Pramana – J. Phys. (2019) 92:63
https://doi.org/10.1007/s12043-019-1728-0

Brief report: Extremely dense general relativistic polytropes of index $n = 1$

RAJAT ROY

Department of Electronics and Electrical Communication Engineering, Indian Institute of Technology, Kharagpur 721 302, India
E-mail: rajatroy@ece.iitkgp.ac.in

MS received 27 April 2018; revised 17 July 2018; accepted 24 August 2018; published online 21 February 2019

Abstract. Polytropic gas spheres of index 1 and extremely high central densities are analysed with the help of general relativistic field equations. Parameters such as the radius and mass are calculated for different central densities. The limiting values of these quantities are obtained and the physical nature of sound waves in these bodies is verified.

Keywords. Polytropes of index 1; general relativity; stellar hydrostatics.

PACS Nos 04.20.−q; 04.40.Dg; 47.35.Rs

1. Introduction

It is a well-known fact from the Newtonian theory of stellar hydrostatic equilibrium [1] that $n = 1$ polytropes have a fixed radius independent of its mass. That is, if one adds more matter which obeys the polytropic $n = 1$ equation of state to an existing star made of the same constituents, the mass of the star will increase without changing its radius. The same conclusion does not hold if the Newtonian description of the star is replaced by a general relativistic one [2,3]. In [2] a numerical approach to the problem has been adopted, which in our opinion is more suited for making actual computations rather than trying to sum up the different terms of a series presented in [3]. The purpose of the present note is to just extend the method of Tooper [2] to extremely high central densities for the $n = 1$ case compared to those given in table 1 of the same paper. We show that although the radius of the star decreases at first after the parameter σ is increased up to a certain value, it becomes almost constant thereafter even if σ is increased by several thousand fold beyond this point. Likewise the total mass also attains a constant value such that the dimensionless quantity $2GM/c^2R$ (M is the total mass including gravitational energy and R is the star's radius) attains a value 0.5768. The material energy (or the total mass minus the gravitational energy) as defined by Weinberg [4] (see eq. (11.1.19)) also remains a constant implying that the star cannot absorb any more in-falling matter without violating its $n = 1$ polytropic equation of state. The central density, which is proportional to σ, however can change, implying that there will be a redistribution of total energy (mass) inside the stellar body keeping its total value intact.

Here, we would like to state that there are other studies (see [5–11]) which are related to polytropes and are of recent origin. Of these, the last three are concerned with the stellar bodies, which have unneutralised charge giving rise to anisotropy in pressure. We think that even if there is some unneutralised charge in stars it can remain only for a short period in which it is exposed to interaction with matter in its surroundings and hence may not be of extreme physical interest. On the other hand, the authors of [5–8] discuss the anisotropy in pressure when its origin may be strong magnetic fields and/or viscosity which are present in neutron stars, white dwarfs, etc. In fact, the papers by Herrera and his co-workers also deal with the question of generalising the polytropic equation of state to the relativistic regime for the isotropic case which is directly linked to the topic discussed in the present note. We strongly feel that Case II (see p. 3 of [6]) is more physical for the following reason. The relations of the type $\rho = P \log \rho_0 + \rho_0$ given on the same page and relevant to Case I can be seen for $n = \infty$ (that is $\gamma = 1$) polytropes to yield $d\rho/d\rho_0 = K(\log \rho_0 + 1) + 1$ which will become negative for any K for ρ_0 very close to zero. Thus, this will create the possibility of having negative (gravitational) mass objects made of diffuse low-density matter.

2. General relativistic equations of stellar equilibrium and its numerical solution

We take the entire formulation in toto from [2] where the quantity σ is defined as $\sigma = K\rho_c^{1/n}/c^2$ (see eq. (2.15) of this reference). Here n takes the value 1 for polytropes of index 1, K is a constant depending on the thermodynamic properties of the constituent matter (and largely independent of its density in a specified range), ρ_c is the density of matter excluding gravitational energy at the stellar centre and c is the velocity of light. For all cases, henceforth, when we refer to the equation numbers of this paper we shall automatically replace the n's by the 1's. Thus, the equation of state is $P = K\rho^2$ (eq. (2.11) of [2]) and the density ρ is expressed in terms of the new function θ of r as $\rho(r) = \rho_c\theta(r)$ (eq. (2.12) of [2]). For the convenience of the reader, we just state that the following equations of [2] are useful in developing the argument below and will not be explicitly stated while writing the mathematical expressions in this section. These are eqs (2.21)–(2.26) and (3.5). Thus, we have the new radial variable ξ expressed as

$$\xi = \left(\frac{2\pi G}{K}\right)^{1/2} r. \tag{1}$$

Another function along with $\theta(\xi)$ is $v(\xi)$ and is defined as

$$v(\xi(r)) = \left(\frac{2\pi G}{K}\right)^{3/2} \int_0^r r'^2\theta(r')\,\mathrm{d}r'. \tag{2}$$

The differential equations governing $\theta(\xi)$ and $v(\xi)$ are

$$\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\frac{1-(4\sigma v/\xi)}{1+\sigma\theta} + v + \sigma\xi\theta\frac{\mathrm{d}v}{\mathrm{d}\xi} = 0 \tag{3}$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = \xi^2\theta, \tag{4}$$

respectively. Our problem is to solve eqs (3) and (4) numerically with the boundary conditions $\theta(0) = 1$ and $v(0) = 0$. The radius of the sphere is $R = (K/(2\pi G))^{1/2}\xi_1$, where ξ_1 is the value of ξ at which the first zero of the function $\theta(\xi)$ occurs while integrating radially outward from the centre. Also, one can compute the value of the quantity $2GM/c^2R = (4\sigma v(\xi_1))/\xi_1$ which is a measure of how close the $n = 1$ stellar body can approach the material description of a black hole. Our computations were extended from the last value of $\sigma = 0.5$, which appears in table 1 of [2] and the exact values of ξ_1 and $2GM/c^2R$ calculated with the help of the respective formulas are reported in §3. It is a little surprising that ξ_1 decreases at first with increasing σ and then marginally increases to become steady at extremely large σ. Similar is the case for $2GM/c^2R$ showing that a change in ρ_c at this stage is not felt by the external observer in the sense of an increase in mass (energy) of the star.

3. Results of numerical integration of eqs (3) and (4) with appropriate boundary conditions

We increase the value of σ from 0.5 to 5 and 13 and then to 130, 1300, 13000 and 130000 and the corresponding values along with those of ξ_1 and $2GM/c^2R = (4\sigma v(\xi_1))/\xi_1$, respectively, are tabulated in table 1. An interesting feature is that in general relativity the stars having different values of σ (and hence of the central density ρ_c) have the same radius R and the same mass energy M.

We also calculated the material energy as defined by eq. (11.1.19) of [4] which is expressed in terms of the notations of [2]:

$$M_{\text{matter}} = \int_0^R 4\pi\,\mathrm{e}^{\nu/2}\,\mathrm{e}^{\lambda/2} r^2\rho_c\theta\,\mathrm{d}r, \tag{5}$$

to find that this quantity is unchanged, exactly like M at all the higher values of σ. Thus general relativistic $n = 1$ polytropes cannot accept any more in-falling matter beyond a certain ρ_c which means that its polytropic equation of state must break down if any more such polytropic gas falls onto the star from space. This thus contradicts the fundamental fact that the equation of state for a particular type of matter is dependent only on its thermodynamic properties within certain limits of densities but should not be determined by the nature of gravitational interaction between the particles of the same matter. On the other hand, a change in ρ_c without a corresponding increase in the material as well as the total mass energy M must be interpreted to mean a redistribution of matter inside the star without a change in the gravitational potential energy, a prospect which is explicitly ruled out by Newtonian gravity. Thus, the general relativistic stars, once they reach a certain degree of compactness, tend to cancel all Newtonian changes within them with the help of higher-order relativistic corrections. This is contrary to a belief that general relativity facilitates the creation of black-hole horizons.

Table 1. Some parameters of dense relativistic $n = 1$ polytropes.

σ	0.5	5	13	130	1300	13000	130000
ξ_1	1.8009	1.2373	1.2316	1.2381	1.2384	1.2384	1.2384
$2GM/c^2R$	0.5532	0.6014	0.5835	0.5769	0.5768	0.5768	0.5768

4. A discussion on the nature of the equation of state at high densities

It is a well-known fact that the monoatomic gaseous matter, which most stars are made of at high temperature in the absence of radiation, will exhibit an adiabatic equation of state with $n = 3/2$. Also, stars containing radiation and matter together in equilibrium will exhibit an equation of state with $n = 3$ [1] (see p. 229 of this reference). All these equations have limited range of applicability and generally tend to get modified at low or high densities. It is not completely unphysical to assume some form of matter to exhibit a $n = 1$ equation of state in at least a part of the range of σ for which the calculations in §3 were carried out. If this turns out to be true then we can have a star-like body when $2GM/c^2R = 0.5768$ with both R and M fixed according to the value of K in eq. (1). However, this stellar body has no unique distribution of mass (both including and excluding the gravitational potential energy) and hence is in some sort of neutral equilibrium rather than being in a stable equilibrium. Of course, at extremely high σ when the polytropic equation of state itself starts breaking down, there is a possibility for the star, depending on the nature of this breakdown, to attain a black-hole configuration. This is different from the breakdown of the polytropic equation of state caused by in-falling matter from space as discussed in the previous section because such an excess matter, even in very minute quantities, is not compatible with the gravitational field equations of hydrostatic equilibrium coupled with the equation of state. Hence, it is not possible to speak of any state of equilibrium breaking down but rather of some sort of explosive event caused by the in-falling matter. Such a possibility is ruled out in Newtonian gravity and as stated in §3, it is purely due to the type of interaction between the stellar matter particles mediated by general relativity.

Here, we would like to give a clear statement on how the present work extends the results of [2] and we would like to state that in [2], the computations were arbitrarily terminated at certain densities due to unphysical superluminal sound speeds at higher levels of compactness. This reasoning is based on incorrect formulas of the speed of sound in superdense objects and is addressed in detail in the next section. As a result of this arbitrary truncation of the sequence of ξ_1 and $2GM/c^2R$, many properties of matter subject to general relativistic interaction could not be ascertained in its entirety.

5. On the question of sound speed at high densities

Before accepting the physical models presented above one must ensure that the velocity of sound waves inside the polytrope may not exceed the speed of light c in free space. We shall first present a non-relativistic derivation of the speed of sound from the equations of hydrodynamics which seem to suggest superluminal sound speeds. However, when we replace these equations with those of relativistic hydrodynamics, our expression for sound speed will yield subluminal values. In the derivation of the non-relativistic equations, we essentially follow the presentation of Christensen-Dalsgaard [12] and the equations of relativistic hydrodynamics are taken from chapter 2 of [4].

The equation of motion in Eulerian form of non-relativistic hydrodynamics is

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P - \rho \nabla \phi \tag{6}$$

and the equation of mass conservation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{7}$$

where ρ and $\vec{v}$ are the density and velocity of matter, respectively, P is the pressure and ϕ is the gravitational potential. Acoustic waves are caused by small deviations of these quantities from their equilibrium values where the effect of gravity on these deviations is neglected, that is $\nabla\phi$ is set equal to zero in the resulting equations. We can make a perturbative expansion of these quantities about their equilibrium values inside the star, e.g., the pressure $P(\vec{r}, t) = P_0(\vec{r}) + p'(\vec{r}, t)$ where the primed quantity is a small deviation from equilibrium. From eq. (6) the perturbation equation is obtained as

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla p'. \tag{8}$$

Taking the divergence of this equation and using eq. (7) while keeping in mind that the equilibrium value of velocity $\vec{v}_0(\vec{r})$ to be zero, we obtain the following wave equation:

$$\frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 p'. \tag{9}$$

In order to solve this equation and obtain a value of sound speed, we have to use the equation of state $P = K\rho^2$ and differentiate it to obtain $\mathrm{d}P = 2K\rho\,\mathrm{d}\rho$ where the following identification with perturbed quantities have to be made: $\mathrm{d}P \leftrightarrow p'$, $\rho \leftrightarrow \rho_0$ and $\mathrm{d}\rho \leftrightarrow \rho'$. Remembering that $P_0 = K\rho_0^2$ we have from eq. (9)

$$\frac{\partial^2 \rho'}{\partial t^2} = \frac{2P_0}{\rho_0}\nabla^2 \rho'. \tag{10}$$

Here the quantity $\sqrt{(2P_0/\rho_0)}$ is the speed of sound and keeping in mind $K = (c^2\sigma)/\rho_c$ we can show that at the stellar centre we have $\sqrt{(2P_0/\rho_0)} = \sqrt{(2P_c/\rho_c)} = c\sqrt{(2\sigma)}$ which obviously exceeds the speed c of light for values more than 0.5 of σ.

Now let us consider the effect of relativistic correction to the equation of hydrodynamics (6) which is given by eq. (2.10.16) of [4] and is stated (keeping in mind our system of units where we retain the velocity of light c explicitly) as

$$\left(\rho + \frac{P}{c^2}\right)\frac{\partial \vec{v}}{\partial t} + \left(\rho + \frac{P}{c^2}\right)\vec{v}\cdot\nabla\vec{v} = -\left(1 - \frac{v^2}{c^2}\right)\left(\nabla P + \frac{\vec{v}}{c^2}\frac{\partial P}{\partial t}\right). \tag{11}$$

Here, we have neglected the contribution of gravity as it does not enter into the perturbation equations. The first point to note is that in the relativistic description of all dynamical processes we have to replace ρ by $(\rho + (P/c^2))$ at all places and this will be true for eq. (7) also if the equilibrium velocity is zero. Thus, proceeding as in the non-relativistic case and making this stated replacement, the velocity of sound will be obtained as $\sqrt{(2P_0/(\rho_0 + (P_0/c^2))}$ in place of $\sqrt{(2P_0/\rho_0)}$. At high values of central densities, even if $P_0 \to \infty$, this velocity can at most become $c\sqrt{2}$ and is independent of σ. Now the bound on σ cannot be 0.5 as is assumed by Tooper for subluminal sound speed but will be higher. In our opinion, if gravitational correction to the acoustic wave equation is taken into account, the limiting value of sound speed can be shown at most to be c.

Before concluding, we would like to discuss a few points on the formulas used by Tooper for calculating the sound speed. First, we quote from the abstract of one of the papers cited by Tooper himself [13] which states 'The correct expression for the velocity of sound in general relativity is not obvious, because of ambiguity in defining the density of matter. The velocity is investigated in part 1 of this paper, and it is found that it never becomes greater than the velocity of light, having the value $3^{-1/2}$ for incompressible matter'. We need to note that the change in specific volume $\mathrm{d}V$ is related to the change in gas density $\mathrm{d}\rho_g$ by $\mathrm{d}V/V = -(\mathrm{d}\rho_g/\rho_g)$ on p. 442 of [2] whereas the first law of thermodynamics as stated for an adiabatic process on this very same page seems to apply to all forms of matter including at least radiation energy and its associated pressure. We feel that such a 'natural relativistic generalisation' of physical laws is inconsistent and hence along with other assumptions leads incorrectly to predictions of superluminal sound speeds in compact objects.

Acknowledgements

The author is thankful to the anonymous referee who asked him to include other related studies (refs [5–11]) which are related to polytropes and are of recent origin.

References

[1] S Chandrasekar, *An introduction to the study of stellar structure* (Dover Publication Inc., Chicago, 1967)
[2] R F Tooper, *Astrophys. J.* **140**, 434 (1964)
[3] T Harko and M K Mak, *Astrophys. Space Sci.* **361**, 283 (2016)
[4] S Weinberg, *Gravitation and cosmology: Principles and applications of the general theory of relativity* (John Wiley and Sons Inc., New York, 1972)
[5] L Herrera and W Barreto, *Phys. Rev. D* **87**, 087303 (2013)
[6] L Herrera and W Barreto, *Phys. Rev. D* **88**, 084022 (2013)
[7] L Herrera, E Fuenmayor and P Leon, *Phys. Rev. D* **93**, 024047 (2016)
[8] S Thirukkanesh and F C Ragel, *Pramana – J. Phys.* **78**, 687 (2012)
[9] P M Takisa and S D Maharaj, *Gen. Rel. Grav.* **45**, 1951 (2013)
[10] S A Ngubelanga and S D Maharaj, *Eur. Phys. J. Plus* **130**, 211 (2015)
[11] S A Ngubelanga, S D Maharaj and S Ray, *Astrophys. Space Sci.* **357**, 74 (2015)
[12] J Christensen-Dalsgaard, *Lecture notes on stellar oscillations*, Chapter 3, 5th edn, http://users-phys.au.dk/jcd/oscilnotes/ (2003)
[13] A R Curtis, *Proc. R. Soc. Lond. A* **200**, 248 (1950)