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Quantum quench dynamics of the one-dimensional Ising model in transverse field

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Abstract. The quantum quench dynamics of the one-dimensional Ising model in transverse field is investigated using the quantum renormalisation group method. The analytic expression of concurrence $C(t)$ is obtained, where the initial state is a superposition state which is constructed from the eigenstates of pure Ising system. The effect of parameter a in the period and range of concurrence are exhibited respectively in the vicinity of the critical point, which show scaling behaviour. When effective magnetic field g is big enough, the maximum limit value is 1.0. However, the minimum value is different, which is also dependent on the evolved time t.

Keywords. Ising model; quantum quench; concurrence; quantum renormalisation group.

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1. Introduction

Quantum quench has been studied intensively in recent years, where a closed system is pushed out of equilibrium by a sudden change in the Hamiltonian which controls its time evolution [1–4]. Entanglement is the property of quantum state as a specific nonlocal quantum correlation, which may be used to describe the evolution of spin systems after the quantum quench [5–9].

The quantum quench of two-spin entanglement with increasing magnetic field in the one-dimensional Ising model is investigated where the initial state is a thermal ground state of the pure Ising model [10,11]. The derivative of entanglement reaches its maximum at the critical point and the scaling behaviour is the same as that of the static ground-state case. Under the influence of local dissipation, the decay rate of concurrence is strongly modified by the non-Markovianity of the evolution [12].

The dynamics of quantum correlations in a class of exactly solvable Ising-type models are investigated, where the time evolution of the initial Bell states are created in a fully polarised background and ground state [13]. Dynamics of the transverse-field Ising model with a time-dependent magnetic field shows that the entanglement entropy crossing quantum phase transition point displays different regimes depending on the value of tuning time-scale and there are multiple crossings at gap-dependent frequency in the entanglement spectrum [14]. The absence of the complete revivals of quantum states for the critical transverse field Ising chain is proved analytically and the numerical results of the XY chain on the critical line is presented after quantum quench, which show that criticality has no significant effect in partial revivals of the ferromagnetic initial states [15].

The influence of Dzyaloshinskii–Moriya (DM) interaction on quantum correlations in two-qubit Werner states and maximally entangled mixed states shows that the state of auxiliary qubit does not affect quantum correlations in both the states and only the strength of DM interaction influences the quantum correlations [16]. In contrast with bipartite partitions of Werner states, tripartite entanglement does not suffer from entanglement sudden death. Meanwhile, bipartite partitions and tripartite entanglement in Greenberger–Horne–Zeilinger states do not feel any influence of DM interaction [17].

Loschmidt echo (LE) of the extended Su–Schrieffer–Heeger model and three-site spin-interacting XY model shows that quantum criticality is neither a sufficient nor a necessary condition for the LE to exhibit an observable revival structure [18]. The time evolution of spin-$1/2$ XXZ models following a quantum quench shows that the entanglement of two-spin qubit can be created and the oscillation frequency of entanglement is dependent strongly on the anisotropic interaction, where the initial states are Neel state and inhomogeneous state [19].

Quantum Ising model is important to investigate the dynamical behaviour of quantum spin system, where the initial states are Bell state, thermal equilibrium state, thermal ground state and so on [10,11,20–22]. The dynamics of the system is strongly dependent on the initial state. In this paper, the quantum quench dynamics of one-dimensional Ising model in transverse field is investigated, where the initial state is a superposition state which is constructed from the eigenstates of the pure Ising model. The organisation of this paper is as follows. In §2, the Hamiltonian and quantum renormalisation group (QRG) functions of one-dimensional Ising model are presented. In §3, the analytic expression of quantum quench dynamics is given using the concurrence. In §4 the results of quantum quench dynamics of the Ising spin model are exhibited and discussed. Finally, conclusions are given in §5.

2. The Hamiltonian and QRG functions of Ising chain

The Hamiltonian of the one-dimensional Ising model in the transverse field is

$$H = -J\sum_{i=1}^{N}(\sigma_i^x\sigma_{i+1}^x + g\sigma_i^z), \tag{1}$$

where σ_i^α refers to the $\alpha(\alpha = x, z)$ component of the Pauli matrix at site i. $J > 0$ is the ferromagnetic exchange coupling and g is the transverse field. It is known that second-order phase transition occurs when $g_c = 1$, where there is order phase for $g < g_c$ and disorder phase for $g > g_c$.

The quantum renormalisation group method is important to study the critical phenomenon and quantum entanglement of spin systems [23–28]. In order to obtain the QRG functions, the Hamiltonian of this system is divided into H^B and H^{BB}, where $H^B = \sum_I h_I^B$ is the two-site block with $h_I^B = -J(\sigma_{I,1}^x\sigma_{I,2}^x + g\sigma_{I,1}^z)$ and $H^{BB} = -J\sum_I(\sigma_{I,2}^x\sigma_{I+1,1}^x + g\sigma_{I,2}^z)$ is the remaining part (see figure 1). The projection operator P is

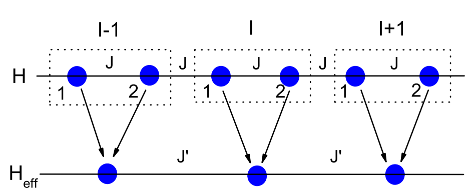

Figure 1. Decomposition of Ising spin chain into blocks. Each block will be represented by an effective spin after the renormalisation with renormalised interactions.

constructed from the lowest two eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ of h_I^B, $P = |\psi_1\rangle\langle\Downarrow| + |\psi_2\rangle\langle\Uparrow|$, where $|\Uparrow\rangle$ and $|\Downarrow\rangle$ are the eigenstates of the effective spin. The effective Hamiltonian $H^{\text{eff}} = P_0(H^B + H^{BB})P_0^\dagger$ is similar to the original one (eq. (1)). The QRG functions are obtained as

$$J' = J\frac{2q}{1+q^2}, \quad q = g + \sqrt{g^2+1}, \quad g' = g^2, \tag{2}$$

where $g\,(g')$ is the original (renormalised) transverse field [10,28], which can be used to calculate the low-energy-state dynamics of entanglement in a large spin system [6,7,10].

3. The analytic expression of time-dependent concurrence

The concurrence is an important quantity to measure quantum entanglement of 1/2-spin system [11,12]. It can be used to describe the dynamic behaviour, which is strongly dependent on the initial state. Compared to the thermal ground state [10,11], the initial state is a superposition state which is constructed from the eigenstates of the pure Ising model.

In order to give the initial state, the two-spin Hamiltonian is considered, which can be written as

$$\bar{H} = -J\begin{pmatrix} 2g & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -2g \end{pmatrix}. \tag{3}$$

The lowest two eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ of the two-spin Hamiltonian are used. Initial state can be written as follows:

$$|\Psi(0)\rangle = \sqrt{a}|\psi_1\rangle + \sqrt{1-a}|\psi_2\rangle, \tag{4}$$

where the parameter a satisfies the relation $0 \le a \le 1$. The eigenstates are $|\psi_1\rangle = |{\leftarrow\leftarrow}\rangle$ and $|\psi_2\rangle = |{\rightarrow\rightarrow}\rangle$ with $|{\leftarrow}\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle + |\downarrow\rangle]$ and $|{\rightarrow}\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle - |\downarrow\rangle]$. Meanwhile, $|\uparrow\rangle$ and $|\downarrow\rangle$ denote the eigenstates of σ^z. If $a = 0$ or 1, the initial state is a product state, otherwise, the initial state is an entangled state.

The density matrix of the initial state is

$$\begin{aligned}\rho(0) &= |\Psi(0)\rangle\langle\Psi(0)| \\ &= a|\psi_1\rangle\langle\psi_1| + \sqrt{a(1-a)}[|\psi_1\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_1|] \\ &\quad + (1-a)|\psi_2\rangle\langle\psi_2|.\end{aligned} \tag{5}$$

Corresponding to the two-spin Hamiltonian eq. (3), the time evolution operator $U(t) = e^{-i\bar{H}t}$ has the following form λ [10]:

$$U(t) = \begin{pmatrix} U_{11}(t) & 0 & 0 & U_{14}(t) \\ 0 & U_{22}(t) & U_{23}(t) & 0 \\ 0 & U_{32}(t) & U_{33} & 0 \\ U_{41}(t) & 0 & 0 & U_{44}(t) \end{pmatrix}, \tag{6}$$

where

$$U_{11}(t) = i\frac{2g}{\sqrt{1+4g}}\sin\left(Jt\sqrt{1+4g^2}\right) + \cos\left(Jt\sqrt{1+4g^2}\right),$$

$$U_{14}(t) = U_{41}(t) = \frac{\mathrm{i}}{\sqrt{1+4g^2}}\sin\left(Jt\sqrt{1+4g^2}\right),$$

$$U_{22}(t) = U_{33}(t) = \cos(Jt),$$

$$U_{23}(t) = U_{32}(t) = \mathrm{i}\sin(Jt),$$

$$U_{44}(t) = U_{11}^*(t).$$

From eqs (5) and (6), the density matrix of the time-evolved state is $\rho(t) = U(t)\rho(0)U^{\dagger}(t)$, where $\rho(0)$ is the density matrix of the initial state in eq. (5). The concurrence of the time-evolved state $\rho(t)$ is defined as

$$C(t) = \max[0, 2\lambda_{\max}(t) - \mathrm{tr}\sqrt{\rho(t)\tilde{\rho}(t)}], \tag{7}$$

where $\tilde{\rho}(t) = (\sigma^y \otimes \sigma^y)\rho^*(t)(\sigma^y \otimes \sigma^y)$ and $\lambda_{\max}$ is the largest eigenvalue of matrix $\sqrt{\rho(t)\tilde{\rho}(t)}$ [5,29,30].

The analytic expression of the concurrence $C(t)$ is

$$C(t) = \Bigg\{4a(1-a) + (1-2a)^2\sin^2(Jt) - \left(1+2\sqrt{a(1-a)}\right)^2\frac{4g^2}{(1+4g^2)^2} \times \sin^4\left(\sqrt{1+4g^2}Jt\right) - \frac{1}{2}(1-2a)^2\frac{\sin(2Jt)\sin(2\sqrt{1+4g^2}Jt)}{\sqrt{1+4g^2}} + (1-2a)^2\frac{\cos(2Jt)\sin^2(\sqrt{1+4g^2}Jt)}{1+4g^2}\Bigg\}^{1/2}, \tag{8}$$

where $a = 0, 1$, the initial state is a product state $C(0) = 0$. For $a = 0.5$,

$$C(t) = \sqrt{1 - \left(\frac{4g}{1+4g^2}\right)^2\sin^4\left(\sqrt{1+4g^2}Jt\right)},$$

which corresponds to the maximum of entangled state, where the period is only dependent on $\sqrt{1+4g^2}J$. Except for some constants, this result is similar to the case in which the initial state is the thermal ground state [10]. If we change a to $(1-a)$, eq. (8) does not change. This means that the concurrence $C(t)$ is symmetric about $a = 0.5$, and so in the following we only concentrate on $0 \le a \le 0.5$. The concurrence $C(t)$ of the initial state is only dependent on parameter a in coefficient of superposition state, which can be written as $C(0) = 2\sqrt{a(1-a)}$.

4. The results of quantum quench dynamics

Quantum quench is the sudden changes of external parameters in the Hamiltonian. Thus, the system will be evolved under the new Hamiltonian of the closed system, which may induce a change of quantum entanglement between two spins [2,3]. A large system $N = 2^{n+1}$ can be effectively described by two sites with renormalised coupling in the nth QRG step. From eq. (8), the concurrence of a large system can be obtained using the QRG functions (eq. (2)), where the coupling constant is seem as the effective coupling constant of the block spin [10].

In figure 2a, the concurrence $C(t)$ vs. a of the two-spin block is exhibited at time $t = 0$. As a increases from 0 to 1.0, the concurrence $C(t)$ increases from 0 to 1 then decreases to 0, which is symmetric about $a = 0.5$. This can be derived from that if we change a to $(1-a)$, the concurrence $C(0) = 2\sqrt{a(1-a)}$ is not changed. The results also exhibit that for $a = 0$, 1,

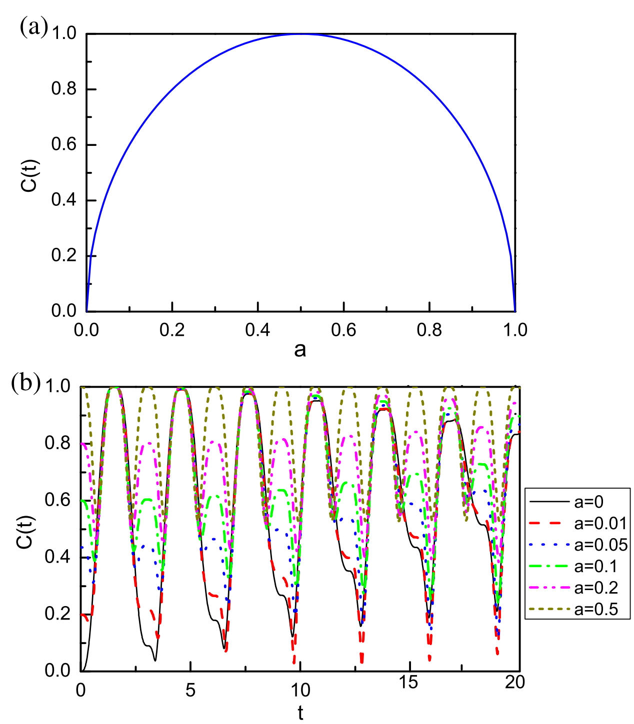

Figure 2. (**a**) The concurrence $C(t)$ vs. parameter a of two-spin block at time $t = 0$ and (**b**) $C(t)$ vs. t with $g = 0.9$ for different values of parameter a.

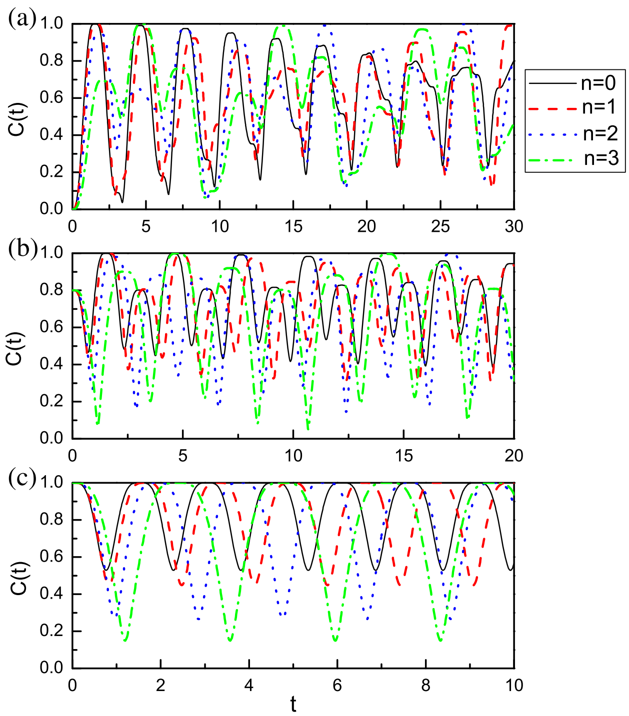

Figure 3. The concurrence $C(t)$ vs. t with parameters $a = 0, 0.2, 0.5$ and $g = 0.9$ on the spin chain in (**a**), (**b**) and (**c**), where n is the number of QRG steps.

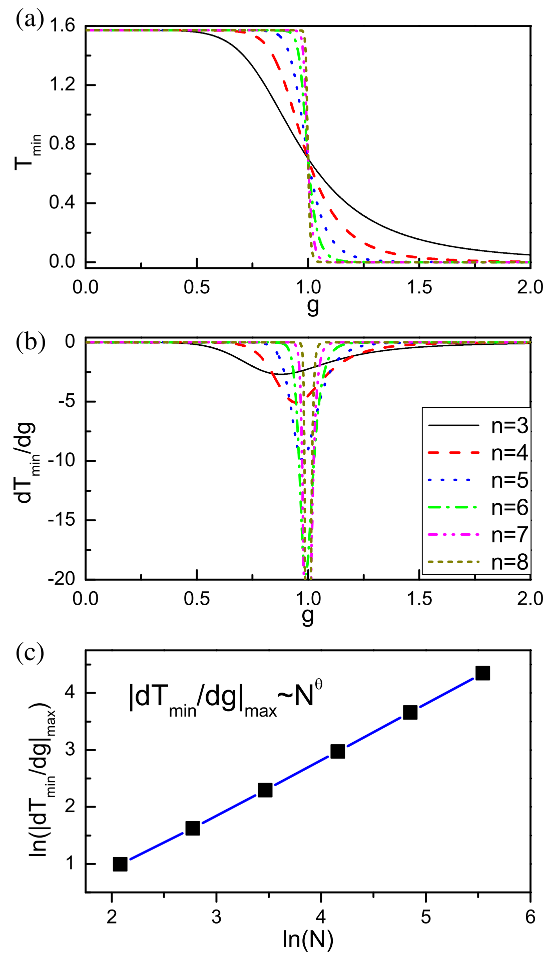

Figure 4. $T_{\min}$ and $\mathrm{d}T_{\min}/\mathrm{d}g$ vs. g with parameter $a = 0.5$ on the spin chain in (**a**) and (**b**), where n is the number of QRG step. The finite scaling behavior is exhibited in (**c**).

when the concurrence $C(0) = 0$, the initial state is a product state and for $a = 0.5$ when the concurrence $C(0) = 1.0$, the initial state is a maximum entangled state. In figure 2b, $C(t)$ vs. t with $g = 0.9$ is exhibited for different parameters $a = 0, 0.01, 0.05, 0.1, 0.2$ and 0.5 of two-spin block. The results show that the concurrence $C(t)$ is of quasiperiodic pattern, where the quasiperiod is $T = 2\pi/\sqrt{1+4g^2} \approx 3.065$. As parameter a increases from $a = 0$ to 0.5, the concurrence $C(0)$ of the initial state increases from 0 to 1. The concurrence $C(t)$ first decreases to a dip, then increase to a peak where the concurrence $C(t) = 1.0$ at $t \approx 1.55$. However, the height of the second peak is different for different values of parameter a. The position of the first dip is not only dependent on g but also dependent on the parameter a, and particularly for $a = 0$, there is no first dip [18].

The evolutions of concurrence $C(t)$ vs. t of the Ising spin chain under QRG transformation are plotted respectively in figures 3a–3c with $a = 0, 0.2, 0.5$ and $g = 0.9$, where n is the number of QRG step. The size of the spin system is $N = 2^{(n+1)}$. When $t = 0$ as a increases from $a = 0$ to 0.5, the initial concurrence $C(t)$ increases from 0 to 1. For different steps of QRG (n), the evolution of concurrence $C(t)$ with $a = 0$ is oscillating and the concurrence $C(t)$ increases from 0 to 1 in figure 3a. In figure 3b, the concurrence $C(t)$ with $a = 0.2$ is exhibited. For different QRG steps n, the concurrence $C(0) = 0.8$. The concurrence $C(t)$ is oscillating, where the range of concurrence $C(t)$ is smaller than the case of $a = 0$ in figure 3a. For different QRG steps n, the concurrence $C(t)$ with $a = 0.5$ exhibits a periodic pattern in figure 3c, where the range and period of concurrence $C(t)$ increase as the QRG step n increases.

From the analytic expression of concurrence $C(t)$ (eq. (8)) with $a = 0.5$, the time corresponding to the first minimum of concurrence $C(t)$ can be obtained ($T_{\min} = \pi/(2\sqrt{1+4g^2})$) in figure 3c. $T_{\min}$ and $\mathrm{d}T_{\min}/\mathrm{d}g$ vs. g of the Ising spin chain under QRG transformation are plotted in figures 4a and 4b. As the number of QRG step n increases, $T_{\min}$ decreases sharply from $T_{\min} = 1.6$ to 0 and $\mathrm{d}T_{\min}/\mathrm{d}g$ is divergent near the critical point at the thermodynamic limit. In figure 4c, the maximum of $|\mathrm{d}T_{\min}/\mathrm{d}g|$ vs. $\ln(N)$ is presented, where $N = 2^{n+1}$ is the total number of spin. The finite scaling behaviour is well exhibited, $|\mathrm{d}T_{\min}/\mathrm{d}g|_{\max} \sim N^{\theta}$, where the entanglement exponent $\theta = 0.9718$. The entanglement

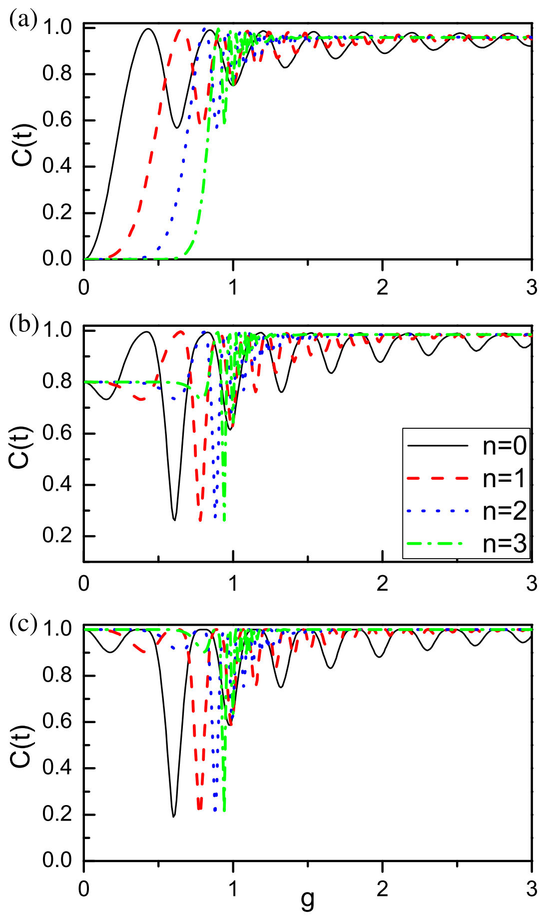

Figure 5. The concurrence $C(t)$ vs. g with time $t = 5.0$ on the spin chain in (**a**), (**b**) and (**c**) for $a = 0, 0.2$ and 0.5, where n is the number of QRG step.

exponent θ is related to the correlation length exponent ν by $\theta = 1/\nu$ [10,28].

If g is big enough, the concurrence $C(t)$ is only dependent on parameters a and t,

$$\lim_{g\to\infty} C(t) = \sqrt{1-(1-2a)^2\cos^2(Jt)}. \qquad (9)$$

The limit value of concurrence $C(t)$ changes from the initial concurrence $2\sqrt{a(1-a)}$ to 1. This means that the limit value is bigger than the initial value, except $a = 0.5$, those two values are equal, where the initial state is the maximum entangled state. Although the concurrence of the initial state is different, the concurrence can be reached to the maximum of concurrence $C(t) = 1.0$ when $\cos(Jt) = 0$. This result is different from that the limit value of concurrence is zero, where the initial state is the thermal ground state [10,11]. The concurrence $C(t)$ vs. g with time $t = 5.0$ on the spin chain are exhibited for $a = 0, 0.2$ and 0.5 in figures 5a–5c. The first dip of concurrence is very deep and the second dip is small. They approach the critical point $g_c = 1.0$ as the number of QRG step n increases. Then, the concurrence oscillates and approaches the limit value of concurrence. The value of concurrence $C(5) = 0.959, 0.985$ and 1.0, corresponding to $a = 0, 0.2$ and 0.5 in figures 5a–5c.

5. Conclusions

In this paper, the quantum quench dynamics of the one-dimensional Ising model in transverse field is investigated using the QRG method, where the initial state is a superposition state which is constructed from the lowest two eigenstates of the pure Ising model. From this initial state $|\Psi(0)\rangle = \sqrt{a}|\psi_1\rangle + \sqrt{1-a}|\psi_2\rangle$, the analytic expression of the concurrence $C(t)$ is presented, which is symmetric about $a = 0.5$. The concurrence $C(t)$ of the two-spin block exhibits a quasiperiodic pattern. As a increases from 0 to 0.5, the range of concurrence decreases except for $a = 0.5$, but the period $T \approx 3.065$ does not change. As the step of QRG n increases, the period and range of concurrence increase except for $a = 0$, and the time $T_{\min}$ sharply decreases at the critical point $g = 1.0$, which is corresponding to the first dip of concurrence with $a = 0.5$. Through the maximum of derivative $|dT_{\min}/dg|$ vs. $\ln(N)$, the scaling behaviour of concurrence is well exhibited. The entanglement exponent is $\theta = 0.9718$. When the effective magnetic field g is big enough, the limit value of the concurrence is between $2\sqrt{a(1-a)}$ and 1. The maximum value of concurrence is $C(t) = 1.0$. However, the minimum value of $C(t)$ is different, which is also dependent on the evolved time t.

Acknowledgements

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