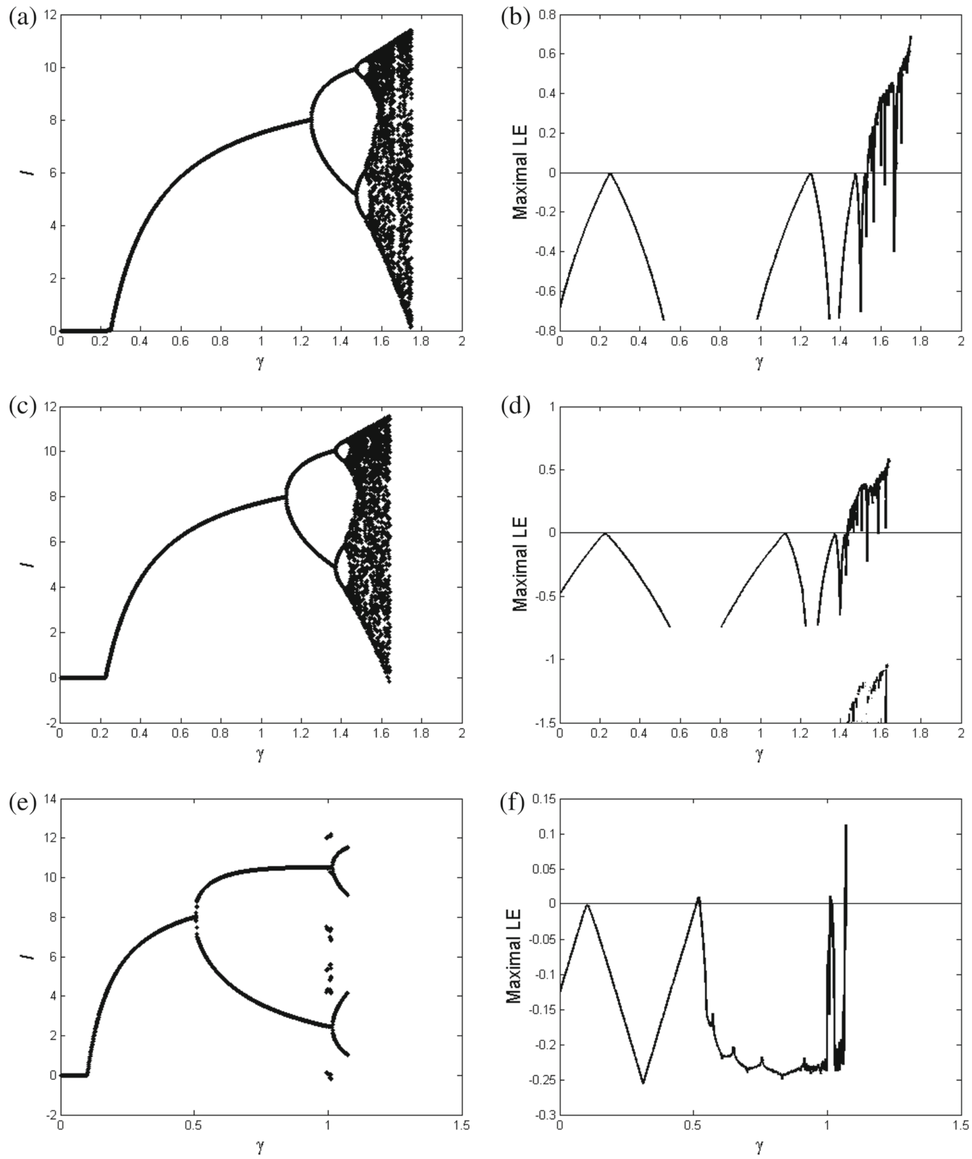

Figure 4. Bifurcation diagrams and the corresponding maximal Lyapunov exponents of model (6) as a function of γ for $q_I = 1, q_L = 1, N = 10, \alpha = 0.2, k = 0.5$. (**a**, **b**) $\beta = 0$, (**c**, **d**) $\beta = 0.1$, and (**e**, **f**) $\beta = 0.6$.

According to the centre manifold theorem, there exists a centre manifold $W_E^*(0,0,0)$ of (29) at E^* in a small neighbourhood of γ^* in the following form:

$$W_E^*(0,0,0) = \{(\tilde{I}, \tilde{L}, \gamma^*) \in R^3, \tilde{L} = \mathcal{N}(\tilde{I}, \gamma_{01}^*), \mathcal{N}(0,0) = 0, D\mathcal{N}(0,0) = 0\},$$

for $\tilde{I}$ and γ^* sufficiently small. Suppose there exists a centre manifold given by

$$\mathcal{N}(\tilde{I}, \gamma^*) = d_0\tilde{I}^2 + d_1\tilde{I}\gamma^* + d_2\gamma^{*2} + O((|\tilde{I}| + |\gamma^*|)^3). \tag{30}$$

The following relation must be satisfied by the centre manifold:

$$\mathcal{N}(-\tilde{I} + \Phi_1(\tilde{I}, \mathcal{N}(\tilde{I}, \gamma^*), \gamma^*), \gamma^*) = \lambda_2\mathcal{N}(\tilde{I}, \gamma^*) + \Phi_2(\tilde{I}, \mathcal{N}(\tilde{I}, \gamma^*), \gamma^*). \tag{31}$$

Substituting (30) into (31), then equating like powers' coefficients in (31), we obtain

$$d_0 = \frac{(1 + a_1)(1 + \lambda_2)(a_{11}a_2^2 + a_{22}a_1)}{a_2(1 + \lambda_2)^2 + a_2b_1^2(\lambda_2 - a_1)^2 - a_2\lambda_2(1 + \lambda_2)^2},$$

$$d_1 = \frac{-a_{13}}{(1 + \lambda_2)(1 + \lambda_2 + a_2^2b_1)},$$

$$d_2 = 0.$$

Now, system (30) restricted to the centre manifold $W_E^*(0,0,0)$ is given by

$$F_{pd} : \tilde{I} \rightarrow q_1\tilde{I} + q_2\tilde{I}\gamma^* + q_3\tilde{I}^2 + q_4\tilde{I}^2\gamma^* + q_5\tilde{I}^3 + O((|\tilde{I}| + |\gamma^*|)^4), \quad (32)$$

where

$$q_1 = -1 + \frac{b_1(\lambda_2 - a_1)}{(\lambda_2 + 1)}, \quad q_2 = \frac{b_1 a_2 c_1(\lambda_2 - a_1)}{a_2(\lambda_2 + 1)},$$

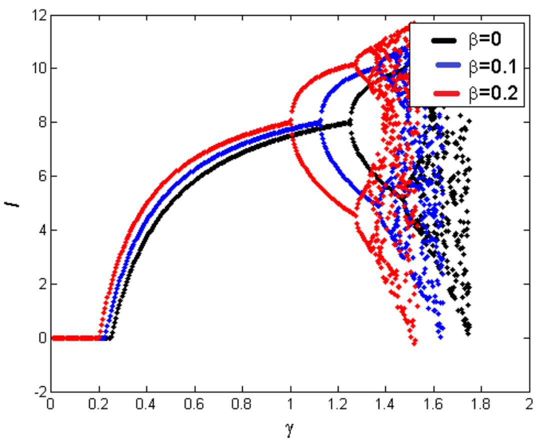

Figure 5. Bifurcation diagram of model (6) for $q_I = q_L = 1, N = 10, \alpha = 0.2$, and $k = 0.5$ for different β.

$$\begin{aligned} q_3 &= \frac{1}{a_2(\lambda_2+1)}(b_1 a_2 c_0(\lambda_2 - a_1) - a_2^3 a_{11} \\ &\quad -2a_2^3 a_{11} c_0 - a_2 a_{22}(1+a_1)^2 \\ &\quad -a_2 a_{22} c_0(\lambda_2 - a_1)), \\ q_4 &= \frac{1}{a_2(\lambda_2+1)}(-2a_2^3 a_{11} c_1 \\ &\quad -2a_2 a_{22} c_1(1+a_1)(\lambda_2 - a_1)), \\ q_5 &= \frac{1}{a_2(\lambda_2+1)}(-2a_2 a_{22} c_0(1+a_1)(\lambda_2 - a_1) \\ &\quad -a_2^4 a_{111} c_0^3 - a_2^4 a_{111} + a_2 a_{222}(1+a_1)^3). \end{aligned} \quad (33)$$

Thus, system (32) admits a flip bifurcation if the quantities α_1^* and α_2^* are not equal to zero, where

$$\alpha_1^* = \left(2\frac{\partial^2 F_{pd}}{\partial\gamma^*\partial\tilde{I}} + \frac{\partial F_{pd}}{\partial\gamma^*}\frac{\partial F_{pd}}{\partial\tilde{I}}\right)_{(0,0)} = 2q_2 \neq 0,$$

$$\alpha_2^* = \left(\frac{1}{2}\left(\frac{\partial^2 F_{pd}}{\partial\tilde{I}^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 F_{pd}}{\partial\tilde{I}^3}\right)\right)_{(0,0)} = 2(q_3^2 + q_5).$$

4. Numerical simulations

In this section, the effect of both deformation parameters q_I and q_L together with the delay strength β on the stability of disease-free fixed point E_0 and the endemic

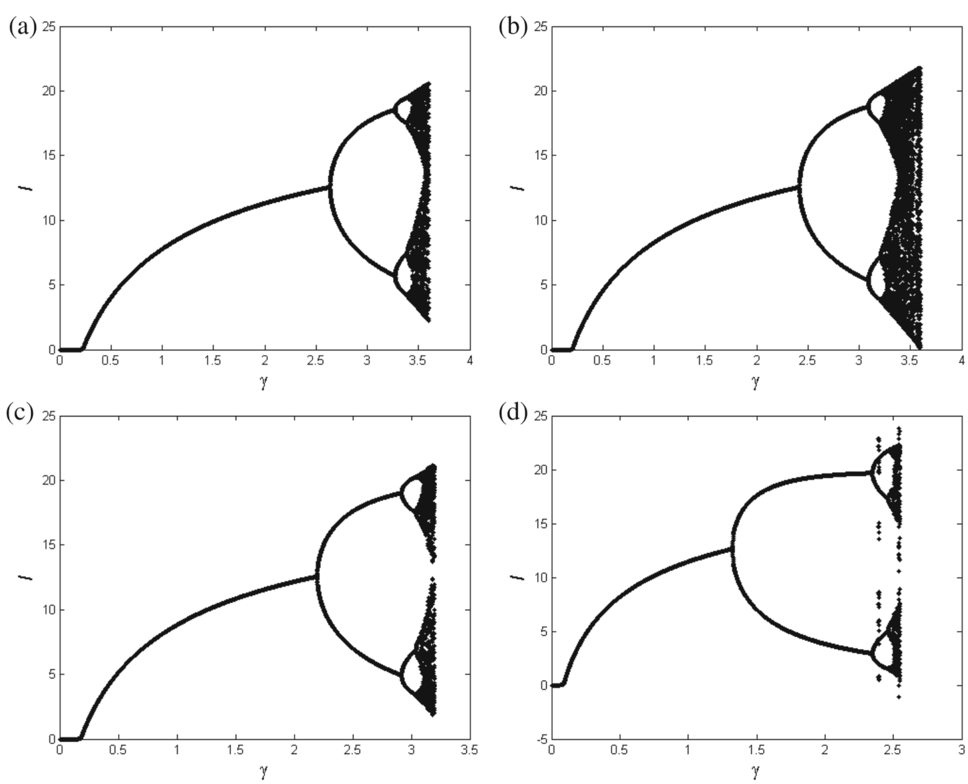

Figure 6. Bifurcation diagrams of model (6) as a function of γ for $q_I = 1.05, q_L = 1.01, N = 10, \alpha = 0.2, k = 0.5$. (**a**) $\beta = 0$, (**b**) $\beta = 0.1$, (**c**) $\beta = 0.2$, and (**d**) $\beta = 0.6$.

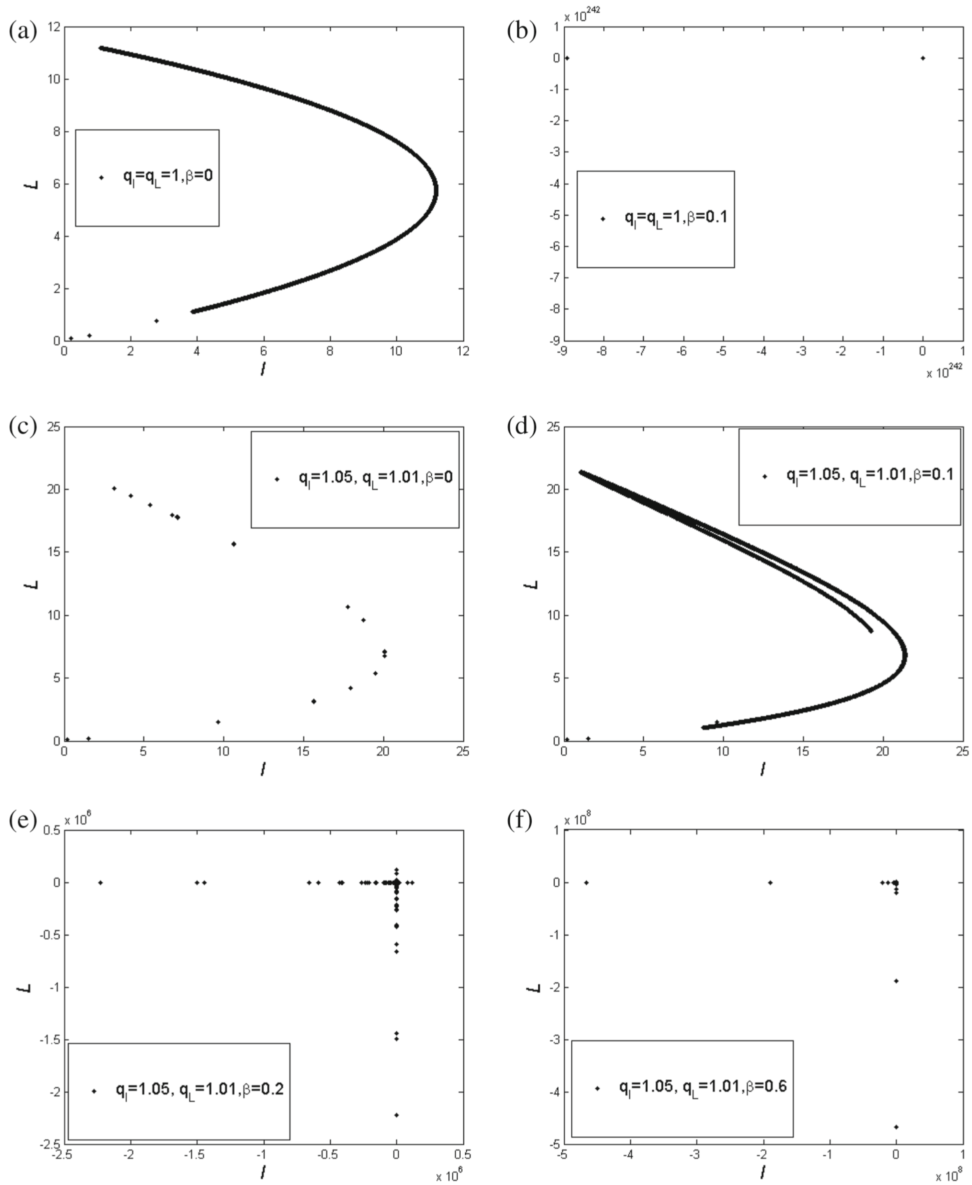

Figure 7. Phase portraits of model (6) for $N = 10, \alpha = 0.2, k = 0.5, \gamma = 3.5$. (**a**) $q_I = q_L = 1, \beta = 0$, (**b**) $q_I = q_L = 1, \beta = 0.1$, (**c**) $q_I = 1.05, q_L = 1.01, \beta = 0$, (**d**) $q_I = 1.05, q_L = 1.01, \beta = 0.1$, (**e**) $q_I = 1.05, q_L = 1.01, \beta = 0.2$, and (**f**) $q_I = 1.05, q_L = 1.01, \beta = 0.6$.

fixed point E^* and on the chaotic behaviour of model (6) is discussed.

Fix the parameters $N = 10, \alpha = 0.2, k = 0.5, \gamma = 0.25$ and start with an initial condition $(I_0, L_0) = (0.2, 0.1)$. If we take $q_I = q_L = 1$, that is there is no deformation in model (6), and consider different values for β, we can see in figures 3a–3c that the presence of the delay strength β definitely increases the number of infected population, where $\beta = 0$ in part (a), $\beta = 0.1$ in part (b), and $\beta = 0.2$ in part (c). On the contrary, if we introduce the deformation parameters, for example, $q_I = -0.95$ and $q_L = -0.8$, we can see as illustrated in figures 3d–3f that the number of infected population is decreased significantly, where $\beta = 0$ in part (d), $\beta = 0.1$ in part (e), and $\beta = 0.2$ in part (f).

Now let us discuss the chaotic behaviour of model (6) in the absence of deformation parameters and in the presence of delay. Consider the parameter values $q_I = 1, q_L = 1, N = 10, \alpha = 0.2$, and $k = 0.5$ with an initial condition $(I_0, L_0) = (0.2, 0.1)$ and suppose γ varies. Figures 4a and 4b represent $\beta = 0$, figures 4c and 4d represent $\beta = 0.1$, and figures 4e and 4f represent $\beta = 0.6$. What can be concluded from these figures is that chaotic behaviour is recovered in the absence of deformation when the value of the delay strength β increases. The corresponding maximal Lyapunov

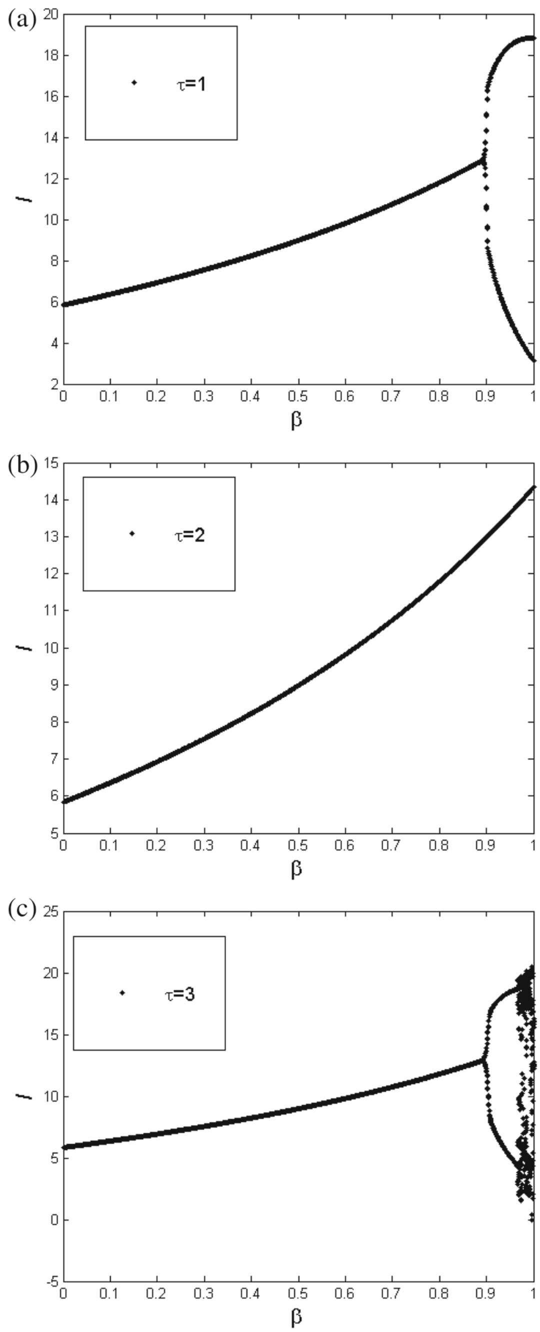

Figure 8. Bifurcation diagram of model (6) as a function of β for $N = 10, \alpha = 0.2, k = 0.5, \gamma = 0.7$ for (**a**) $\tau = 1$, (**b**) $\tau = 2$, and (**c**) $\tau = 3$.

exponents are plotted in figures 4b, 4d and 4f. Chaos recovery can also be shown in figure 5 where there are no deformations but the delay strength parameter β takes the value 0, 0.1, and 0.2, respectively.

Let us now investigate the effect of deformation on the chaotic behaviour of model (6). Fix $q_I = 1.05$, $q_L = 1.01, N = 10, \alpha = 0.2, k = 0.5$, and vary γ. Figures 6a–6d show that increasing the value of the delay strength β suppresses the chaotic behaviour. We start with $\beta = 0$ in figure 6a, then $\beta = 0.1$ in figure 6b, followed by $\beta = 0.2$ in figure 6c, and we end with $\beta = 0.6$ in figure 6d.

Figures 7a–7f show the phase portraits of model (6) for $N = 10, \alpha = 0.2, k = 0.5$, and $\gamma = 3.5$. First of all take $q_I = q_L = 1$, that is, no deformation exists in the model. Increasing the delay strength β from 0 to 0.1 in parts (a), (b), transform the model from the strange attractor to a period-two orbit. Then, consider the deformation values $q_I = 1.05$ and $q_L = 1.01$. Start with $\beta = 0$ in (c), then $\beta = 0.1$ in (d), followed by $\beta = 0.2$ in (e), and end with $\beta = 0.6$ in (f). We see clearly the suppression of chaotic behaviour of model (6) as a result of increasing the delay strength β.

Bifurcation diagrams with respect to the delay strength β for increasing time delay τ are depicted in figures 8a–8c. Starting with $\tau = 1$, some values of β put the model in a period-two point as seen in figure 8a. For $\tau = 2$, a steady state is obtained as shown in figure 8b. Finally, for $\tau = 3$ the model is in a chaotic state for certain values of β as seen in figure 8c.

5. Conclusion

In this paper, a delayed q-deformed discrete SIS epidemic model has been studied. In fact, we extended the q-deforamtion scheme of Jaganathan and Sinha [3] to the discrete epidemic SIS model described by Tassier [19]. q-deformation of variables is employed in this work to model the reshaping of cells infected by the disease. Existence and local stability analysis at disease-free fixed point E_0 and endemic fixed point E^* were discussed. We performed a codimension-one bifurcation analysis at both fixed points via the bifurcation theory and the centre manifold theorem. The model exhibits transcritical, pitchfork, and flip bifurcations at E_0 and a flip bifurcation at E^*. Depending on the value of q, it has been shown that chaotic behaviour will be enhanced or suppressed. In the canonical model (1), we can see from figure 1 that the infected fraction of population increases quickly and then settles down at a fixed value. Meanwhile, the fraction of the susceptible individuals decreases quickly and settle down at the susceptible fixed point. In the modified model (6), we can see that the number of infected individuals decreases in a long-term behaviour. With delay and in the absence of deformation, the peak of infection is increased. On the other hand, by introducing deformation to the model and increasing the delay strength value β, the peak of infection is eradicated. Moreover, chaotic behaviour is recovered in the absence of deformation by increasing the value of β. Comparing figures 3a–3c with figures 3d–3f clearly indicates that the deformation of the dynamic equation is strongly effective in decreasing the infective population. Figure 8a implies that a delay has a destabilising effect, while figure 8c shows that a delay can be a source for chaotic behaviour. Finally, considering deformations in the model and increasing the value of β, suppresses the chaotic behaviour. It can be said that introducing both deformation and delay is a route to suppression of chaos. The main thrust in the considered model is that the disease will die out by increasing the value of both deformation parameters q_I and q_L and delay strength parameter β. We strongly believe that such a study of q-deformation together with delay can be used beneficially in modelling several phenomena.

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