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On invariant analysis and conservation laws for degenerate coupled multi-KdV equations for multiplicity $l = 3$

R K GUPTA[1] and MANJIT SINGH[2],*

[1]Centre for Mathematics and Statistics, School of Basic and Applied Sciences, Central University of Punjab, Bathinda 151 001, India
[2]Yadavindra College of Engineering, Punjabi University, Guru Kashi Campus, Talwandi, Sabo 151 302, India
*Corresponding author. E-mail: manjitcsir@gmail.com

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Abstract. The degenerate coupled multi-Korteweg–de Vries equations for coupled multiplicity $l = 3$ are studied. The equations, also known as three-field Kaup–Boussinesq equations, are considered for invariant analysis and conservation laws. The classical Lie's symmetry method is used to analyse the symmetries of equations. Based on the Killing's form, which is invariant of adjoint action, the full classification for Lie algebra is presented. Further, one-dimensional optimal group classification is used to obtain invariant solutions. Besides this, using general theorem proved by Ibragimov, we find several non-local conservation laws for these equations. The conserved currents obtained in this work can be useful for the better understanding of some physical phenomena modelled by the underlying equations.

Keywords. Lie symmetries; optimal system; exact solutions; conservation laws.

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1. Introduction

Recently, degenerate coupled Korteweg–de Vries (KdV) equations for coupled multiplicity $l = 2, 3, 4$ have been considered for travelling wave solutions by Gürses and Pekcan [1,2]. The multisystem of Kaup–Boussinesq equations is given by (see [3,4])

$$
\begin{aligned}
u_t &= \frac{3}{2}uu_x + q_x^2,\\
q_t^2 &= q^2u_x + \frac{1}{2}uq_x^2 + q_x^3,\\
&\vdots\\
q_t^{l-1} &= q^{l-1}u_x + \frac{1}{2}uq_x^{l-1} + v_x,\\
v_t &= -\frac{1}{4}u_{xxx} + vu_x + \frac{1}{2}uv_x,
\end{aligned}
\tag{1}
$$

where $q^1 = u$ and $q^l = v$. For $l = 2$, system (1) reduces to

$$
\begin{aligned}
u_t &= \frac{3}{2}uu_x + v_x,\\
v_t &= -\frac{1}{4}u_{xxx} + vu_x + \frac{1}{2}uu_x.
\end{aligned}
\tag{2}
$$

System (2) has been studied in detail by Gürses and Pekcan [1] for travelling wave solutions, and they have proved that there exists no asymptotically vanishing travelling wave solution of system (2). Wazwaz [5] has studied generalised version of system (2) for multiple-soliton solutions. It is pertinent to mention that system (2), which is also known as Kaup–Boussinesq system, exhibits the same shallow water wave characteristics in the same approximation as the well-known Boussinesq equation in the lowest order in small parameters controlling weak dispersion and nonlinearity effects [6–9]. Moreover, the function $v(x, t)$ denotes the height of the water surface above a horizontal bottom, whereas the function $u(x, t)$ denotes its velocity averaged over depth. The Kaup–Boussinesq system (2) corresponds to the case when the gravity force dominates over the capillary one and it is completely integrable [10–12].

For $l = 3$, system (1) has the following form:

$$
\begin{aligned}
&u_t - \frac{3}{2}uu_x - v_x = 0,\\
&v_t - vu_x - \frac{1}{2}uv_x - w_x = 0,\\
&w_t + \frac{1}{4}u_{xxx} - wu_x - \frac{1}{2}uw_x = 0.
\end{aligned}
\tag{3}
$$

The system of eq. (3), which is also known as three-field Kaup–Boussinesq equations [13], has been discussed for travelling wave solutions [2], wherein authors have given a general approach to solve eq. (1) for $l \geq 3$. Subsequently, by using the bifurcation analysis, Li and Chen [14] have given complete parametric representations of travelling wave solutions of system (1) for $l = 2, 3, 4$, which was missing in the work of Gürses and Pekcan [1,2]. In the literature, we have noticed that eq. (3) has not been completely analysed, and so in this work, we propose to investigate eq. (3) for Lie's symmetry analysis and for conservation laws using the recently proposed new theorem by Ibragimov.

The paper is organised as follows. In §2, based on the classical Lie symmetry analysis, we have obtained four-dimensional Lie algebra. Starting with a brief discussion about classification techniques, Lie algebra is then classified into mutually conjugate classes by identifying the Killing's form which is invariant of full adjoint action. Reductions are also presented corresponding to every conjugate class and exact solutions are also obtained. In §3, based on a new theorem proposed by Ibragimov, several non-local conservation laws are also constructed. Finally, in §4, the conclusion is drawn.

2. Lie symmetry analysis of Kaup–Boussinesq equation (3)

In order to identify Lie point symmetries for eq. (3), we follow the standard procedure given in [15–18]. The procedure is so algorithmic that it has been successfully implemented in symbolic languages such as 'Maple' and 'Mathematica'. The Maple package 'PDEtools' written by Terrab [19] is quite interactive and efficient. It becomes indispensable for researchers in the field of partial differential equations (PDEs). In the following, we have used this Maple package to find out Lie symmetries for eq. (3). So, we consider one-parameter local Lie group of point transformations:

$$\begin{aligned}
\tilde{x} &= x + \epsilon\xi(x,t,u,v,w) + O(\epsilon^2),\\
\tilde{t} &= t + \epsilon\tau(x,t,u,v,w) + O(\epsilon^2),\\
\tilde{u} &= u + \epsilon\eta_1(x,t,u,v,w) + O(\epsilon^2),\\
\tilde{v} &= v + \epsilon\eta_2(x,t,u,v,w) + O(\epsilon^2),\\
\tilde{w} &= w + \epsilon\eta_3(x,t,u,v,w) + O(\epsilon^2),
\end{aligned} \tag{4}$$

where ϵ is the group parameter. The invariance of eq. (3) under symmetry transformations (4) gives rise to overdetermined system of linear partial differential equations in $\xi, \tau, \eta_1, \eta_2$ and η_3. Such overdetermined system may be derived by considering the associated vector field, which may be expressed as

$$V = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta_1\frac{\partial}{\partial u} + \eta_2\frac{\partial}{\partial v} + \eta_3\frac{\partial}{\partial w}. \tag{5}$$

Third-order prolongation of vector field (5) when applied in the following manner

$$V^{(3)}(\Delta)\big|_{(3)} = 0, \text{ here } \Delta \text{ is system (3)}, \tag{6}$$

will give infinitesimals of symmetry transformation as follows:

$$\begin{aligned}
\xi &= -\frac{5c_3}{6}t + \frac{3c_1}{5}x + c_4, \quad \tau = c_1 t + c_2,\\
\eta_1 &= -\frac{2c_1}{5}u + c_3, \quad \eta_2 = -\frac{4c_1}{5}v - \frac{2c_3}{3}u,\\
\eta_3 &= -\frac{c_3}{3}v - \frac{6c_1}{5}w.
\end{aligned} \tag{7}$$

Infinitesimals (7) give the following four-dimensional Lie algebra:

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial t},\ V_2 = \frac{\partial}{\partial x} \quad \text{(translation)},\\
V_3 &= -\frac{5t}{6}\frac{\partial}{\partial x} + \frac{\partial}{\partial u} - \frac{2u}{3}\frac{\partial}{\partial v} - \frac{v}{3}\frac{\partial}{\partial w} \quad \text{(Galilean boost)},\\
V_4 &= \frac{3x}{5}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2u}{5}\frac{\partial}{\partial u}\\
&\quad - \frac{4v}{5}\frac{\partial}{\partial v} - \frac{6w}{5}\frac{\partial}{\partial w} \quad \text{(dilation)}.
\end{aligned} \tag{8}$$

The non-zero Lie commutations of Lie algebra (8) are obtained as follows:

$$\begin{aligned}
&[V_1, V_3] = -\frac{5V_2}{6}, \quad [V_1, V_4] = V_1,\\
&[V_2, V_4] = \frac{3V_2}{5}, \quad [V_3, V_4] = -\frac{2V_3}{5}.
\end{aligned} \tag{9}$$

The non-zero Lie brackets (9) show that the Lie algebra (8) is solvable.

2.1 *Construction of optimal system for Lie algebra* (8)

In the symmetry analysis, it is well-known that whenever PDEs or system of PDEs admit the symmetry group (or group of invariant transformations), then one can find group invariant solution corresponding to each subgroup by reducing the number of independent variables in the original system. There exist infinitely many such subgroups and hence infinitely many group invariant solutions. But, most of these group invariant solutions would be equivalent by some transformation in the full symmetry group. In order to minimise the search of inequivalent group invariant solutions under transformations in the full symmetry group, the concept of optimal system is introduced. Although the classification of Lie algebras by using adjoint transformations was known to Lie himself, it was Ovsiannikov [15] who first used the Lie group classification to derive inequivalent group invariant solutions. Ovsiannikov used a global adjoint matrix to construct optimal systems and he further extended his technique to derive multidimensional optimal systems. In the construction of the two-dimensional optimal system, Galas and Richter [20] made some modifications in the technique of Ovsiannikov by selecting elements from the normaliser of the one-dimensional optimal system.

Apart from these techniques, the method of classifying the subalgebras proposed by Patera *et al* [21] is par excellence (see [22] for recent applications) and in their subsequent work [23] they have classified all the real Lie algebras of dim ≤ 4 under the group of inner automorphisms. It is worth mentioning that the Lie algebra of dimension greater than 4 has also been fully classified. For example, Turkowski [24] has classified all six-dimensional solvable Lie algebras containing four-dimensional nil radical. He has also classified all the Lie algebras up to dimension 9. Chou *et al* [25] and Chou and Qu [26] suggested a slightly modified technique of classifying Lie algebras. They have constructed different varieties of invariants of the group of inner automorphism including numerical and conditional invariants. Despite the early work [27] on group classification by using adjoint actions (or identification of equivalence classes based on the sign of Killing's form), the work of Chou *et al* is very useful as their additional invariants help to confirm optimality, i.e. their technique confirms the completeness and mutual inequivalence of the representatives of subalgebras. In the present work, we shall use only the Killing's form as invariant for detecting all the representatives of subalgebras. Before going any further, in this subsection, we shall first introduce some definitions and lemmas.

DEFINITION 1

The symmetric bilinear form φ on the space of Lie algebra $\mathscr{L}$, that is, the mapping $\varphi : \mathscr{L} \times \mathscr{L} \to \mathbb{R}$ is called *invariant* (relative to group Int $\mathscr{L}$) if for any inner automorphism $A \in \text{Int}\,\mathscr{L}$, and for any $V_1, V_2 \in \mathscr{L}$

$$\varphi\langle A\langle V_1\rangle, A\langle V_2\rangle\rangle = \varphi\langle V_1, V_2\rangle.$$

In terms of adjoint representation Ad_g, the real function φ on Lie algebra $\mathscr{L}$ is invariant if and only if $\varphi(\text{Ad}_g(V)) = \varphi(V)$ for all $V \in \mathscr{L}$ and $g \in G$ (group generated by $\mathscr{L}$).

DEFINITION 2

Let $\mathscr{L}$ be a Lie algebra and $V \in \mathscr{L}$. Then the adjoint transformation defined by V is the linear transformation $\text{ad}(V) : \mathscr{L} \to \mathscr{L}$ defined by

$$\text{ad}(V)(W) = [V, W], \text{ for all } W \in \mathscr{L},$$

where $[\cdot\,, \cdot]$ is the usual Lie bracket. The exponential of $\text{ad}(X)$, usually denoted by $\text{Ad}(X)$, is a Lie algebra isomorphism. The symmetric bilinear form $K : \mathscr{L} \times \mathscr{L} \to \mathbb{R}$ defined by

$$K\langle V, W\rangle = \text{tr}(\text{ad}(V) \circ \text{ad}(W))$$

is called the Killing's form. This Killing's form is the invariant of the group of inner automorphism Int $\mathscr{L}$, the importance of which we shall realise during the construction of optimal system of subalgebras.

Lemma 1. *Let* $V = \sum_{i=1}^{4} a_i V_i$ *be the general element of Lie algebra* $\mathscr{L}^4$ *given by* (8) *and* $a_1, \ldots, a_4 \in \mathbb{R}$. *The invariant function* φ *is of the form* $f(a_4)$. *Here* f *is an arbitrary function.*

Proof. The general invariant function φ can be obtained by solving the system of linear partial differential equations given by

$$a_4 \frac{\partial\varphi}{\partial a_2} = 0, \quad a_4 \frac{\partial\varphi}{\partial a_1} - \frac{5a_3}{6}\frac{\partial\varphi}{\partial a_2} = 0,$$
$$-\frac{2a_4}{5}\frac{\partial\varphi}{\partial a_3} + \frac{5a_1}{6}\frac{\partial\varphi}{\partial a_2} = 0,$$
$$-a_1\frac{\partial\varphi}{\partial a_1} + \frac{2a_3}{5}\frac{\partial\varphi}{\partial a_3} - \frac{3a_2}{5}\frac{\partial\varphi}{\partial a_2} = 0. \tag{10}$$

A straightforward solution of system (10) is $f(a_4)$ for arbitrary function f. The procedure for constructing the system of PDEs (10) is discussed in detail in [28]. □

Lemma 2. *The Killing's form is particularly invariant which can be derived from the general solution of system* (10), *for Lie algebra* (8) $K\langle V, V\rangle = \frac{38}{25}a_4^2$.

Proof. The direct computations show that

$$\text{ad}(V) = \begin{bmatrix} -a_4 & 0 & 0 & a_1 \\ 5a_3/6 & -3a_4/5 & -5a_1/6 & 3a_2/5 \\ 0 & 0 & 2a_4/5 & -2a_3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

By definition of the Killing's form, $K\langle V, V\rangle = \text{tr}(\text{ad}(V) \circ \text{ad}(V)) = \frac{38}{25}a_4^2$. □

In search of the group invariant solutions, one ought to be careful of instances where two group invariant solutions can be recovered from each other by some transformation in the full symmetry group. For example, the two group invariant solutions Ψ_1 and Ψ_2 are called essentially inequivalent if it is impossible to connect these solutions by some four parameter group transformation $\tilde{\psi} = \exp[\sum_{i=1}^{4} a_i V_i]\psi$. In this manner, the group invariant solutions separate into equivalence classes, and the collection of generators corresponding to these classes would constitute an optimal system. In order to find such equivalence classes, we define the following adjoint operator:

$$\mathrm{Ad}_{\exp(\epsilon V)}(W) = \exp(-\epsilon V) W \exp(\epsilon V) = \tilde{W}(\epsilon). \quad (12)$$

The adjoint transformation (12) can be written through Lie brackets using the Campbell–Hausdorff formula as follows:

$$\mathrm{Ad}_{\exp(\epsilon V)}(W) = W - \epsilon[V, W] + \frac{\epsilon^2}{2}[V, [V, W]] - \cdots, \quad (13)$$

where $[\cdot, \cdot]$ is the Lie bracket defined by (9). Let $V = \sum_{i=1}^{4} a_i V_i$, and based on this Lie bracket and formula (13), the straightforward calculations show that

$$\mathrm{Ad}_{\exp(\epsilon_3 v_3)}\, \mathrm{Ad}_{\exp(\epsilon_4 v_4)}\, \mathrm{Ad}_{\exp(\epsilon_1 v_1)}\, \mathrm{Ad}_{\exp(\epsilon_2 v_2)}(V) = \sum_{i=1}^{4} \tilde{a}_i V_i. \quad (14)$$

The full adjoint transformation (14) in the matrix notation is

$$A = \begin{bmatrix} e^{\epsilon_4} & 0 & 0 & -\epsilon_1 e^{\epsilon_4} \\ -\frac{5\epsilon_3}{6} e^{\epsilon_4} & e^{\frac{3\epsilon_4}{5}} & \frac{5\epsilon_1}{6} e^{\frac{3\epsilon_4}{5}} & -\frac{3\epsilon_2}{5} e^{\frac{3\epsilon_4}{5}} + \frac{5\epsilon_1\epsilon_3}{6} e^{\epsilon_4} \\ 0 & 0 & e^{-\frac{2\epsilon_4}{5}} & \frac{2\epsilon_3}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (15)$$

The construction of adjoint matrix (15) is discussed in [15], where the coefficients $\tilde{a}_1, \ldots, \tilde{a}_4$ in (14) are given by the following relations:

$$\begin{aligned} \tilde{a}_1 &= -a_4\epsilon_1 e^{\epsilon_4} + a_1 e^{\epsilon_4}, \\ \tilde{a}_2 &= \frac{5a_3\epsilon_1}{6} e^{\frac{3\epsilon_4}{5}} - \frac{3a_4\epsilon_2}{5} e^{\frac{3\epsilon_4}{5}} + a_2 e^{\frac{3\epsilon_4}{5}} + \frac{5a_4\epsilon_1\epsilon_3}{6} e^{\epsilon_4} - \frac{5a_1\epsilon_3}{6} e^{\epsilon_4}, \\ \tilde{a}_3 &= \frac{2a_4\epsilon_3}{5} + a_3 e^{-\frac{2\epsilon_4}{5}}, \\ \tilde{a}_4 &= a_4. \end{aligned} \quad (16)$$

The last equation in (16) agrees with the invariance of the Killing's form under full adjoint transformation (14).

Theorem 3. *The one-dimensional optimal system corresponding to Lie algebra* (7) *is* $\{V_1, V_2, V_3, V_4, \alpha V_1 \pm V_3\}$.

Proof. Let $V = \sum_{i=1}^{4} a_i V_i$ and $K = \frac{38}{25} a_4^2$. We have the following cases for K.

Case 1. For $K \neq 0$, we take $a_4 = 1$. Choosing $\epsilon_4 = 0$, system of eq. (16) becomes

$$\begin{aligned} \tilde{a}_1 &= -\epsilon_1 + a_1, \\ \tilde{a}_2 &= \frac{5a_3\epsilon_1}{6} - \frac{3\epsilon_2}{5} + a_2 + \frac{5\epsilon_1\epsilon_3}{6} - \frac{5a_1\epsilon_3}{6}, \\ \tilde{a}_3 &= \frac{2\epsilon_3}{5} + a_3, \\ \tilde{a}_4 &= 1. \end{aligned}$$

The selection

$$\epsilon_1 = a_1, \quad \epsilon_2 = \frac{25\, a_3 a_1}{18} + \frac{5a_2}{3}, \quad \epsilon_3 = -\frac{5a_3}{2}$$

gives $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = 0$, and we obtain the simplification $V = V_4$.

Case 2. For $K = 0$, we have to take $a_4 = 0$.

(1) $a_3 = 1$. Choosing $\epsilon_4 = \frac{5}{2}\ln(a_3)$ gives $\tilde{a}_3 = \pm 1$, and appropriate selection of ϵ_1, ϵ_3 gives $\tilde{a}_1 = a_1 {a_3}^{5/2}$ and $\tilde{a}_2 = 0$. We obtain the simplification $V = \alpha V_1 \pm V_3$, $\alpha = a_1 {a_3}^{5/2}$.

(2) $a_3 = 0$. System (16) reduces to $\tilde{a}_1 = a_1 e^{\epsilon_4}$, $\tilde{a}_2 = a_2 e^{3\epsilon_4/5} - ((5a_1\epsilon_3/6)\, e^{\epsilon_4})$. By taking $\epsilon_3 = (6a_2 e^{-(2\epsilon_4/5)})/5a_1$, we obtain the simplification $V = V_1$.

(3) $a_3 = 0, a_1 = 0$. In this case we obtain the straightforward simplification $V = V_2$.

(4) $a_3 \neq 0, a_1 = 0$.

$$\tilde{a}_2 = \frac{5a_3\epsilon_1}{6} e^{\frac{3\epsilon_4}{5}} + a_2 e^{\frac{3\epsilon_4}{5}}, \quad \tilde{a}_3 = a_3 e^{-\frac{2\epsilon_4}{5}}.$$

By taking $\epsilon_1 = -(6a_2/5a_3)$ we obtain the simplification $V = V_3$. □

2.2 *Symmetry reductions and invariant solutions*

By virtue of vector fields V_1 and V_2, we can see that eq. (3) admits symmetry in space and time translation. So by letting $\xi = x - ct$, we have similarity transformations $u = F(\xi)$, $v = G(\xi)$ and $w = H(\xi)$. Substituting into system (3), we obtain

$$-cF_\xi - \frac{3}{2}FF_\xi - G_\xi = 0, \tag{17a}$$

$$-cG_\xi - GF_\xi - \frac{1}{2}FG_\xi - H_\xi = 0, \tag{17b}$$

$$-cH_\xi + \frac{1}{4}F_{\xi,\xi,\xi} - HF_\xi - \frac{1}{2}FH_\xi = 0. \tag{17c}$$

Integrating (17a) with respect to ξ gives

$$G = -cF - \frac{3}{4}F^2 + d_1, \tag{18}$$

here d is a constant of integration.

Again substituting this G into (17b) gives

$$H_\xi = (c^2 - d_1)F_\xi + 3cFF_\xi + \frac{3}{2}F^2F_\xi,$$

followed by integrating once with respect to ξ

$$H = (c^2 - d_1)F + \frac{3c}{2}F^2 + \frac{1}{2}F^3 + d_2. \tag{19}$$

Substituting (18) and (19) into (17c) gives

$$-c^3F_\xi + cd_1F_\xi - \frac{9}{2}c^2FF_\xi - \frac{9}{2}cF_\xi F^2 + \frac{1}{4}F_{\xi,\xi,\xi} + \frac{3}{2}d_1F_\xi F - \frac{5}{4}F_\xi F^3 - d_2F_\xi = 0. \tag{20}$$

Integrating (20) with respect to ξ and then second integration after using the integrating factor F_ξ gives

$$(F_\xi)^2 = \frac{1}{2}F^5 + 3cF^4 + (6c^2 - 2d_1)F^3 + 4(c^3 - cd_1 + d_2)F^2 + 8d_3F + 8d_4, \tag{21}$$

where c, d_1, d_2, d_3, d_4 are constants of integration. Detailed discussion about the solution of (21) can be seen in [2]. The reductions corresponding to the rest of the vector fields have been classified in the following cases.

Case 3. Reduction under the subalgebra V_3.

- Similarity variables:

$$\xi = t,$$
$$u = F(t) - \frac{6x}{5t},$$
$$v = G(t) + \frac{4ux}{5t} + \frac{12x^2}{25t^2},$$
$$w = H(t) + \frac{2vx}{5t} - \frac{4ux^2}{25t^2} - \frac{8x^3}{125t^3}. \tag{22}$$

- Reduced system: Substituting (22) into (3), the reduced system is obtained as follows:

$$tF_t + F = 0,$$
$$2tF^2 - 5t^2G_t - 4txF_t - 4xF - 4tG = 0,$$
$$4txF^2 + 5t^2FG - 25t^3H_t - 10t^2xG_t - 4tx^2F_t - 4x^2F - 8txG - 30t^2H = 0. \tag{23}$$

- Similarity solutions:

$$u = \frac{c_1}{t} - \frac{6x}{5t}, \tag{24}$$

$$v = -\frac{{c_1}^2}{3t^2} + \frac{c_2}{t^{4/5}} + \frac{4c_1x}{5t^2} - \frac{12x^2}{25t^2}, \tag{25}$$

$$w = -\frac{c_1c_2}{3t^{9/5}} + \frac{{c_1}^3}{27t^3} + \frac{c_3}{t^{6/5}} - \frac{2x{c_1}^2}{15t^3} \tag{26}$$

$$+ \frac{2xc_2}{5t^{9/5}} + \frac{4c_1x^2}{25t^3} - \frac{8x^3}{125t^3}. \tag{27}$$

Case 4. Reduction under subalgebra V_4.

- Similarity variables:

$$\xi = \frac{t}{x^{5/3}}, \quad u = \frac{1}{x^{2/3}}F(\xi), \quad v = \frac{1}{x^{4/3}}G(\xi),$$
$$w = \frac{1}{x^2}H(\xi). \tag{28}$$

- Reduced system: Substituting (28) into (3), the reduced system is obtained as follows:

$$15F\xi F_\xi + 6F^2 + 10\xi G_\xi + 8G + 6F_\xi = 0,$$
$$5F\xi G_\xi + 10G\xi F_\xi + 8FG + 10\xi H_\xi + 12H + 6G_\xi = 0,$$
$$125\xi^3F_{\xi,\xi,\xi} - 90F\xi H_\xi - 180H\xi F_\xi + 750\xi^2F_{\xi,\xi} - 180FH + 830\xi F_\xi + 80F - 108H_\xi = 0. \tag{29}$$

- Similarity solutions:

$$u = \frac{1}{x^{2/3}}\sum_{n=0}^{\infty}P_n\left(\frac{t}{x^{5/3}}\right)^n,$$
$$v = \frac{1}{x^{4/3}}\sum_{n=0}^{\infty}Q_n\left(\frac{t}{x^{5/3}}\right)^n,$$
$$w = \frac{1}{x^2}\sum_{n=0}^{\infty}R_n\left(\frac{t}{x^{5/3}}\right)^n, \tag{30}$$

where the coefficients P_n, Q_n and R_n are obtained in Theorem 4.

Case 5. Reduction under subalgebra $\alpha V_1 + V_3$.

- Similarity variables:

$$\xi = \frac{12\alpha x}{5} + t^2,$$
$$u = -F(\xi) - \frac{t}{\alpha},$$
$$v = G(\xi) - \frac{2tF(\xi)}{3\alpha} + \frac{4x}{5\alpha},$$
$$w = H(\xi) - \frac{tG(\xi)}{3\alpha} - \frac{4xF(\xi)}{15\alpha} - \frac{4tx}{45\alpha^2} - \frac{2t\xi}{27\alpha^3}. \quad (31)$$

- Reduced system: Substituting (31) into (3), the reduced system is obtained as follows:

$$18F\alpha^2F_\xi + 12\alpha^2G_\xi - 1 = 0,$$
$$9F\alpha^3G_\xi + 18G\alpha^3F_\xi + 18\alpha^3H_\xi + 4\alpha\xi F_\xi + 6F\alpha = 0,$$
$$3888\alpha^6F_{\xi,\xi,\xi} - 1350F\alpha^4H_\xi - 2700H\alpha^4F_\xi + 450F\alpha^2\xi F_\xi + 150F^2\alpha^2 - 375G\alpha^2 - 125\xi = 0. \quad (32)$$

- Similarity solutions:

$$u = -\sum_{n=0}^{\infty} P_n\left(\frac{12\alpha x}{5} + t^2\right)^n - \frac{t}{\alpha},$$
$$v = \sum_{n=0}^{\infty} Q_n\left(\frac{12\alpha x}{5} + t^2\right)^n - \frac{2t}{3\alpha}\sum_{n=0}^{\infty} P_n\left(\frac{12\alpha x}{5} + t^2\right)^n + \frac{4x}{5\alpha},$$
$$w = \sum_{n=0}^{\infty} R_n\left(\frac{12\alpha x}{5} + t^2\right)^n - \frac{t}{3\alpha}\sum_{n=0}^{\infty} Q_n\left(\frac{12\alpha x}{5} + t^2\right)^n - \frac{4x}{15\alpha}\sum_{n=0}^{\infty} P_n\left(\frac{12\alpha x}{5} + t^2\right)^n - \frac{4tx}{45\alpha^2} - \frac{2t\left(\frac{12\alpha x}{5} + t^2\right)}{27\alpha^3}, \quad (33)$$

where the coefficients P_n, Q_n and R_n are obtained in Theorem 5.

For similarity solutions of reductions corresponding to vector V_4 and $\alpha V_1 + V_3$, we seek power series solution of the form

$$F = \sum_{n=0}^{\infty} P_n\xi^n, \quad G = \sum_{n=0}^{\infty} Q_n\xi^n, \quad H = \sum_{n=0}^{\infty} R_n\xi^n, \quad (34)$$

where P_n, Q_n and R_n are unknown coefficients of power series that need to be determined later. On substituting (34) into reductions corresponding to respective vector fields we have the following theorems:

Theorem 4. *Substitution of power series* (34) *into reductions corresponding to vector field* V_4 *gives the following recurrence relations*:

$$P_{n+1} = -\frac{1}{6(n+1)}\left(10nQ_n + 15\sum_{k=0}^{n}(n-k)P_kP_{n-k} + 6\sum_{k=0}^{n}P_kP_{n-k} + 8Q_n\right),$$
$$Q_{n+1} = -\frac{1}{6(n+1)}\left(10nR_n + 5\sum_{k=0}^{n}(n-k)P_kQ_{n-k} + 10\sum_{k=0}^{n}(n-k)Q_kP_{n-k} + 8\sum_{k=0}^{n}P_kQ_{n-k} + 12R_n\right),$$
$$R_{n+1} = \frac{-1}{108(n+1)}\left(-125n^3P_n - 375n^2P_n - 330nP_n + 180\sum_{k=0}^{n}P_kR_{n-k} + 90\sum_{k=0}^{n}(n-k)P_kR_{n-k} + 180\sum_{k=0}^{n}(n-k)R_kP_{n-k} - 80P_n\right), \quad (35)$$

where P_0, Q_0, R_0 *ought to be taken as arbitrary, and*

$$P_1 = -{P_0}^2 - \frac{4Q_0}{3}, \quad Q_1 = -\frac{4P_0Q_0}{3} - 2R_0,$$
$$R_1 = -\frac{5P_0R_0}{3} + \frac{20P_0}{27}. \quad (36)$$

Proof. For brevity, we have omitted detailed calculations and results in the form of power series solutions for system (3) corresponding to reductions under vector field V_4 are interpreted in Case 4. □

Theorem 5. *Substitution of power series* (34) *into reductions corresponding to vector field* $\alpha V_1 + V_3$ *gives the following recurrence relations*:

$$P_{n+3} = \frac{-1}{1296\,\alpha^4\big(n^3+6\,n^2+11\,n+6\big)} \times \Big(-450\,\alpha^2 \sum_{k=0}^{n} (n-k+1)\,P_k\,R_{n-k+1} - 900\,\alpha^2 \sum_{k=0}^{n} (n-k+1)\,R_k\,P_{n-k+1} + 50 \sum_{k=0}^{n} P_k\,P_{n-k} + 150 \sum_{k=0}^{n} (n-k)\,P_k\,P_{n-k} - 125 Q_n\Big),$$

$$Q_{n+1} = \frac{-3}{2(n+1)} \sum_{k=0}^{n} (n-k+1)\,P_k\,P_{n-k+1},$$

$$R_{n+1} = \frac{-1}{18\,\alpha^2(n+1)} \Big(9\alpha^2 \sum_{k=0}^{n} (n-k+1)\,P_k Q_{n-k+1} + 18\,\alpha^2 \sum_{k=0}^{n} (n-k+1)\,Q_k\,P_{n-k+1} + 4\,n\,P_n + 6P_n\Big)$$

and

$$P_3 = \frac{1}{31104\,\alpha^4}(1350\,\alpha^2 {P_0}^3 P_1 - 1800\,\alpha^2 P_0 P_1 Q_0 + 3600\,\alpha^2 P_1 R_0 - 875 P_0^2 + 500 Q_0),$$

$$Q_1 = \frac{1-18\,\alpha^2 P_0 P_1}{12\alpha^2},$$

$$R_1 = \frac{6\alpha^2 P_0^2 P_1 - 8\alpha^2 P_1 Q_0 - 3P_0}{8\alpha^2},$$

$$P_4 = \frac{1}{186624\,\alpha^6}(4050\alpha^4 P_0^3 P_2 + 2025\alpha^4 P_0^2 P_1^2 - 4050\alpha^4 P_0 P_1 Q_1 - 5400\,\alpha^4 P_0 P_2 Q_0 + 8100\alpha^4 P_1 R_1 + 10800\,\alpha^4 P_2 R_0 - 3000\alpha^2 P_0 P_1 + 750\,\alpha^2 Q_1 + 250),$$

$$Q_2 = -\frac{3 P_0 P_2}{2} - \frac{3 P_1^2}{4},$$

$$R_2 = \frac{1}{72\alpha^2}(54\alpha^2 P_0^2 P_2 + 27\alpha^2 P_0 P_1^2 - 54\,\alpha^2 P_1 Q_1 - 72\,\alpha^2 P_2 Q_0 - 20\,P_1), \quad (37)$$

where P_0, Q_0, R_0, P_1, P_2 are ought to be taken as arbitrary.

Proof. Again, for brevity, we have omitted detailed calculations and results in the form of power series solutions for system (3) corresponding to reductions under vector field $\alpha V_1 + V_3$ are interpreted in Case 5. □

3. Conservation laws

In physics, the conservation laws are fundamental laws which ensure that certain physical quantity will not change with time during the course of physical process [29]. Some of the well-known conservation laws in physics are conservation of mass, momentum, energy, electric charge, etc. It is a well-known fact that the Noether's theorem gives conservation laws for a system only when it has variational principle. To establish conservation laws for a system without variational structure, Ibragimov [30] has given a new theorem based on the concept of adjoint equations for nonlinear equations. In the recent literature, many authors have applied the theorem of Ibragimov to derive conservation laws. For instance, in [31] it was proved that the Camassa–Holm is strictly self-adjoint and conservation laws were also obtained without classical Lagrangians. Freire and Sampaio [32] have constructed some conservation laws for the nonlinear self-adjoint class of the generalised fifth-order equation, such as a general Kawahara equation, modified Kawahara equation and simplified modified Kawahara equation. Johnpillai and Khalique [33] have applied the same theorem to derive conservation laws for the generalised KdV equation of time-dependent variable coefficients. For further details about the application of theorem by Ibragimov, see [22,34–40].

Based on the theory developed in [30] and notations adopted therein, we define formal Lagrangian for eq. (3) in the following manner:

$$I = \phi(x,t)\Big(u_t - \frac{3}{2}u u_x - v_x\Big) + \psi(x,t)\Big(v_t - v u_x - \frac{1}{2}u v_x - w_x\Big) + \theta(x,t)\Big(w_t + \frac{1}{4}u_{xxx} - w u_x - \frac{1}{2}u w_x\Big), \quad (38)$$

where $\phi(x,t)$, $\psi(x,t)$ and $\theta(x,t)$ are new dependent variables. The adjoint equations for (3) can be written as

$$F^* = \frac{\delta I}{\delta u} = 0,\ G^* = \frac{\delta I}{\delta v} = 0,\ H^* = \frac{\delta I}{\delta w} = 0, \quad (39)$$

where we have used the variational derivative $\delta/\delta u^\alpha$ defined by the relation

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_s}}. \quad (40)$$

Substituting Lagrangian (38) into (39) and using relation (40), we obtain the adjoint equations:

$$F^* = \frac{1}{2}\psi v_x + \frac{1}{2}\theta w_x + \frac{3}{2}\phi_x u + \psi_x v + \theta_x w - \theta_t - \frac{1}{4}\theta_{xxx} = 0,$$

$$G^* = -\frac{1}{2}\psi u_x + \phi_x + \frac{1}{2}\psi_x u - \psi_t = 0,$$

$$H^* = -\frac{1}{2}\theta u_x + \psi_x + \frac{1}{2}\theta_x u - \theta_t = 0. \quad (41)$$

For conservation laws we shall use the following theorem proved in [30].

Theorem 6. *Any infinitesimal symmetry (Lie point, Lie Bäcklund, non-local)*

$$V = \xi^i(x, u, u_{(1)}, \dots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \dots)\frac{\partial}{\partial u^\alpha}$$

of eq. (3) leads to conservation laws $D_i(C^i) = 0$ constructed by the formula

$$C^i = \xi^i I + W^\alpha\left[\frac{\partial I}{\partial u_i^\alpha} - D_j\left(\frac{\partial I}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial I}{\partial u_{ijk}^\alpha}\right) - \cdots\right] + D_j(W^\alpha)\left[\frac{\partial I}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial I}{\partial u_{ijk}^\alpha}\right) + \cdots\right] + D_j D_k(W^\alpha)\left[\frac{\partial I}{\partial u_{ijk}^\alpha} - \cdots\right], \quad (42)$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$ and I is the Lagrangian defined by (38).

Relation (42) can be simplified by writing Lagrangian I with respect to all mixed derivative $u_{ij}^\alpha, u_{ijk}^\alpha, \dots$ in a symmetric manner. We obtain

$$C^x = \xi I + W^{(1)}\left[\frac{\partial I}{\partial u_x} - D_x\left(\frac{\partial I}{\partial u_{xx}}\right) + D_x^2\left(\frac{\partial I}{\partial u_{xxx}}\right)\right] + W^{(2)}\frac{\partial I}{\partial v_x} + W^{(3)}\frac{\partial I}{\partial w_x} + D_x(W^{(1)}) \times\left[\left(\frac{\partial I}{\partial u_{xx}}\right) - D_x\left(\frac{\partial I}{\partial u_{xxx}}\right)\right] + D_x^2(W^{(1)})\frac{\partial I}{\partial u_{xxx}},$$

$$C^t = \tau I + W^{(1)}\frac{\partial I}{\partial u_t} + W^{(2)}\frac{\partial I}{\partial v_t} + W^{(3)}\frac{\partial I}{\partial w_t}, \quad (43)$$

where D_i denotes the operator of total differentiation:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha\frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha} + \cdots,$$

and rest of the details about notations can be seen in [30].

In the following cases, we shall find conserved currents (43) corresponding to every symmetry generator of optimal system obtained in Theorem 3.

Case 1. For the generator $V_1 = \partial/\partial t$, the Lie's characteristic functions are obtained as follows:

$$W^{(1)} = -u_t,\ W^{(2)} = -v_t,\ W^{(3)} = -w_t. \quad (44)$$

Substituting (44) into (43) yields the following conserved currents:

$$C^x = \frac{3}{2}u_t\phi u + u_t\psi v + u_t\theta w - \frac{1}{4}u_t\theta_{xx} + v_t\phi + \frac{1}{2}v_t\psi u + w_t\psi + \frac{1}{2}w_t\theta u + \frac{1}{4}u_{xt}\theta_x - \frac{1}{4}u_{xxt}\theta,$$

$$C^t = -\frac{3}{2}\phi u u_x - \phi v_x - \psi v u_x - \frac{1}{2}\psi u v_x - \psi w_x + \frac{1}{4}\theta u_{xxx} - \theta w u_x - \frac{1}{2}\theta u w_x, \quad (45)$$

where $\phi(x,t)$, $\psi(x,t)$ and $\theta(x,t)$ are arbitrary solutions of adjoint equation (41).

Case 2. For the generator $V_2 = \partial/\partial x$, the Lie's characteristic functions are obtained as follows:

$$W^{(1)} = -u_x,\ W^{(2)} = -v_x,\ W^{(3)} = -w_x. \quad (46)$$

Substituting (46) into (43) yields the following conserved currents:

$$C^x = u_t\phi + v_t\psi + w_t\theta - \frac{1}{4}u_x\theta_{xx} + \frac{1}{4}u_{x,x}\theta_x,$$

$$C^t = -\phi u_x - \psi v_x - \theta w_x, \quad (47)$$

where $\phi(x,t)$, $\psi(x,t)$ and $\theta(x,t)$ are the arbitrary solutions of adjoint equation (41).

Case 3. For the generator

$$V_3 = -\frac{5t}{6}\frac{\partial}{\partial x} + \frac{\partial}{\partial u} - \frac{2u}{3}\frac{\partial}{\partial v} - \frac{v}{3}\frac{\partial}{\partial w},$$

the Lie's characteristic functions are obtained as follows:

$$W^{(1)} = 1 + \frac{5t}{6}u_x,\ W^{(2)} = -\frac{2u}{3} + \frac{5t}{6}v_x,$$

$$W^{(3)} = -\frac{v}{3} + \frac{5t}{6}w_x. \quad (48)$$

Substituting (48) into (43) yields the following conserved currents:

$$C^x = \frac{1}{3}\psi u^2 - \frac{5}{6}tu_t\phi - \frac{5}{6}tv_t\psi - \frac{5}{6}tw_t\theta + \frac{5tu_x\theta_{xx}}{24} + \frac{1}{6}v\theta u - \frac{5tu_{xx}\theta_x}{24} + \frac{1}{4}\theta_{xx} - \frac{2}{3}\psi v - \theta w - \frac{5}{6}\phi u,$$

$$C^t = \phi + \frac{5}{6}\phi t u_x - \frac{2}{3}\psi u + \frac{5}{6}\psi t v_x - \frac{1}{3}v\theta + \frac{5}{6}\theta t w_x, \tag{49}$$

where $\phi(x,t)$, $\psi(x,t)$ and $\theta(x,t)$ are arbitrary solutions of adjoint equation (41).

Case 4. For the generator

$$V_4 = \frac{3x}{5}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2u}{5}\frac{\partial}{\partial u} - \frac{4v}{5}\frac{\partial}{\partial v} - \frac{6w}{5}\frac{\partial}{\partial w},$$

the Lie's characteristic functions are obtained as follows:

$$W^{(1)} = -\frac{2u}{5} - \frac{3x}{5}u_x - tu_t,$$

$$W^{(2)} = -\frac{4v}{5} - \frac{3x}{5}v_x - tv_t,$$

$$W^{(3)} = -\frac{6w}{5} - \frac{3x}{5}w_x - tw_t. \tag{50}$$

Substituting (50) into (43) yields the following conserved currents:

$$C^x = \frac{3}{5}\phi u^2 - \frac{1}{10}u\theta_{xx} + \frac{4}{5}v\phi + \frac{6}{5}w\psi + \frac{1}{4}\theta_x u_x - \frac{2}{5}\theta u_{xx} + \frac{3}{2}tu_t\phi u + tu_t\psi v + tu_t\theta w + \frac{1}{2}tv_t\psi u + \frac{1}{2}tw_t\theta u + tv_t\phi + u\theta w - \frac{1}{4}tu_t\theta_{xx} + \frac{3\theta_x x u_{xx}}{20} + \frac{1}{4}\theta_x t u_{xt} - \frac{1}{4}\theta t u_{xxt} + \frac{4}{5}u\psi v + \frac{3}{5}x\psi v_t + \frac{3}{5}x\phi u_t - \frac{3xu_x\theta_{xx}}{20} + \frac{3}{5}x\theta w_t + tw_t\psi,$$

$$C^t = -\frac{3}{2}t\phi u u_x - t\phi v_x - t\psi v u_x - \frac{1}{2}t\psi u v_x - t\psi w_x + \frac{1}{4}tu_{xxx}\theta - t\theta w u_x - \frac{1}{2}t\theta u w_x - \frac{2}{5}\phi u - \frac{3}{5}\phi x u_x - \frac{4}{5}\psi v - \frac{3}{5}\psi x v_x - \frac{6}{5}\theta w - \frac{3}{5}\theta x w_x, \tag{51}$$

where $\phi(x,t)$, $\psi(x,t)$ and $\theta(x,t)$ are arbitrary solutions of adjoint equation (41).

Case 5. For the generator

$$\alpha V_1 + V_3 = -\frac{5t}{6}\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial t} + \frac{\partial}{\partial u} - \frac{2u}{3}\frac{\partial}{\partial v} - \frac{v}{3}\frac{\partial}{\partial w},$$

the Lie's characteristic functions are obtained as follows:

$$W^{(1)} = 1 + \frac{5t}{6}u_x - \alpha u_t,$$

$$W^{(2)} = -\frac{2u}{3} + \frac{5t}{6}v_x - \alpha v_t,$$

$$W^{(3)} = -\frac{v}{3} + \frac{5t}{6}w_x - \alpha w_t. \tag{52}$$

Substituting (52) into (43) yields the following conserved currents:

$$C^x = \frac{3}{2}\alpha u_t\phi u + \alpha u_t\psi v + \alpha u_t\theta w + \frac{1}{2}\alpha v_t\psi u + \frac{1}{2}\alpha w_t\theta u + \frac{1}{6}v\theta u + \frac{1}{4}\theta_x\alpha u_{xt} + \frac{5tu_x\theta_{xx}}{24} - \frac{5}{6}tw_t\theta + \alpha v_t\phi - \frac{1}{4}\alpha u_t\theta_{xx} - \frac{1}{4}\theta\alpha u_{xxt} - \frac{5}{6}tv_t\psi - \frac{5}{6}tu_t\phi - \frac{5tu_{xx}\theta_x}{24} + \alpha w_t\psi + \frac{1}{4}\theta_{xx} - \frac{5}{6}\phi u - \frac{2}{3}\psi v - \theta w + \frac{1}{3}\psi u^2,$$

$$C^t = -\frac{3}{2}\alpha\phi u u_x - \alpha\phi v_x - \alpha\psi v u_x - \frac{1}{2}\alpha\psi u v_x - \alpha\psi w_x + \frac{1}{4}\alpha\theta u_{xxx} - \alpha\theta w u_x - \frac{1}{2}\alpha\theta u w_x + \phi + \frac{5}{6}\phi t u_x - \frac{2}{3}\psi u + \frac{5}{6}\psi t v_x - \frac{1}{3}v\theta + \frac{5}{6}\theta t w_x, \tag{53}$$

where $\phi(x,t)$, $\psi(x,t)$ and $\theta(x,t)$ are arbitrary solutions of adjoint equation (41). In a similar manner, the conserved currents corresponding to the generator

$$\alpha V_1 - V_3 = \frac{5t}{6}\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial t} - \frac{\partial}{\partial u} + \frac{2u}{3}\frac{\partial}{\partial v} + \frac{v}{3}\frac{\partial}{\partial w}$$

can also be calculated.

Remark 1. Despite the huge success of new conservation theorem of Ibragimov, the recent comments from Anco [41] confirm the incompleteness of the theorem. In particular, the formulation proposed by Ibragimov can generate trivial conservation laws and does not always yield non-trivial conservation laws. But fortunately, in the present case, all the conservation laws given in (45), (47), (49), (51) and (53) are not trivial. Rather, these conservation laws are non-local.

4. Conclusion

Using the classical Lie symmetry analysis we have analysed three-field Kaup–Boussinesq system (3) in a comprehensive manner. Based on the Killing's form derived in Lemma 2, the complete classification of Lie algebra (8) is obtained in Theorem 3. Similarity reductions and invariant solutions using the power series method are also presented. Apart from this usual symmetry analysis, we have demonstrated the construction of several non-local conservation laws based on the theory of a new conservation theorem [30]. The work presented here emphasised the relevance of new conservation theorem by Ibragimov for the construction of conservation from Lie symmetries without the formulation of classical Lagrangian.

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