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On the integrability of some two-component Camassa–Holm-type systems

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Abstract. Some two-component Camassa–Holm-type systems are proposed. These systems are shown to be integrable with Lax pairs and bi-Hamiltonian structures. We construct dual hierarchies of these two-component Camassa–Holm-type systems via the tri-Hamiltonian duality method, and derive spectral problems of these dual hierarchies.

Keywords. Camassa–Holm-type system; Lax pair; bi-Hamiltonian structure.

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1. Introduction

In 1993, Camassa and Holm [1,2] derived the celebrated Camassa–Holm (CH) equation

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}, \tag{1}$$

from an asymptotic approximation to the Hamiltonian for the Green–Naghdi equations in shallow water theory. The CH equation is completely integrable with a Lax pair and a bi-Hamiltonian structure [1,2]. Moreover, the CH equation has many non-standard properties: (1) it possesses peaked soliton solutions which are weak solutions with discontinuous first derivatives [1–4], and the equation admitting the peakon solution is called CH-type equation and (2) it admits the weak Painlevé property [5]. Interestingly, there is a reciprocal transformation connecting the CH equation to the first negative flow of the Korteweg–de Vries (KdV) hierarchy [6], which displays the standard Painlevé test.

Subsequently, many CH-type systems are discovered and studied [7–13]. Recently, two integrable equations with peakons and cubic nonlinearity have been found. One is the Novikov's cubic nonlinear equation [14]

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}, \tag{2}$$

which was discovered by Novikov in a symmetry classification of non-local partial differential equations with quadratic or cubic nonlinearity. The Novikov equation was shown to be integrable with matrix Lax pair, bi-Hamiltonian structure and infinitely many conserved quantities, and related by a reciprocal transformation to a negative flow in the Sawada–Kotera hierarchy [15]. The other is the Fokas–Olver–Rosenau–Qiao (FORQ) equation [16–18]

$$m_t + [m(u^2 - u_x^2)]_x = 0, \quad m = u - u_{xx}, \tag{3}$$

which was shown to be integrable with the Lax pair and bi-Hamiltonian structure in [19] and corresponds to a negative flow in the (modified) KdV hierarchy by a reciprocal transformation [15].

Qu *et al* [20] recently introduced the generalised two-component CH-type system

$$\begin{aligned} m_t &= [m(u^2+v^2-u_x^2-v_x^2)]_x - 2m(uu_x + vv_x),\\ n_t &= [n(u^2+v^2-u_x^2-v_x^2)]_x - 2n(uu_x + vv_x),\\ m &= u - u_{xx}, \ n = v - v_{xx}, \end{aligned} \tag{4}$$

which is Lax integrable. Very recently, Xia *et al* [21] proposed the following generalised two-component CH-type system:

$$\begin{aligned} m_t &= (m\mathcal{H})_x + m\mathcal{H} - \tfrac{1}{2}m(u-u_x)(v+v_x),\\ n_t &= (n\mathcal{H})_x - n\mathcal{H} + \tfrac{1}{2}n(u-u_x)(v+v_x),\\ m &= u - u_{xx}, \quad n = v - v_{xx}, \end{aligned} \tag{5}$$

where $\mathcal{H}$ is an arbitrary smooth function of u, v and their derivatives. Different choices of $\mathcal{H}$ lead to different peakon equations, which include the CH equation (1), the FORQ equation (3) and Song–Qu–Qiao system [22]. Furthermore, they presented the Lax pair and infinitely many conservation laws of the system for the general $\mathcal{H}$, and discussed the bi-Hamiltonian structures of the system for the special choices of $\mathcal{H}$.

The purpose of this paper is to propose some two-component generalisations of the FORQ equation (3), which are

$$
\begin{aligned}
&m_t = -[m(u^2 + v^2 - u_x^2 - v_x^2) - 2n(u_x v_x - uv)]_x,\\
&n_t = -[n(u^2 + v^2 - u_x^2 - v_x^2) - 2m(u_x v_x - uv)]_x,\\
&m = u - u_{xx}, \quad n = v - v_{xx},
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
&m_t = -[m(u^2 - v^2 - u_x^2 + v_x^2) - 2n(uv - u_x v_x)]_x,\\
&n_t = -[n(u^2 - v^2 - u_x^2 + v_x^2) - 2m(u_x v_x - uv)]_x,\\
&m = u - u_{xx}, \quad n = v - v_{xx}
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
&m_t = -[m(u^2 - u_x^2)]_x,\\
&n_t = -[2m(uv - u_x v_x) + n(u^2 - u_x^2)]_x,\\
&m = u - u_{xx}, \quad n = v - v_{xx}.
\end{aligned}
\tag{8}
$$

Importantly, after direct calculation, we find that there are no transformations for system (4) and no functions $\mathcal{H}$ in (5) which make them transform to any of our above systems. Systems (6)–(8) reduce to eq. (3) when $v = 0$. Moreover, as $v = u$ and $v = \mathrm{i}u$, the first two systems respectively reduce to eq. (3) after a scaling transformation. We prove the integrability of systems (6)–(8) by providing their Lax pairs and bi-Hamiltonian structures. The latter is used to derive dual hierarchies (the terminology in [16]) of these systems by the tri-Hamiltonian duality method and construct the spectral problems of these dual hierarchies.

The paper is organised as follows. In §2 and §3, the Lax pairs and the bi-Hamiltonian structures of systems (6)–(8) are presented, respectively. In §4, the dual hierarchies of systems (6)–(8) are constructed. Some concluding remarks are given in §5.

2. Lax pairs of systems (6)–(8)

It was shown in [19] that the FORQ equation (3) arises as a zero curvature equation

$$F_t - G_x + [F, G] = 0, \tag{9}$$

this being the compatibility condition for the linear system

$$\phi_x = F\phi, \quad \phi_t = G\phi, \tag{10}$$

with

$$\phi = (\phi_1,\ \phi_2)^{\mathrm{T}},$$

$$F = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda m \\ -\frac{1}{2}\lambda m & \frac{1}{2} \end{pmatrix},$$

$$G = \begin{pmatrix} \frac{1}{\lambda^2} + \frac{1}{2}(u^2 - u_x^2) & -\frac{u - u_x}{\lambda} - \frac{1}{2}\lambda m(u^2 - u_x^2) \\ \frac{u + u_x}{\lambda} + \frac{1}{2}\lambda m(u^2 - u_x^2) & -\frac{1}{\lambda^2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}. \tag{11}$$

Furthermore, eq. (3) can be written in the bi-Hamiltonian form $m_t = \mathcal{K}(\delta H_2/\delta m) = \mathcal{J}(\delta H_1/\delta m)$, using the Hamiltonian operators

$$\mathcal{K} = \partial - \partial^3, \quad \mathcal{J} = \partial m \partial^{-1} m \partial, \tag{12}$$

and the associated Hamiltonian functionals

$$H_1 = -\int mu\,\mathrm{d}x,\ H_2 = -\frac{1}{4}\int u^4 + 2u^2 u_x^2 - \frac{u_x^4}{3}\mathrm{d}x. \tag{13}$$

By a direct generalisation of the FORQ equation's spectral problem (10) and (11), one can get the 4×4 matrix spectral problem

$$\Phi_x = F\Phi, \quad \Phi_t = G\Phi, \tag{14}$$

where $\Phi = (\Phi_1,\ \Phi_2)^{\mathrm{T}}$, Φ_1 and Φ_2 are two-dimensional row vectors and

$$F = \begin{pmatrix} -\frac{I}{2} & \frac{1}{2}\lambda M \\ -\frac{1}{2}\lambda M & \frac{I}{2} \end{pmatrix},$$

$$G = \begin{pmatrix} \frac{I}{\lambda^2} + \frac{1}{2}A & -\frac{U - U_x}{\lambda} - \frac{1}{2}\lambda B \\ \frac{U + U_x}{\lambda} + \frac{1}{2}\lambda C & -\frac{I}{\lambda^2} - \frac{1}{2}D \end{pmatrix}, \tag{15}$$

where I is the 2×2 identity matrix, $U,\ M$ are the 2×2 matrix potentials and $M = U - U_{xx}$. $A,\ B,\ C,\ D$ are functions of U and its derivatives. Substituting the expressions of F and G into (9), we find that

$$
\begin{aligned}
&A = (U - U_x)(U + U_x),\\
&D = (U + U_x)(U - U_x),
\end{aligned}
\tag{16}
$$

$$MC = BM, \quad MB = CM, \tag{17}$$

$$M_t = -B_x - B + \tfrac{1}{2}(MD + AM), \tag{18}$$

$$M_t = -C_x + C - \tfrac{1}{2}(MA + DM). \tag{19}$$

Conditions (17) demand that the matrices B and C can only be chosen from $(U - U_x)M(U + U_x)$ and $(U + U_x)M(U - U_x)$, and so we must restrict the choice of admissible matrix U. Setting

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the sufficient condition (17) can be satisfied by taking

$$a=d,\quad bc_x=b_xc,\quad b_xc_{xx}=b_{xx}c_x,$$
$$b_{xx}c_{xxx}=b_{xxx}c_{xx}.$$

In this paper, we take the following three possibilities to fulfil (17):

$$U=\begin{pmatrix} u & v\\ v & u\end{pmatrix},\ U=\begin{pmatrix} u & v\\ -v & u\end{pmatrix},$$
$$U=\begin{pmatrix} u & v\\ 0 & u\end{pmatrix} \tag{20}$$

and the corresponding expressions of M are

$$M=\begin{pmatrix} m & n\\ n & m\end{pmatrix},\ M=\begin{pmatrix} m & n\\ -n & m\end{pmatrix},$$
$$M=\begin{pmatrix} m & n\\ 0 & m\end{pmatrix}. \tag{21}$$

Here $m=u-u_{xx}$, $n=v-v_{xx}$. Then $B=C=(U-U_x)M(U+U_x)=(U+U_x)M(U-U_x)$ and one can get the definite expression of G as

$$G=\begin{pmatrix} \frac{I}{\lambda^2}+\frac{1}{2}(U-U_x)(U+U_x) & G_1\\ G_2 & -\frac{I}{\lambda^2}-\frac{1}{2}(U+U_x)(U-U_x)\end{pmatrix}, \tag{22}$$

with $G_1=-((U-U_x)/\lambda)-\frac{1}{2}\lambda(U-U_x)M(U+U_x)$ and $G_2=((U+U_x)/\lambda)+\frac{1}{2}\lambda(U+U_x)M(U-U_x)$, and then the matrix generalisation of the FORQ equation (3) is

$$M_t+[M(U^2-U_x^2)]_x=0,\quad M=U-U_{xx}. \tag{23}$$

Substituting different choices of U in (20) into (23), respectively, eq. (9) is equivalent to nothing but systems (6)–(8).

3. Bi-Hamiltonian structures of systems (6)–(8)

To derive the bi-Hamiltonian structures of systems (6) and (7), we first introduce the new variables $\tilde{u}=u+v$, $\tilde{v}=u-v$ and $\tilde{u}=u+\mathrm{i}v$, $\tilde{v}=u-\mathrm{i}v$ respectively and then both (6) and (7) reduce to the following two decoupled FORQ equation (3) in the new variables:

$$\tilde{m}_t=-[\tilde{m}(\tilde{u}^2-\tilde{u}_x^2)]_x,\quad \tilde{m}=\tilde{u}-\tilde{u}_{xx},$$
$$\tilde{n}_t=-[\tilde{n}(\tilde{v}^2-\tilde{v}_x^2)]_x,\quad \tilde{n}=\tilde{v}-\tilde{v}_{xx}. \tag{24}$$

It is not hard to construct and test the diagonal bi-Hamiltonian structure for the coupled system (24) with elements given by (12) in the variables $\tilde{m}$ and $\tilde{n}$, namely

$$\begin{pmatrix}\tilde{m}_t\\ \tilde{n}_t\end{pmatrix}=\tilde{\mathcal{K}}\begin{pmatrix}\delta\tilde{H}_2/\delta\tilde{m}\\ \delta\tilde{H}_2/\delta\tilde{n}\end{pmatrix}=\tilde{\mathcal{J}}\begin{pmatrix}\delta\tilde{H}_1/\delta\tilde{m}\\ \delta\tilde{H}_1/\delta\tilde{n}\end{pmatrix} \tag{25}$$

with the diagonal Hamiltonian operators

$$\tilde{\mathcal{K}}=\begin{pmatrix}\partial-\partial^3 & 0\\ 0 & \partial-\partial^3\end{pmatrix},$$
$$\tilde{\mathcal{J}}=\begin{pmatrix}\partial\tilde{m}\partial^{-1}\tilde{m}\partial & 0\\ 0 & \partial\tilde{n}\partial^{-1}\tilde{n}\partial\end{pmatrix} \tag{26}$$

and the Hamiltonian functionals

$$\tilde{H}_1=-\int \tilde{m}\tilde{u}+\tilde{n}\tilde{v}\,\mathrm{d}x, \tag{27}$$

$$\tilde{H}_2=-\frac{1}{4}\int \tilde{u}^4+2\tilde{u}^2\tilde{u}_x^2-\frac{\tilde{u}_x^4}{3} -\left(\tilde{v}^4+2\tilde{v}^2\tilde{v}_x^2-\frac{\tilde{v}_x^4}{3}\right)\mathrm{d}x. \tag{28}$$

Using the usual results given in [23], one can get the relationship between the bi-Hamiltonian operators of systems (6) and (24) as

$$\tilde{\mathcal{K}}=Q\mathcal{K}Q^*,\quad \tilde{\mathcal{J}}=Q\mathcal{J}Q^*, \tag{29}$$

where Q is the transformation matrix between the variables $\tilde{u}$, $\tilde{v}$ and u, v, i.e.

$$Q=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

and the symbol $*$ denotes the adjoint of an operator.

After direct calculation, we find that system (6) may be reformulated as a bi-Hamiltonian system

$$\begin{pmatrix}m_t\\ n_t\end{pmatrix}=\mathcal{K}\begin{pmatrix}\delta H_2/\delta m\\ \delta H_2/\delta n\end{pmatrix}=\mathcal{J}\begin{pmatrix}\delta H_1/\delta m\\ \delta H_1/\delta n\end{pmatrix}, \tag{30}$$

using the Hamiltonian operators

$$\mathcal{K}=\begin{pmatrix}0 & \partial-\partial^3\\ \partial-\partial^3 & 0\end{pmatrix}, \tag{31}$$

$$\mathcal{J}=\begin{pmatrix}\partial m\partial^{-1}m\partial+\partial n\partial^{-1}n\partial & \partial m\partial^{-1}n\partial+\partial n\partial^{-1}m\partial\\ \partial m\partial^{-1}n\partial+\partial n\partial^{-1}m\partial & \partial m\partial^{-1}m\partial+\partial n\partial^{-1}n\partial\end{pmatrix} \tag{32}$$

and the Hamiltonian functionals

$$H_1 = -\int um + vn\,\mathrm{d}x,$$

$$H_2 = -\int (u^2+v^2)(uv+u_xv_x) + (u_x^2+v_x^2)\left(uv-\frac{1}{3}u_xv_x\right)\mathrm{d}x.$$

In a similar way, system (7) is also found to have the bi-Hamiltonian form (30). Here the Hamiltonian operator $\mathcal{K}$ is the same as operator (31) and

$$\mathcal{J} = \begin{pmatrix} \partial m\partial^{-1}m\partial - \partial n\partial^{-1}n\partial & \partial m\partial^{-1}n\partial + \partial n\partial^{-1}m\partial \\ \partial m\partial^{-1}n\partial + \partial n\partial^{-1}m\partial & -\partial m\partial^{-1}m\partial + \partial n\partial^{-1}n\partial \end{pmatrix} \tag{33}$$

and the Hamiltonian functionals are

$$H_1 = -\int um - vn\,\mathrm{d}x,$$

$$H_2 = -\int (u^2-v^2)(uv+u_xv_x) + (u_x^2-v_x^2)\left(uv-\frac{1}{3}u_xv_x\right)\mathrm{d}x.$$

System (8) can be rewritten as

$$m_t = -[m(u^2-u_x^2)]_x = K(u),$$
$$n_t = K'(u)[v],$$
$$m = u - u_{xx}, \quad n = v - v_{xx}, \tag{34}$$

where $K'(u)[v] = (\mathrm{d}/\mathrm{d}\epsilon)K(u+\epsilon v)|_{\epsilon=0}$ is the Fréchet derivative of $K(u)$ in the direction v. Using the formulas in [24], bi-Hamiltonian structure of system (8) or (34) can be derived directly from the bi-Hamiltonian structure of the FORQ equation (3) as

$$\mathcal{K} = \begin{pmatrix} 0 & \underline{\mathcal{K}} \\ \underline{\mathcal{K}} & \underline{\mathcal{K}}'(u)[v] \end{pmatrix},$$

$$\mathcal{J} = \begin{pmatrix} 0 & \underline{\mathcal{J}} \\ \underline{\mathcal{J}} & \underline{\mathcal{J}}'(u)[v] \end{pmatrix},$$

$$H = \underline{H}'(u)[v].$$

and

$$\mathcal{J} = \begin{pmatrix} 0 & \partial m\partial^{-1}m\partial \\ \partial m\partial^{-1}m\partial & \partial m\partial^{-1}n\partial + \partial n\partial^{-1}m\partial \end{pmatrix} \tag{35}$$

and the associated Hamiltonian functionals are

$$H_1 = -\int um + vn\,\mathrm{d}x,$$

$$H_2 = -\int u^2(uv+u_xv_x) + u_x^2v\left(u-\frac{1}{3}u_xv_x\right)\mathrm{d}x.$$

Remark 1. We employ the analogous methods presented in [25] to study the Lax pairs and bi-Hamiltonian structures of the two-component CH-type systems (6)–(8).

4. Dual hierarchies of systems (6)–(8)

By the matrix generalisations of the spectral problem of FORQ equation (3), we get the two-component CH-type systems (6)–(8). If the dual hierarchies of these two-component systems are the corresponding matrix generalisations of the modified Korteweg–de Vries (mKdV) hierarchy, then a nature idea arises. The answer is not apparent. As we know, the CH and FORQ equations are reciprocally linked to the first negative flow of the KdV and mKdV hierarchies respectively, the latter two hierarchies are connected by a Miura transformation. The relation between the CH and FORQ equations, however, is a reciprocal transformation [26]. Therefore, it is necessary to study the dual hierarchies of systems (6)–(8).

Through the tri-Hamiltonian duality method proposed in [16], we have the duality of the Hamiltonian pair $\mathcal{K}$ and $\mathcal{J}$ of the CH-type system (6) as

$$\hat{\mathcal{K}} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{36}$$

and

$$\hat{\mathcal{J}} = \begin{pmatrix} \partial m\partial^{-1}m\partial + \partial n\partial^{-1}n\partial & \partial^3 + \partial m\partial^{-1}n\partial + \partial n\partial^{-1}m\partial \\ \partial^3 + \partial m\partial^{-1}n\partial + \partial n\partial^{-1}m\partial & \partial m\partial^{-1}m\partial + \partial n\partial^{-1}n\partial \end{pmatrix}, \tag{37}$$

Here the symbol $-$ refers to the objects of the scalar FORQ equation (3). Therefore, system (8) can be rewritten as the bi-Hamiltonian form (30). Now the Hamiltonian operator $\mathcal{K}$ is also equal to operator (31)

which is just the bi-Hamiltonian operators of a generalised mKdV hierarchy. Applying the recursion operator $\hat{\mathcal{R}} = \hat{\mathcal{J}}\hat{\mathcal{K}}^{-1}$ to the symmetry $(m_x, n_x)^{\mathrm{T}}$, one has the two-component system

$$m_t = m_{xxx} + \frac{3}{2}(m^2+n^2)n_x + 3mnm_x,$$
$$n_t = n_{xxx} + \frac{3}{2}(m^2+n^2)m_x + 3mnn_x. \quad (38)$$

Setting $m = n$, the above system reduces to the mKdV equation

$$m_t = m_{xxx} + 6m^2 m_x, \quad (39)$$

which admits the spectral problem

$$\phi_x = F\phi, \quad F = \begin{pmatrix} \lambda & m \\ -m & -\lambda \end{pmatrix}. \quad (40)$$

Applying similar matrix generalisation of spectral problem of the FORQ equation (3) to spectral problem (40), one has the 4 × 4 spectral problem

$$\Phi_x = F\Phi, \quad F = \begin{pmatrix} \lambda I & V \\ -V & -\lambda I \end{pmatrix} \quad (41)$$

with

$$V = \begin{pmatrix} v_1 & v_2 \\ v_2 & v_1 \end{pmatrix}. \quad (42)$$

According to the procedure for deriving the bi-Hamiltonian operators of system (6) presented in §3, we obtain bi-Hamiltonian operators for the generalised mKdV hierarchy with the spectral problem (41) and (42) as

$$\bar{\mathcal{K}} = \hat{\mathcal{K}} \quad (43)$$

and

$$\bar{\mathcal{J}} = \begin{pmatrix} \partial^3 + \partial v_1 \partial^{-1} v_1 \partial + \partial v_2 \partial^{-1} v_2 \partial & \partial v_1 \partial^{-1} v_2 \partial + \partial v_2 \partial^{-1} v_1 \partial \\ \partial v_1 \partial^{-1} v_2 \partial + \partial v_2 \partial^{-1} v_1 \partial & \partial^3 + \partial v_1 \partial^{-1} v_1 \partial + \partial v_2 \partial^{-1} v_2 \partial \end{pmatrix}. \quad (44)$$

Through formula (29), we find that the operator $\bar{\mathcal{J}}$ can be transformed to the operator $\hat{\mathcal{J}}$ under the variable transformation

$$v_1 = \frac{\mathrm{i}+1}{2}m + \frac{1-\mathrm{i}}{2}n,$$
$$v_2 = \frac{1-\mathrm{i}}{2}m + \frac{\mathrm{i}+1}{2}n. \quad (45)$$

Then from the above transformation, one can easily get the spectral problem of the dual hierarchy of the two-component system (6) as the linear system (41) with

$$V = \begin{pmatrix} \frac{\mathrm{i}+1}{2}m + \frac{1-\mathrm{i}}{2}n & \frac{1-\mathrm{i}}{2}m + \frac{\mathrm{i}+1}{2}n \\ \frac{1-\mathrm{i}}{2}m + \frac{\mathrm{i}+1}{2}n & \frac{\mathrm{i}+1}{2}m + \frac{1-\mathrm{i}}{2}n \end{pmatrix}.$$

Using the above method, one can derive the spectral problem of the dual hierarchy of system (7), which is the linear system (41) with

$$V = \begin{pmatrix} \frac{\sqrt{2}}{2}(m+n) & \frac{\sqrt{2}}{2}(m-n) \\ -\frac{\sqrt{2}}{2}(m-n) & \frac{\sqrt{2}}{2}(m+n) \end{pmatrix}.$$

It is not hard to find that the duality of the Hamiltonian pair $\mathcal{K}$ and $\mathcal{J}$ of system (8) is just that of the corresponding matrix generalisation of the mKdV equation (39). So the spectral problem for its dual hierarchy is the linear system (41) with

$$V = \begin{pmatrix} m & n \\ 0 & m \end{pmatrix}.$$

5. Conclusions and discussions

In this paper, we present three two-component CH-type systems, and provide their Lax pairs and bi-Hamiltonian structures. The dual hierarchies of these two-component CH-type systems are just the corresponding matrix generalisations of mKdV hierarchy after the variable transformations, which lead to the spectral problems of the dual hierarchies.

A remarkable property of the CH-type equation is that it possesses peakon solutions. How to derive the peakon solutions of our two-component CH-type systems (6)–(8) and other related issues will be discussed in future research.

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