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# Minimal length Schrödinger equation via factorisation approach

S A KHORRAM-HOSSEINI[1], S ZARRINKAMAR[2],* and H PANAHI[1]

[1]Physics Department, University of Guilan, Rasht, Iran
[2]Department of Basic Sciences, Garmsar Branch, Islamic Azad University, Garmsar, Iran
*Corresponding author. E-mail: zarrinkamar.s@gmail.com

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**Abstract.** The fourth-order modified Schrödinger equation due to the generalised uncertainty principle is considered in one dimension with a box problem. The factorisation of fourth-order self-adjoint differential equations is then discussed and thereby the wave functions and energy spectra of the modified Schrödinger equation are derived.

**Keywords.** Minimal length; generalised uncertainty principle; Planck scale; Schrödinger equation; factorisation; particle in box.

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## 1. Introduction

The goal of unification of quantum mechanics and gravity has been investigated by many theories including loop quantum gravity, double special relativity and string theory [1–5]. Although such scenarios are based on different assumptions and approaches, all imply the generalisation of the Heisenberg uncertainty principle. The so-called generalised uncertainty principle (GUP) corresponds to the existence of a minimal length of other Planck length $l_{\rm p} = \sqrt{\hbar G/c^3}$ [3]. The generalisation modifies the wave function's form and therefore the basic characteristics of the equation, i.e. the wave function and the energy eigenvalue are altered. It goes without saying that the order of energy at such theories is much higher than the order of our present experiments and we have to work on theoretical bases. The existing literature indicates that all basic equations of quantum mechanics including non-relativistic Schrödinger, relativistic Klein–Gordon, Dirac and Duffin–Kemmer–Petiau equations have been analysed within this modified framework with various interactions ([6–18] and references therein). The arising equation, in its simplest form, i.e. the one-dimensional Schrödinger equation, appears in the sixth-order form. The latter is frequently approximated by a fourth-order differential equation. This is just when the problem arises; unlike the second-order differential equations, we know so little about higher-order equations and their analytical study. In the present work, we first review the elegant idea of Caruntu for the factorisation of fourth- and sixth-order ordinary differential equations [19,20]. Next, we apply their novel idea to the fourth-order minimal length Schrödinger equation with box example and obtain the wave function and the energy spectra.

## 2. Factorisation of self-adjoint differential equations

Let us consider a self-adjoint ordinary differential operator of the form [19,20]

$$L_{(2n)} = \frac{1}{\rho}\frac{{\rm d}^n}{{\rm d}x^n}\left(\rho\beta^n\frac{{\rm d}^n}{{\rm d}x^n}\right), \tag{1}$$

where the scalar functions $\rho(x)$, $\beta(x)$ and $\alpha(x)$ satisfy

$$\frac{1}{\rho}\frac{{\rm d}\rho}{{\rm d}x} = \frac{\alpha}{\beta}, \tag{2}$$

$$\frac{{\rm d}^2\alpha}{{\rm d}x^2} + \frac{{\rm d}^3\beta}{{\rm d}x^3} = 0. \tag{3}$$

For $n = 2$, eq. (1) can be factorised as

$$L_4 = L_2(L_2 - \delta_2), \tag{4}$$

where

$$\delta_k = (k-1)\frac{{\rm d}\alpha}{{\rm d}x} + \frac{k(k-1)}{2}\frac{{\rm d}^2\beta}{{\rm d}x^2}. \tag{5}$$

In the fourth-order case, we have

$$(L_{2n} - \mu)[y] = 0 \tag{6}$$

and the equation is factorised as [19,20]

$$\prod_{k=1}^{n}(L_2-\lambda_k)[y]=0. \tag{7}$$

The constants $\lambda_k$ are given by

$$\begin{cases}\lambda_1+\lambda_2+\cdots+\lambda_n=\delta_2+\cdots+\delta_n,\\ \lambda_1\lambda_2+\lambda_1\lambda_3+\cdots+\lambda_{n-1}\lambda_n\\ \quad=\delta_2\delta_3+\cdots+\delta_{n-1}\delta_n,\\ \cdots\\ \lambda_1\lambda_2\cdots\lambda_n=(-1)^{n-1}\mu.\end{cases} \tag{8}$$

The general solution of the spectral-type equation (6) is given by

$$y=\sum_{k=1}^{n}y_k, \tag{9}$$

where $y_k$ is obtained from

$$L_2[y_k]-\lambda_k y_k=0,\quad k=1,2,\ldots,n. \tag{10}$$

The extended form of eq. (9) is

$$\beta(x)\frac{\mathrm{d}^2y_k}{\mathrm{d}x^2}+[\alpha(x)+\beta'(x)]\frac{\mathrm{d}y_k}{\mathrm{d}x}-\lambda_k y_k=0. \tag{11}$$

In summary, we may write [19,20]

$$L_4=\beta^2\frac{\mathrm{d}^4}{\mathrm{d}x^4}+2\beta(\alpha+2\beta')\frac{\mathrm{d}^3}{\mathrm{d}x^3}+\{[\beta(\alpha+2\beta')]'\\ +\alpha(\alpha+2\beta')\}\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \tag{12}$$

which gives [19,20]

$$\frac{1}{\rho}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\rho\beta^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)=\left[\beta\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(\alpha+\beta')\frac{\mathrm{d}}{\mathrm{d}x}\right]\\ \times\left[\beta\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(\alpha+\beta')\frac{\mathrm{d}}{\mathrm{d}x}-(\alpha'+\beta')\right], \tag{13}$$

where $\delta_2=\alpha'+\beta''$. Therefore, the factorisation appears as

$$\left[\beta\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(\alpha+\beta')\frac{\mathrm{d}}{\mathrm{d}x}-\lambda_1\right]\\ \times\left[\beta\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(\alpha+\beta')\frac{\mathrm{d}}{\mathrm{d}x}-\lambda_2\right]=0, \tag{14a}$$

where $\lambda_1$ and $\lambda_2$ are given by

$$\lambda_1=\frac{\delta_2+\sqrt{\delta_2^2+4\mu}}{2},\\ \lambda_2=\frac{\delta_2-\sqrt{\delta_2^2+4\mu}}{2}. \tag{14b}$$

## 3. Generalised uncertainty principle

We consider a GUP of the form [3,4]

$$\Delta x\Delta p\geq\frac{\hbar}{2}(1+\beta(\Delta p)^2+\gamma), \tag{15}$$

where $\beta$ and $\gamma$ are positive constants. Such a form corresponds to the minimal length $(\Delta x)_{\min}=\hbar\sqrt{\beta}$, supposed to be of the Planck length $l_{\mathrm{p}}=\sqrt{\hbar G/c^3}$ order. Equation (15) implies the GUP [3,4] $[x,p]=i\hbar(1+\beta p^2)$. For non-Hermitian quantum mechanics with minimal length, the interested reader may find useful content in [21,22]. Up to the first order in $\beta$, the modified GUP implies [3,4]

$$P_{\mathrm{op}}=\frac{\hbar}{i}\frac{\partial}{\partial x}\left[1+\frac{\beta}{3}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\right]. \tag{16}$$

Let us now consider the so-called box example

$$V(x)=\begin{cases}0, & 0<x<a,\\ \infty, & \text{elsewhere.}\end{cases} \tag{17}$$

In this case, the resulting modified Schrödinger equation appears as

$$\frac{\partial^4\psi(x)}{\partial x^4}-\frac{3}{2\beta\hbar^2}\frac{\partial^2\psi(x)}{\partial x^2}-\frac{3mE'}{\beta\hbar^4}\psi(x)=0. \tag{18}$$

Equation (18) can be factorised as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\lambda_1\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\lambda_2\right)\psi(x)=0 \tag{19a}$$

with

$$\lambda_1=\frac{3+\sqrt{48\beta mE'+9}}{4\beta\hbar^2},\\ \lambda_2=-\frac{12mE'}{\hbar^2(3+\sqrt{48\beta mE'+9})}. \tag{19b}$$

For the basic concepts of factorisation, we refer to the instructive book of Dong [23]. Also, some related aspects to supersymmetry quantum mechanics can be found in [18,24] and references therein. Now, eq. (18) can be factorised as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{3+\sqrt{48\beta mE'+9}}{4\beta\hbar^2}\right) \times \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{12mE'}{\hbar^2(3+\sqrt{48\beta mE'+9})}\right)\psi(x) = 0 \tag{20}$$

and the general solution of eq. (4) is given by

$$\psi(x) = \psi_1(x) + \psi_2(x), \tag{21}$$

where $\psi_1(x)$ and $\psi_2(x)$ are the general solutions of the second-order differential equations obtained as

$$\psi_1(x) = c_1 \sinh\left(\sqrt{\frac{3+\sqrt{48\beta mE'+9}}{4\beta\hbar^2}}x\right) + c_2 \cosh\left(\sqrt{\frac{3+\sqrt{48\beta mE'+9}}{4\beta\hbar^2}}x\right), \tag{22}$$

$$\psi_2(x) = c_3 \sin\left(\sqrt{\frac{12mE'}{\hbar^2(3+\sqrt{48\beta mE'+9})}}x\right) + c_4 \cos\left(\sqrt{\frac{12mE'}{\hbar^2(3+\sqrt{48\beta mE'+9})}}x\right). \tag{23}$$

Considering the boundary condition at the origin removes the terms $\cosh x$ and $\cos x$. On the other hand, as the wave function vanishes at $x = a$, the term $\sinh x$ is neglected and the wave function takes the form

$$\Psi(x) = c_3 \sin(jx) \tag{24a}$$

with

$$\sqrt{\frac{12mE'}{\hbar^2(3+\sqrt{48\beta mE'+9})}} = j. \tag{24b}$$

The boundary condition at $x = a$ implies $\sin ja = 0$ where $j = n\pi/a$ and $n = 1, 2, 3, \ldots$. Using the normalisation condition $\int_0^a |c_3|^2 \sin^2((n\pi x)/a) = 1$, we simply find

$$\psi(x) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}. \tag{25}$$

The energy condition

$$\sqrt{\frac{12mE'}{\hbar^2(3+\sqrt{48\beta mE'+9})}} = \frac{n\pi}{a} \tag{26}$$

leads to

$$E'_n = \frac{n^2\pi^2\hbar^2}{2ma^2} + \frac{\beta}{3}\frac{n^4\pi^4\hbar^4}{ma^4}, \tag{27}$$

which reduces to the well-known case of ordinary quantum mechanics in the case of vanishing minimal length parameter.

## 4. Conclusions

We considered the one-dimensional modified Schrödinger equation due to minimal length with the problem of a particle in a box. We considered the approximate case in which the arising Schrödinger equation appears in the fourth-order form. We next considered the factorisation of the arising fourth-order equation and thereby reported the wave function and energy spectra. The idea looks interesting when we bear in mind the very few analytical works in the field and the deep insight the analytical approaches provide us. We hope to generalise the idea to more realistic interactions.

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