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All single travelling wave patterns to fractional Jimbo–Miwa equation and Zakharov–Kuznetsov equation

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Abstract. By the complete discrimination system of polynomial method, we obtain the classification and representation of all single travelling wave solutions to (3 + 1)-dimensional conformal fractional Jimbo–Miwa equation and fractional Zakharov–Kuznetsov equation. These solutions show rich evolution patterns of models described by these two equations.

Keywords. Conformal fractional derivative; complete discrimination system for polynomial method; travelling wave solution; fractional Jimbo–Miwa equation; fractional Zakharov–Kuznetsov equation.

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1. Introduction

The differential equations, which are the most important models, play key roles in simulating many physical and social phenomena. Soliton equations, such as the famous Korteweg–de Vries (KdV) equation, are a special kind of nonlinear partial differential equations which are characterised by having solitary wave solutions. Finding exact solutions of such differential equations is still an important and difficult problem. Many useful methods, such as inverse scattering method [1], bilinear method [2], Backlund transformation method [3], homogeneous balance method [4,5], canonical-like transformation method [6], Riemann–Hilbert method [1], Jacobi elliptic function expansion method [7,8], the complete discrimination system for polynomial method [9–15] and other direct or indirect methods (see [16–31] and the references therein) have been proposed to find exact solutions to these soliton equations. In this paper, we use the complete discrimination system of the polynomial method to study conformal fractional differential equations. As fractional derivative has more advantages in modelling the practice problems as it can offer a variable index α to describe more subtle processes, many papers on studies on fractional differential equations are published. Recently, a new fractional derivative, namely the conformal fractional derivative, is proposed to study some physical problems [32,33], leading to many related studies from theory and applications (see e.g. [34–39]).

For convenience, we give a simple introduction of conformal fractional derivative. For a function $\phi = \phi(t)$ defined on $(0, +\infty)$ and for a given $\alpha \in (0, 1]$, the conformal fractional derivative is defined by [32]

$$D_t^{\alpha}(\phi(t)) = \lim_{h\to 0} \frac{\phi(t + ht^{1-\alpha}) - \phi(t)}{h}. \tag{1}$$

If the above limitation exists, the function $\phi(t)$ is called α-derivable at t. The basic properties of the conformal fractional derivative can be given as follows [32,33]:

(i) $D_t^{\alpha}(\phi(t) \pm \eta(t)) = D_t^{\alpha}(\phi(t)) \pm D_t^{\alpha}(\eta(t))$,
(ii) $D_t^{\alpha}(\phi(t)\eta(t)) = D_t^{\alpha}(\phi(t))\eta(t) + D_t^{\alpha}(\eta(t))\phi(t)$,
(iii) $D_t^{\alpha}(\phi(t)/\eta(t)) = \frac{D_t^{\alpha}(\phi(t))\eta(t) - D_t^{\alpha}(\eta(t))\phi(t)}{\eta^2(t)}$,
(iv) $D_t^{\alpha}(\phi(\eta(t))) = t^{1-\alpha}\eta^{\alpha-1}\eta'(t)D_{\eta}^{\alpha}\phi(\eta)$,
(v) $D_t^{\alpha}\phi(t) = t^{1-\alpha}\phi'(t)$,

where $\eta(t)$ is a suitable function in each case, e.g. $\eta(t) \neq 0$ and $D_t^{\alpha}(\eta(t))$ exists in Case (iii). The detailed proofs can be found in [32,33]. Other related studies can be seen in [34–39].

In this paper, we consider two nonlinear fractional differential equations, namely fractional Jimbo–Miwa (JM) equation and fractional Zakharov–Kuznetsov (ZK) equation. The JM equation describing some interesting

(3 + 1)-dimensional wave phenomenon as the second member in the entire Kadomtsev–Petviashvili (KP) hierarchy was first investigated by JM and its soliton solutions were obtained in [40]. Its exact solutions and other properties were extensively studied in a series of papers [41–48].

The ZK equation arises from plasma physics. In particular, there exist plasmas with high-energy ion beams in the Van Allen radiation belts and in the plasma sheet boundary layer of the Earth's magnetosphere. Hence, the important aim is to study the case of finite ion temperature. First, Kadomtsev and Petviashvili [49] tried to model a soliton in a two-dimensional system. And then Zakharov and Kuznetsov (ZK) modelled a soliton in a three-dimensional system [50]. They obtained a three-dimensional differential equation now called the ZK equation for a non-relativistic magnetised plasma with $Ti = 0$. Other studies can be found in the existing references (see e.g. [51–59]).

Although there are many studies related to the solutions and the symmetry of the JM equation and the ZK equation, to our knowledge, the complete classification of all travelling wave solutions to fractional JM equation and fractional ZK equation is still not there. In this paper, we give the classification and representation of all single travelling wave solutions to the conformal fractional JM equation (JM, for simplicity) and fractional ZK equation (ZK, for simplicity). These solutions include solitary wave solutions, rational solution and elliptic function solutions. Our result shows that the solutions of JM and ZK equations have rich evolution patterns.

2. Classification of solutions to fractional JM equation

The conformal fractional JM equation [40] reads as

$$rD_t^{\alpha}(v_y) + pv_yv_{xx} + qv_xv_{xy} + v_{xxxy} - sv_{yz} = 0, \quad (2)$$

where p, q, r and s are real constants. Take the travelling wave transformation

$$\xi = k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^{\alpha},$$
$$v(x, y, z, t) = v(\xi), \quad (3)$$

where k_1, k_2, k_3 and ω are real numbers. Substituting it into the JM equation gives an ordinary differential equation:

$$-\omega k_2 r v'' + pk_1^2k_2v''v' + qk_1^2k_2v''v' + k_1^3k_2v^{(4)} - sk_2k_3v'' = 0, \quad (4)$$

where the prime represents the derivative with respect to ξ. Further, integrating it yields

$$-\omega r v' + k_1^3v''' + \frac{1}{2}pk_1^2(v')^2 + \frac{1}{2}qk_1^2(v')^2 - sk_3v' = c_1, \quad (5)$$

where c_1 is an integral constant. By letting

$$u = v' \quad (6)$$

and integrating once again, we get

$$(u')^2 = a_3u^3 + a_2u^2 + a_1u + a_0, \quad (7)$$

where

$$a_3 = -\frac{p+q}{3k_1}, \quad (8)$$

$$a_2 = \frac{\omega r + sk_3}{k_1^3}, \quad (9)$$

and a_1 and a_0 are two arbitrary constants.

Let

$$w = (a_3)^{1/3}u, \quad d_2 = a_2(a_3)^{-2/3},$$
$$d_1 = a_1(a_3)^{-1/3}, \quad d_0 = a_0. \quad (10)$$

Then eq. (7) becomes

$$\int \frac{1}{\sqrt{w^3 + d_2w^2 + d_1w + d_0}}\,\mathrm{d}w = \pm(a_3)^{1/3}(\xi - \xi_0). \quad (11)$$

We denote

$$F(w) = w^3 + d_2w^2 + d_1w + d_0. \quad (12)$$

Then

$$\Delta = -27\left(\frac{2d_2^3}{27} + d_0 - \frac{d_1d_2}{3}\right)^2 - 4\left(d_1 - \frac{d_2^2}{3}\right)^3, \quad (13)$$

$$D = d_1 - \frac{d_2^2}{3}, \quad (14)$$

make up a complete discrimination system for $F(w)$ [15]. We have the following cases:

I: $\Delta = 0, D < 0$. Then $F(w) = (w - \alpha_0)^2(w - \beta), \alpha_0 \neq \beta$. When $w > \beta$, from (6), we have

$$v_1 = (a_3)^{-1/3}\left\{\alpha_0\left(k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^{\alpha} - \xi_0\right) - \frac{2\beta}{\sqrt{\alpha_0 - \beta}(a_3)^{1/3}\{\exp(\frac{\sqrt{\alpha_0-\beta}}{2}(a_3)^{1/3}(k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^{\alpha} - \xi_0)) - 1\}}\right\}, \quad \alpha_0 > \beta, \quad (15)$$

$$v_2 = (a_3)^{-1/3}\left\{\alpha_0\left(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha-\xi_0\right) + \frac{2\beta}{\sqrt{\alpha_0-\beta}(a_3)^{1/3}\{\exp(\frac{\sqrt{\alpha_0-\beta}}{2}(a_3)^{1/3}(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha-\xi_0))+1\}}\right\},\quad \alpha_0>\beta, \tag{16}$$

$$v_3 = (a_3)^{-1/3}\alpha_0\left(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha-\xi_0\right) + 4\sqrt{\alpha_0-\beta}\tan\left(\frac{\sqrt{\beta-\alpha_0}}{2}(a_3)^{1/3}\left(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha-\xi_0\right)\right),\quad \alpha_0<\beta. \tag{17}$$

From the above solutions, we can see that solutions (15) and (16) are two solitary solutions, and solution (17) is a periodic solution.

II: $\Delta = 0,\ D = 0$. Then $F(w) = (w-\alpha_0)^3$. We get a rational solution

$$v_4 = \alpha_0\left(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha-\xi_0\right) - \frac{4}{(a_3)^{2/3}(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha-\xi_0)}. \tag{18}$$

This is a rational solution which also is a singular solution with a movable singular point depending on the initial condition.

III: $\Delta > 0,\ D < 0$. Then $F(w) = (w-\alpha_0)(w-\beta)(w-\gamma)$. Without loss of generality, we suppose $\alpha_0 < \beta < \gamma$. When $\alpha_0 < w < \beta$, we have the Jacobian elliptic function solutions

The solution is also a double periodic solution, but it will not be continuous in the whole domain because the solution will be infinity when the denominator becomes zero.

According to expressions (15)–(21), we have given the corresponding classification of all single travelling wave solutions v_i $(i = 1,\ldots,7)$ to JM equation (2). We can see that these solutions include solitary wave solutions in terms of trigonometric functions, rational solutions and double periodic solutions in terms of Jacobian elliptic functions. In the next section, we give the concrete representations of these solutions so that we can see that these solutions can be realised under the concrete conditions.

Remark 1. When $\alpha = 1$, we get classification of all single travelling wave solutions to the usual JM equation.

$$v_5 = \int \frac{(\beta-\alpha_0)\,\mathrm{sn}^2(\frac{\sqrt{\gamma-\alpha_0}}{2}(a_3)^{1/3}(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha-\xi_0),m)+\alpha_0}{(a_3)^{1/3}}\,\mathrm{d}\xi, \tag{19}$$

$$v_6 = \int \frac{\gamma-\beta\,\mathrm{sn}^2(\frac{\sqrt{\gamma-\alpha_0}}{2}(a_3)^{1/3}(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha-\xi_0),m)}{(a_3)^{1/3}\,\mathrm{cn}^2(\frac{\sqrt{\gamma-\alpha_0}}{2}(a_3)^{1/3}(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha-\xi_0),m)}\mathrm{d}\xi. \tag{20}$$

These two solutions are double periodic solutions. And the last solution is not continuous in the whole domain because it will become infinity when the denominator takes zero.

IV: $\Delta < 0$. Then $F(w) = (w-\alpha_0)(w^2+p_0w+q_0)$ and $p_0^2 - 4q_0 < 0$. We have the following Jacobian elliptic function solution:

3. Representation of solutions to JM equation

According to the above classification of all single travelling wave solutions to JM equation, we give the corresponding representation of these solutions. By taking concrete parameter values and conditions, we give concrete solutions. This means that all these solutions can be realised.

$$v_7 = \int\left\{\frac{\alpha_0-\sqrt{\alpha_0^2+p_0\alpha_0+q_0}}{(a_3)^{1/3}} + \frac{2\sqrt{\alpha_0^2+p_0\alpha_0+q_0}}{(a_3)^{1/3}\{1+\mathrm{cn}((\alpha_0^2+p_0\alpha_0+q_0)^{1/4}(a_3)^{1/3}(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha-\xi_0),m)\}}\right\}\mathrm{d}\xi. \tag{21}$$

Family 1. Take $a_3 = 1, a_2 = -1, a_1 = -1, a_0 = 1$ and other parameters satisfy the following conditions:

$$k_1 = -\frac{p+q}{3}, \tag{22}$$

$$\omega = \frac{-k_1^3 - sk_3}{r}. \tag{23}$$

Then we have $\alpha_0 = 1, \beta = -1, \alpha_0 > \beta$ and hence the solutions are given by

$$v_1 = k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0 + \frac{2}{\sqrt{2}\{\exp(\frac{\sqrt{2}}{2}(k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0)) - 1\}}, \tag{24}$$

$$v_2 = k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0 - \frac{2}{\sqrt{2}\{\exp(\frac{\sqrt{2}}{2}(k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0)) + 1\}}. \tag{25}$$

Family 2. Take $a_3 = 1, a_2 = 1, a_1 = -1, a_0 = -1$ and other parameters satisfy the following conditions (it is easy to see that the conditions are easily satisfied):

$$k_1 = -\frac{p+q}{3}, \tag{26}$$

$$\omega = \frac{k_1^3 - sk_3}{r}. \tag{27}$$

Then we have $\alpha_0 = -1, \beta = 1, \alpha_0 < \beta$ and hence the solutions are given by

$$v_3 = -\left(k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0\right) + 4\sqrt{2}\tan\left(\frac{\sqrt{2}}{2}\left(k_1x+k_2y+k_3z-\frac{\omega}{\alpha}t^\alpha - \xi_0\right)\right). \tag{28}$$

Family 3. Take $a_3 = 1, a_2 = 3, a_1 = 3, a_0 = 1$ and other parameters satisfy the following conditions:

$$k_1 = -\frac{p+q}{3}, \tag{29}$$

$$\omega = \frac{3k_1^3 - sk_3}{r}. \tag{30}$$

Then we have $\alpha_0 = -1$, and hence the solution is given by

$$v_4 = -\left(k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0\right) - \frac{4}{k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0}. \tag{31}$$

Family 4. Take $a_3 = 1, a_2 = -2, a_1 = -1, a_0 = 2$ and other parameters satisfy the following conditions:

$$k_1 = -\frac{p+q}{3}, \tag{32}$$

$$\omega = \frac{-2k_1^3 - sk_3}{r}. \tag{33}$$

Then we have $\alpha_0 = -1, \beta = 1, \gamma = 2, \alpha_0 < \beta < \gamma$, and hence the solutions are given by

$$v_5 = \int \left(2\,\mathrm{sn}^2\left(\frac{\sqrt{3}}{2}\left(k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0\right), m\right) + \alpha_0\right) \mathrm{d}\xi, \tag{34}$$

$$v_6 = \int \frac{2 - \mathrm{sn}^2(\frac{\sqrt{3}}{2}(k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0)), m)}{\mathrm{cn}^2(\frac{\sqrt{3}}{2}(k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0)), m)} \mathrm{d}\xi, \tag{35}$$

where $m^2 = 2/3$.

Family 5. Take $a_3 = 1, a_2 = 2, a_1 = 2, a_0 = 1$ and other parameters satisfy the following conditions:

$$k_1 = -\frac{p+q}{3}, \tag{36}$$

$$\omega = \frac{2k_1^3 - sk_3}{r}. \tag{37}$$

Then we have $\alpha_0 = -1, p_0 = 1, q_0 = 1$, and hence the solution is given by

$$v_7 = -2\left(k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha - \xi_0\right) + \int \frac{2}{1+\mathrm{cn}((k_1x+k_2y + k_3z-\frac{\omega}{\alpha}t^\alpha - \xi_0), m)} \mathrm{d}\xi, \tag{38}$$

where $m^2 = 3/4$.

From the above discussions, we can see that all these solutions can be realised under the concrete parameters. This means that the fractional JM equation shows rich travelling wave patterns.

4. Representation and classification of solutions to ZK equation

Similarly, we can give the classification and representation of all single travelling wave solutions to fractional ZK equation

$$D_t^\alpha u + puu_x + gu_{zzz} + ru_{xxz} + su_{yyz} = 0, \tag{39}$$

where p, q, r and s are real constants. For simplicity, we only give the corresponding representation and omit the detailed classification.

Take the travelling wave transformation

$$\xi = k_1x + k_2y + k_3z - \frac{\omega}{\alpha}t^\alpha, \tag{40}$$

where k_1, k_2, k_3 and ω are real numbers. Substituting it into the ZK equation and integrating it give an ordinary differential equation

$$(u')^2 = a_3u^3 + a_2u^2 + a_1u + a_0, \tag{41}$$

where

$$a_3 = \frac{pk_1}{3(qk_3^3 + rk_1^2k_3 + sk_2^2k_3)}, \tag{42}$$

$$a_2 = -\frac{\omega}{qk_3^3 + rk_1^2k_3 + sk_2^2k_3}, \tag{43}$$

and a_1 and a_0 are two arbitrary constants.

Similar to the JM equation, we have the following representations of solutions to ZK equation under concrete parameters:

I. Take $a_3 = 1, a_2 = -1, a_1 = -1, a_0 = 1$ and $pk_1/3 = qk_3^3 + rk_1^2k_3 + sk_2^2k_3, \omega = -(pk_1/3)$. The solutions are given by

$$u_1 = \tanh^2\left(\frac{\sqrt{2}}{2}\left(k_1x + k_2y + k_3z + \frac{pk_1}{3\alpha}t^\alpha - \xi_0\right)\right), \tag{44}$$

$$u_2 = \coth^2\left(\frac{\sqrt{2}}{2}\left(k_1x + k_2y + k_3z + \frac{pk_1}{\alpha}t^{3\alpha} - \xi_0\right)\right). \tag{45}$$

II. Take $a_3 = 1, a_2 = 1, a_1 = -1, a_0 = -1$ and $(pk_1/3) = qk_3^3 + rk_1^2k_3 + sk_2^2k_3, \omega = pk_1/3$. The solutions are given by

$$u_3 = -1 + 2\sec^2\left(\frac{\sqrt{2}}{2}\left(k_1x + k_2y + k_3z - \frac{pk_1}{3\alpha}t^\alpha - \xi_0\right)\right). \tag{46}$$

III. Take $a_3 = 1, a_2 = 3, a_1 = 3, a_0 = 1$ and $(pk_1/3) = qk_3^3 + rk_1^2k_3 + sk_2^2k_3, \omega = pk_1$. Then we have $\alpha_0 = -1$, and hence the solution is given by

$$u_4 = -1 + \frac{4}{(k_1x + k_2y + k_3z - \frac{pk_1}{\alpha}t^\alpha - \xi_0)^2}. \tag{47}$$

IV. Take $a_3 = 1, a_2 = -2, a_1 = -1, a_0 = 2$ and $(pk_1/3) = qk_3^3 + rk_1^2k_3 + sk_2^2k_3, \omega = -(2pk_1/3)$. The solutions are given by

$$u_5 = 2\,\mathrm{sn}^2\left(\frac{\sqrt{3}}{2}\left(k_1x + k_2y + k_3z + \frac{2pk_1}{3\alpha}t^\alpha - \xi_0\right), m\right) - 1 \tag{48}$$

and

$$u_6 = \frac{2 - \mathrm{sn}^2(\frac{\sqrt{3}}{2}(k_1x + k_2y + k_3z + \frac{2pk_1}{3\alpha}t^\alpha - \xi_0), m)}{\mathrm{cn}^2(\frac{\sqrt{3}}{2}(k_1x + k_2y + k_3z + \frac{2pk_1}{3\alpha}t^\alpha - \xi_0), m)}, \tag{49}$$

where $m^2 = 2/3$.

V. Take $a_3 = 1, a_2 = 2, a_1 = 2, a_0 = 1$ and $(pk_1/3) = qk_3^3 + rk_1^2k_3 + sk_2^2k_3, \omega = 2pk_1/3$. The solution is given by

$$u_7 = -2 + \frac{2}{1 + \mathrm{cn}((k_1x + k_2y + k_3z - \frac{2pk_1}{3\alpha}t^\alpha - \xi_0), m)}, \tag{50}$$

where $m^2 = 3/4$.

These solutions show the concrete representations of rich travelling wave patterns of fractional ZK equation. This is a complete classification of all single travelling wave patterns of the ZK equation.

Remark 2. For the conformal fractional modified Zakharov–Kuznetsov (mZK) equation,

$$D_t^\alpha u + pu^2u_x + gu_{zzz} + ru_{xxz} + su_{yyz} = 0, \tag{51}$$

under the travelling wave transformation, it can be reduced to the following ordinary differential equation (ODE):

$$(u')^2 = a_4u^4 + a_2u^2 + a_1u + a_0. \tag{52}$$

According to [13,15], there are nine cases to give classification to all single travelling wave solutions of eq. (52). Here we omit the discussion for simplicity.

5. Conclusions

We used the complete discrimination system of polynomial method to (3+1)-dimensional conformal fractional JM equation and fractional ZK equation, and obtained the classification of their single travelling wave solutions in varied forms. By taking $\alpha = 1$, we also obtained the corresponding solutions to the usual JM and ZK equations. These results are complete. As a result, we can see that for conformal fractional differential equations, the method is also a useful tool to study their travelling wave solutions. Our results show that the evolution patterns of the conformal fractional JM and ZK equations are rich and varied.

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