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Truncated q-deformed fermion algebras and phase transition

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Abstract. In this paper, we apply the q-deformed fermion theory to the phase transition from the ordinary fermion into the truncated q-deformed fermion at the critical temperature T_c.

Keywords. q-deformed fermion algebra; phase transition; quantum group.

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1. Introduction

Over the last two decades, quantum group or q-deformed algebra has been the subject of intensive research in physics and mathematics. From the mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2]. The notion of quantum group is easily approached through one of the quantum algebra, which corresponds to a deformation, depending on a certain parameter, of a Lie algebra.

The classical bosonic oscillator algebra allows different types of deformations [3–10]. The first deformation was accomplished by Arik and Coon [3]. They used q-calculus which was originally introduced by Jackson in the early 20th century [11]. In the study of the basic hypergeometric function, Jackson invented the Jackson derivative and integral, which is now called q-derivative and q-integral. Jackson's pioneering research enabled theoretical physicists and mathematicians to study the new physics or mathematics related to the q-calculus. Much was accomplished in this direction and work is under way to find the meaning of the deformed theory.

Arik and Coon's q-boson algebra is given by

$$aa^\dagger - qa^\dagger a = 1, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a. \quad (1)$$

The q-boson algebra with $q \leftrightarrow q^{-1}$ duality was introduced by Macfarlane [4] and Biedenharn [5]. Their q-oscillator algebra is given by

$$aa^\dagger - qa^\dagger a = q^{-N}, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a. \quad (2)$$

Algebra (2) was shown to be related to the deformation of the uncertainty relation which is called a generalised uncertainty relation suggested by Kempf [12,13]. It is thought that the generalised uncertainty relation results from the quantum gravity effect.

The q-deformed fermion algebra was first introduced in [14,15] as follows:

$$bb^\dagger + qb^\dagger b = 1, \quad [N, b^\dagger] = b^\dagger, \quad [N, b] = -b. \quad (3)$$

One year later, another type of q-fermion algebra was introduced [16] as follows:

$$bb^\dagger + qb^\dagger b = q^{-N}, \quad [N, b^\dagger] = b^\dagger, \quad [N, b] = -b. \quad (4)$$

The above algebra does not possess $q \leftrightarrow q^{-1}$ duality like its bosonic version (2).

In this paper, we apply q-deformed fermion theory to the phase transition from the ordinary fermion into the truncated q-deformed fermion at the critical temperature T_c. This paper is organised as follows: in §2 we discuss the extension of Pauli exclusion principle. In §3 we discuss the truncated q-deformed fermion algebra and phase transition. In §4 we discuss the Case I phase transition. In §5 we discuss the Case II phase transition.

2. Extension of Pauli exclusion principle

Ordinary fermion is governed by the fermion algebra

$$bb^\dagger + b^\dagger b = 1, \quad [N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \quad (5)$$

where the number operator N is given by $N = b^\dagger b$. This algebra gives two-dimensional Fock space spanned by the number eigenstates $\{|0\rangle, |1\rangle\}$, where $N|n\rangle = n|n\rangle,\ n = 0, 1$. This fact says that the ordinary fermion obeys the Pauli exclusion principle.

2.1 *Pauli exclusion principle*

No two identical particles having spin equal to half an odd integer can be in the same quantum state.

The Hamiltonian for fermion is then given by

$$H = \frac{w}{2}(b^\dagger b - bb^\dagger), \tag{6}$$

which can be derived from the supersymmetry and we set $\hbar = 1$. The partition function is

$$Z = 2\cosh\left(\frac{\beta w}{2}\right). \tag{7}$$

The internal energy is

$$U = -\frac{w}{2}\tanh\left(\frac{\beta w}{2}\right) = w\left(\frac{1}{e^{\beta w}+1} - \frac{1}{2}\right) = w\left(\langle b^\dagger b\rangle - \frac{1}{2}\right), \tag{8}$$

where

$$\langle b^\dagger b\rangle = \frac{1}{e^{\beta w}+1} \tag{9}$$

is the Fermi–Dirac distribution and $-w/2$ is the zero-point energy. The corresponding specific heat is

$$C_V = \frac{w^2}{kT^2}\left(\frac{1}{e^{\beta w}+1}\right)^2. \tag{10}$$

Here let us consider the phase transition for fermions. We shall consider a situation in which the particle breaks the Pauli exclusion principle below some temperature called the critical temperature T_c; that is to say, the particles behave like ordinary fermions above this temperature but below this temperature they behave like exotic particles which are neither bosons nor fermions. At this stage, we consider the case that this exotic particle obeys the generalised Pauli exclusion principle which is stated below.

2.2 *Generalised Pauli exclusion principle*

No M identical exotic particles having spin equal to $1/M$ can be in the same quantum state where M is an integer larger than 2.

3. Truncated q-deformed fermion algebra and phase transition

Now, let us assume that the particle obeys the truncated q-deformed fermion algebra below the critical temperature. This algebra is defined as

$$bb^\dagger + Q(T)b^\dagger b = 1 - P(T)|M-1\rangle\langle M-1|,$$
$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \tag{11}$$

where $Q(T)$ and $P(T)$ are the functions depending on temperature which should obey

$$Q(T) = \begin{cases} 1, & T > T_c \\ q(T), & T < T_c \end{cases} \tag{12}$$

and

$$P(T) = \begin{cases} 0, & T > T_c \\ p(T), & T < T_c \end{cases} \tag{13}$$

and $P(T)$ should be determined so that algebra (11) gives M-dimensional Fock space. Here we know that $N \neq b^\dagger b$, which implies that N is the same for $T < T_c$ and $T > T_c$. The values of $Q(T)$ and $P(T)$ at the critical temperature T_c depend on the continuity of these functions. In the case of continuous $Q(T)$ and $P(T)$, we have $Q(T_c) = 1$ and $P(T_c) = 0$ while in the case of discontinuous $Q(T)$ and $P(T)$ we adopt $Q(T_c) = \frac{1}{2}(1 + q(T_c))$ and $P(T_c) = \frac{1}{2}p(T_c)$. Here, truncation is necessary for the convergence of partition function when we adopt the Hamiltonian for q-fermion as

$$H = \frac{w}{2}(b^\dagger b - bb^\dagger). \tag{14}$$

Now let us find the Fock representation for algebra (11). The action of N is standard in the sense that

$$N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \ldots, M-1, \tag{15}$$

while the action of the remaining operators is given by

$$b|n\rangle = c_n|n-1\rangle,$$
$$b^\dagger|n\rangle = c_{n+1}|n+1\rangle. \tag{16}$$

Because $|M-1\rangle$ is the highest state, we need

$$c_M = 0. \tag{17}$$

Thus, we know

$$c_n^2 = \frac{1-(-q)^n}{1+q}, \quad n = 0, 1, \ldots, M-1 \tag{18}$$

and

$$P(T) = \frac{1+(-q)^{M-1}}{1+q}. \tag{19}$$

Thus, the algebra can be rewritten as

$$bb^\dagger + Q(T)b^\dagger b = 1 - P(T)|M-1\rangle\langle M-1|,$$

$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \tag{20}$$

where

$$Q(T) = 1 + (q(T) - 1)\theta(T_c - T) \tag{21}$$

and

$$P(T) = \left(\frac{1 + (-q(T))^{M-1}}{1 + q(T)}\right)\theta(T_c - T). \tag{22}$$

Here we assume that the difference between 1 and $q(T)$ is sufficiently small. Through this paper we can consider the following two cases:

Case I: Discontinuous $Q(T)$

$$Q(T) = \begin{cases} 1, & T > T_c \\ q(T) = q = 1 - \epsilon, & T < T_c, \end{cases}$$

where we set $0 < q < 1$ and assume that $\epsilon > 0$ is sufficiently small.

Case II: Continuous $Q(T)$

$$Q(T) = \begin{cases} 1, & T > T_c \\ q(T) = 1 - \left(1 - \frac{T}{T_c}\right)^a \epsilon, & T < T_c, \end{cases}$$

where we assume that $\epsilon > 0$ is sufficiently small and a is real.

4. Case I phase transition

In this case, below the critical temperature, we have the truncated q-fermion algebra defined by

$$bb^\dagger + qb^\dagger b = 1 - \left(\frac{1 + (-q)^{M-1}}{1 + q}\right)|M - 1\rangle\langle M - 1|,$$
$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b. \tag{23}$$

The action of N is standard in the sense that

$$N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \ldots, \tag{24}$$

while the action of the remaining operators is given by

$$b|n\rangle = \sqrt{\{n\}_q}|n - 1\rangle, \quad n = 0, 1, \ldots, M - 1,$$
$$b^\dagger|n\rangle = \sqrt{\{n + 1\}_q}|n + 1\rangle,$$
$$n = 0, 1, \ldots, M - 2, \quad b^\dagger|M - 1\rangle = 0, \tag{25}$$

where the q-number is defined as

$$\{n\}_q = \frac{1 - (-q)^n}{1 + q}. \tag{26}$$

The q-number can also be written as

$$\{2m\}_q = \frac{1 - q^{2m}}{1 + q} \tag{27}$$

and

$$\{2m + 1\}_q = \frac{1 + q^{2m+1}}{1 + q}. \tag{28}$$

The q-number is non-negative for $0 < q < 1$. For Hamiltonian (14), we have the following energy level:

$$E_n = -\frac{w}{2}(-q)^n, \quad n = 0, 1, 2, \ldots, M - 2,$$
$$E_{M-1} = \frac{w}{2}\left(\frac{1 - (-q)^{M-1}}{1 + q}\right). \tag{29}$$

For $M = 2$ we have

$$E_0 = -\frac{w}{2}, \quad E_1 = \frac{w}{2}, \tag{30}$$

for $M = 3$ we have

$$E_0 = -\frac{w}{2}, \quad E_1 = \frac{qw}{2}, \quad E_2 = \frac{w}{2}(1 - q) \tag{31}$$

and for $M = 4$ we have

$$E_0 = -\frac{w}{2}, \quad E_1 = \frac{qw}{2},$$
$$E_2 = -\frac{q^2 w}{2}, \quad E_3 = \frac{w}{2}(1 - q + q^2). \tag{32}$$

Thus, the partition function for the q-fermion depends on whether M is even or odd.

4.1 *Case that M is even*

In this case, we can set

$$M = 2p, \quad p = 1, 2, \ldots. \tag{33}$$

The Fock space consists of p even number states and p odd number states. The partition function is then given by

$$Z_p = \sum_{m=0}^{p-1} \exp\left(\frac{\beta w}{2} q^{2m}\right) + \sum_{m=0}^{p-2} \exp\left(-\frac{\beta w}{2} q^{2m+1}\right)$$
$$+ \exp\left[-\frac{\beta w}{2}\left(\frac{1 + q^{2p-1}}{1 + q}\right)\right]. \tag{34}$$

We restrict our discussion to the case that for $q = 1 - \epsilon$, ϵ is positive and sufficiently small and we consider the terms which are first order in ϵ. Up to a first order in ϵ, we get

$$Z_p \approx p\left[\left(1 - \frac{1}{2}\beta w\epsilon(p - 1)\right)e^{\beta w/2}\right.$$
$$\left. + \left(1 + \frac{1}{2}\beta w\epsilon(p - 1)\right)e^{-\beta w/2}\right]. \tag{35}$$

This case has the fermionic limit $p = 1$ (or $M = 2$). The internal energy is given by

$$U = -\frac{w}{2}\left[\tanh\left(\frac{1}{2}\beta w\right) - \frac{\epsilon(p-1)(\beta w + \sinh(\beta w))}{1+\cosh(\beta w)}\right]. \quad (36)$$

We know that at low temperature, the internal energy behaves as $U \sim -w/2 + \epsilon(p-1)w/2$. The specific heat is

$$C_{\mathrm{V}} = \frac{w^2}{4kT^2}\left[\mathrm{sech}^2\left(\frac{w}{2kT}\right) + \frac{2\epsilon(p-1)\,(-2+(w/kT)\tanh(w/2kT))}{1+\cosh\,(w/kT)}\right]. \quad (37)$$

One can easily find that at low temperature we have $C_{\mathrm{V}} \sim 0$ as in the ordinary fermion case.

4.2 *Case that M is odd*

In this case we can set

$$M = 2p+1, \quad p = 1, 2, \ldots. \quad (38)$$

The Fock space consists of p even number states and p odd number states. The partition function is then given by

$$Z_p = \sum_{m=0}^{p-1}\exp\left(\frac{\beta w}{2}q^{2m}\right) + \sum_{m=0}^{p-1}\exp\left(-\frac{\beta w}{2}q^{2m+1}\right) + \exp\left[-\frac{\beta w}{2}\left(\frac{1-q^{2p}}{1+q}\right)\right]. \quad (39)$$

For $q = 1-\epsilon$, ϵ is positive and sufficiently small we consider the terms which are first order in ϵ. Up to a first order in ϵ, we get

$$Z_p \approx p\left(1 - \frac{1}{2}\beta w\epsilon(p-1)\right)\mathrm{e}^{\beta w/2} + p\left(1+\frac{p}{2}\beta w\epsilon\right)\mathrm{e}^{-\beta w/2} + 1 - \frac{p}{2}\beta w\epsilon. \quad (40)$$

This case does not have fermionic limit for any p. The internal energy is given by

$$U = -\frac{pw\sinh(\beta w/2)}{1+2p\cosh(\beta w/2)} + \frac{pw\epsilon}{4(\mathrm{e}^{\beta w/2}+p+p\mathrm{e}^{\beta w})^2} \times(-2p^2 + U_1\mathrm{e}^{2\beta w} + U_2\mathrm{e}^{\beta w/2} + U_3\mathrm{e}^{3\beta w/2} + U_4\mathrm{e}^{\beta w}), \quad (41)$$

where

$$U_1 = 2p(p-1),$$
$$U_2 = 2p\beta w,$$
$$U_3 = -(2-4p+\beta w),$$
$$U_4 = 2+2p(-1+(2p-1)\beta w). \quad (42)$$

The specific heat is

$$C_{\mathrm{V}} = \frac{w^2}{2kT^2}\left[\frac{p\cosh(\beta w/2)}{1+2p\cosh(\beta w/2)} - \frac{2p^2\sinh^2(\beta w/2)}{(1+2p\cosh(\beta w/2))^2} + \epsilon F(T)\right], \quad (43)$$

where

$$F(T) = \frac{1}{4(1+2p\cosh(\beta w/2))^3} \times(C_1\mathrm{e}^{-\beta w/2} + C_2\cosh(\beta w/2) + C_3\cosh(\beta w) + C_4\cosh(3\beta w/2) + C_5\sinh(\beta w/2) + C_6\sinh(\beta w) + C_7\sinh(\beta w/3)) \quad (44)$$

with

$$C_1 = 2p[4-\beta w + 2p(2p-1)(2+\beta w)],$$
$$C_2 = -\,p[4p(-8+\beta w) + 3(4+\beta w)],$$
$$C_3 = -\,[4+\beta w + 4p^2(2p-1)(\beta w - 2)],$$
$$C_4 = p(\beta w - 4),$$
$$C_5 = -\,p(12 - 16p + 3\beta w + 8p\beta w),$$
$$C_6 = 4+\beta w + 4p^2(2p-1)(\beta w - 2),$$
$$C_7 = p(\beta w - 4). \quad (45)$$

5. Case II phase transition

In the previous section, we discussed Case I phase transition for M odd or even. We found that the even M case involves the fermionic limit while the odd M case does not. In Case II phase transition, the deformation function is continuous at the critical temperature, which implies that the odd M case does not arise. Hence, we will restrict our discussion to the even M case, which gives the following phase transition rule.

'The ordinary fermion can be transformed into the truncated q-fermion with even M while the transformation into the truncated q-fermion with odd M is not allowed.'

In this case, the internal energy is given by

$$U = -\frac{w}{2}\tanh\left(\frac{1}{2}\beta w\right) + \frac{(p-1)w\epsilon(\beta + (a-1)\beta_{\mathrm{c}})}{2(\beta - \beta_{\mathrm{c}})}\left(1 - \frac{\beta_{\mathrm{c}}}{\beta}\right)^a, \quad (46)$$

where $\beta_{\mathrm{c}} = 1/(kT_{\mathrm{c}})$. We know that at low temperature, the internal energy behaves as $U \sim -w/2 + \epsilon(p-1)w/2$. The specific heat is

$$C_V = \frac{w^2}{4kT^2}\left[\operatorname{sech}^2\left(\frac{w}{2kT}\right) + \frac{2\epsilon(p-1)a(1-a)\beta_c^2}{\beta w(\beta-\beta_c)^2}\left(1-\frac{\beta_c}{\beta}\right)^a\right]. \quad (47)$$

One can easily find that at low temperature we have

$$C_V \sim \left(\frac{\epsilon(p-1)a(1-a)w}{2T_c^2}\right)T. \quad (48)$$

From eq. (47) we know that the specific heat below the critical temperature behaves like

$$C_V \sim (T_c - T)^a, \quad (49)$$

where a can be interpreted as the critical exponent of the Landau phase transition theory. Thus, in Case II phase transition, near the critical temperature, we have

$$C_V \sim \begin{cases} \text{constant}, & T > T_c, \\ (T_c - T)^a, & T < T_c. \end{cases} \quad (50)$$

6. Conclusion

In this paper, we applied q-deformed fermion theory to the phase transition from the ordinary fermion into the truncated q-deformed fermion at the critical temperature T_c. We adopted truncation because the partition function for the q-fermion diverges and so it induces unphysical thermodynamic functions. We used the truncated q-deformed fermion algebra to discuss the microscopic structure for the exponent appearing in the Landau phase transition theory. It is well known that phase transitions, continuous or not, are characterised by the fact that certain quantities show very large fluctuation as the critical point is approached, and some may even diverge. The heat capacity behaves like $C_V \sim |T_c - T|^{-\alpha}$, where α is the critical exponent. In this paper we obtained similar results with Landau's phase transition model for two types of transition. Case I transition could be regarded as a kind of first-order transition because the value of q is changed discontinuously near the critical temperature while Case II transition could be regarded as a kind of second-order transition because the value of q is changed continuously near the critical temperature. For Case II we discovered the critical exponent. For the phase transition, our model gives microscopic results because the energy spectrum of q-algebra is microscopic which is different from Landau's model where macroscopic results are obtained.

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