Pramana – J. Phys. (2019) 92:7
https://doi.org/10.1007/s12043-018-1673-3

Periodic solution of the cubic nonlinear Klein–Gordon equation and the stability criteria via the He-multiple-scales method

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MS received 23 April 2018; revised 28 May 2018; accepted 4 June 2018; published online 31 October 2018

Abstract. The current work demonstrated a new technique to improve the accuracy and computational efficiency of the nonlinear partial differential equation based on the homotopy perturbation method (HPM). In this proposal, two different homotopy perturbation expansions, the outer expansion and the inner one, are introduced based on two different homotopy parameters. The multiple-scale homotopy technique (He-multiple-scalas method) is applied as an outer perturbation for the nonlinear Klein–Gordon equation. A highly accurate periodic temporal solution has been derived from three orders of perturbation. The amplitude equation, which is imposed as a uniform condition, is of the fourth-order cubic–quintic nonlinear Schrödinger equation. The standard HPM with another homotopy parameter has been used as an inner perturbation to obtain a spatial solution of the nonlinear Schrödinger equation. The cubic–quintic Landau equation is obtained in the inner perturbation technique. Finally, the approximate solution is derived from the temporal and spatial solutions. Further, two different tools are used to obtain the same stability conditions. One of them is a new tool based on the HPM, by constructing the nonlinear frequency. The method adopted here is important and powerful for solving partial differential nonlinear oscillator systems arising in nonlinear science and engineering.

Keywords. Multiple scales; homotopy perturbation method; cubic nonlinear Klein–Gordon equation; cubic–quintic nonlinear Schrödinger equation; nonlinear Landau equation; stability analysis.

PACS Nos 02.60.Lj; 02.70.Wz; 02.30.Jr; 45.10.Hj; 46.40.Ff

1. Introduction

The Klein–Gordon equation which was proposed by Oskar Klein and Walter Gordon in 1926 is a relativistic version of the Schrödinger equation describing free particles. It has many applications in physics and engineering such as quantum field theory, relativistic physics, dispersive wave phenomena, plasma physics and nonlinear optics.

Many efforts have been made to obtain approximate and numerical solutions of linear Klein–Gordon and nonlinear Klein–Gordon equations. Deeba and Khuri [1], El-Sayed [2], Kaya and El-Sayed [3] and Wazwaz [4] used the Adomian decomposition method developed by Adomian [5]. Yusufoglu [6] and Batiha [7] used the variational iteration method developed by He [8] to obtain an approximate solution of the nonlinear Klein–Gordon equation. Khan [9] and Rabie [10] used the modified Laplace decomposition method proposed by Khuri [11] to solve Klein–Gordon equations. Keskin *et al* [12] applied a reduced differential transform method to calculate the approximate analytical solution of the Klein–Gordon equations. Based on the homotopy perturbation method (HPM), Odibat and Momani [13] and Kumar *et al* [14] found approximate solutions of the nonlinear Klein–Gordon equations with initial conditions. The Klein–Gordon equation is an important group of partial differential equations and is present in relativistic quantum mechanics and field theory, which is immensely important for high-energy physicists and is employed for modelling various phenomena, including the propagation of dislocations in crystals and the behaviour of elementary particles [15]. On the other hand, the one-dimensional Klein–Gordon equation is given through partial differential equation [16].

The nonlinear Klein–Gordon equation appears in many types of nonlinearities. The Klein–Gordon equation plays a significant role in many scientific applications [17] and in mathematical physics [18–20]. Several methods such as Weierstrass elliptic function method, elliptic equation rational expansion method and the extended F-function method (see [21] and references therein) are developed to solve Klein–Gordon-type equations. Sirendaoreji [21] used the auxiliary equation method to construct more types of new exact travelling wave solutions of the nonlinear Klein–Gordon equation with quadratic and cubic nonlinearity.

In recent years, the application of the HPM in nonlinear problems has been developed by scientists and engineers [22–28] because this method continuously converts the difficult problem under study into a simple problem which is easy to solve. The HPM was proposed first by He [29] and was further developed and improved by him [30–37]. This method has been improved by many mathematicians and engineers to solve various functional equations. He [38] modified the HPM by introducing an auxiliary term in order to introduce a non-zero auxiliary parameter. This modification allows solving many nonlinear equations with no natural frequency. A combination of the Laplace transform with the HPM is called the He–Laplace method. This coupling has successfully been applied to solve many types of linear and nonlinear ordinary and partial differential equations (for examples, see [39–41]). The parameterised HPM has been addressed by Adamu and Ogenyi [24] for a modification of the HPM. They introduce a new parameter, α, which can be optimal. When it is equal to unity, it turns to its classic version of the homotopy parameter.

The main purpose of the present work is to derive a periodic solution and to find stability criteria of the cubic nonlinear Klein–Gordon equation. In this proposal, the modification of the homotopy perturbation [25–27] is applicable. This modification is called the He-multiple-scales method

2. The cubic nonlinear Klein–Gordon equation

In the present paper, we are concerned with the analytical approximation of the following nonlinear Klein–Gordon equation:

$$y_{tt} - P y_{xx} + m^2 y + Q y^3 = 0, \quad y = y(x,t), \tag{1}$$

where P, m^2 and Q are known physical constants. It is subject to the initial conditions

$$y(x,0) = f(x), \quad y_t(x,0) = g(x).$$

The analysis is based on the modification of HPM by multiple-scale properties [26,27].

Choosing the two operators $L(y)$ and $N(y)$ as

$$L(y) = \left(\partial_{tt} + m^2\right) y \quad \text{and} \quad N(y) = Q y^3 - P y_{xx} \tag{2}$$

the homotopy equation is constructed in the following form:

$$H(y,\rho) = \left(\partial_{tt} + m^2\right) y + \rho\left(Q y^3 - P y_{xx}\right) = 0, \quad \rho \in [0,1]. \tag{3}$$

Consider the three time-scales T_0, T_1 and T_2; $T_n = \rho^n t$, so that

$$\partial_t \to D_0 + \rho D_1 + \rho^2 D_2, \tag{4}$$

$$\partial_{tt} \to D_0^2 + 2\rho D_0 D_1 + \rho^2\left(D_1^2 + 2 D_0 D_2\right),$$

$$D_n \equiv \frac{\partial}{\partial T_n}. \tag{5}$$

Expand the function $y(x,t;\rho)$ as

$$y(x,t;\rho) = y_0(x,T_0,T_1,T_2) + \rho y_1(x,T_0,T_1,T_2) + \rho^2 y_2(x,T_0,T_1,T_2) + \cdots, \tag{6}$$

where $y_n(x,T_0,T_1,T_2)$, $n = 0,1,2,\ldots$, are unknowns to be determined later.

Employing (5) and (6) into (3) and equating the terms with identical powers of ρ yields

$$\rho^0 : \left(D_0^2 + m^2\right) y_0(x,T_0,T_1,T_2) = 0, \tag{7}$$

$$\rho^1 : \left(D_0^2 + m^2\right) y_1(x,T_0,T_1,T_2) = -2 D_0 D_1 y_0 + P y_{0xx} - Q y_0^3, \tag{8}$$

$$\rho^2 : \left(D_0^2 + m^2\right) y_2(x,T_0,T_1,T_2) = -2 D_0 D_1 y_1 - \left(2 D_0 D_2 + D_1^2\right) y_0 + P y_{1xx} - 3 Q y_0^2 y_1. \tag{9}$$

The solution of eq. (7) can be sought as

$$y_0(x,T_0,T_1,T_2) = u(x,T_1,T_2)\,\mathrm{e}^{\mathrm{i} m T_0} + \bar{u}(x,T_1,T_2)\,\mathrm{e}^{-\mathrm{i} m T_0}. \tag{10}$$

Inserting (10) into (8) results in

$$\left(D_0^2 + m^2\right) y_1(x,T_0,T_1,T_2) = \left(-2\mathrm{i} m D_1 u + P u_{xx} - 3 Q u^2 \bar{u}\right) \mathrm{e}^{\mathrm{i} m T_0} - Q u^3 \mathrm{e}^{3\mathrm{i} m T_0} + \text{c.c.} \tag{11}$$

The uniform solution requires the following solvability condition:

$$2\mathrm{i} m D_1 u - P u_{xx} + 3 Q u^2 \bar{u} = 0. \tag{12}$$

This equation is the cubic nonlinear Schrödinger equation in the slow time-scale T_1.

The resulting solution of (11), without secular terms, is given as

$$y_1(x, T_0, T_1, T_2) = -\frac{Q}{8m^2}\left(u^3\,(x, T_1, T_2)\,\mathrm{e}^{3\mathrm{i}mT_0} + \bar{u}^3(x, T_1, T_2)\,\mathrm{e}^{-3\mathrm{i}mT_0}\right). \quad (13)$$

Employing (10) and (13) into (9) using (12) yields the following uniform solution:

$$y_2(x, T_0, T_1, T_2) = -\frac{3Q}{64m^4}\left[2P\left(u^2u_{xx} - uu_x^2\right) - 7Qu^4\bar{u}\right]\mathrm{e}^{3\mathrm{i}mT_0} - \frac{Q^2}{64m^4}u^5\mathrm{e}^{5\mathrm{i}mT_0} + \text{c.c.} \quad (14)$$

The condition for obtaining this uniform solution is

$$\left(2\mathrm{i}mD_2 + D_1^2\right)u - \frac{3Q^2}{8m^2}u^3\bar{u}^2 = 0. \quad (15)$$

This equation contains derivatives with respect to T_1 as well as T_2. Removing the derivative with respect to T_1 with the help of eq. (12) yields

$$2\mathrm{i}mD_2u - \frac{P^2}{4m^2}u_{xxxx} + \frac{3PQ}{4m^2}\left(u^2\bar{u}\right)_{xx} - \frac{3QP}{4m^2}u^2\bar{u}_{xx} + \frac{3QP}{2m^2}u\bar{u}u_{xx} - \frac{21Q^2}{8m^2}u^3\bar{u}^2 = 0. \quad (16)$$

Now, we have two amplitude equations: one with T_1 derivative and the other with T_2 derivative. Multiplying eq. (12) by ρ and adding to eq. (16) multiplied by ρ^2 and using the multiple-scale expansion of the first-order derivative (4), to return to the original variable t, we obtain

$$2\mathrm{i}m\frac{\partial}{\partial t}u - \rho\left(Pu_{xx} - 3Qu^2\bar{u}\right) + \rho^2\left(-\frac{P^2}{4m^2}u_{xxxx} + \frac{3PQ}{4m^2}\left(u^2\bar{u}\right)_{xx} - \frac{3QP}{4m^2}u^2\bar{u}_{xx} + \frac{3QP}{2m^2}u\bar{u}u_{xx} - \frac{21Q^2}{8m^2}u^3\bar{u}^2\right) = 0. \quad (17)$$

At the final form, where $\rho \to 1$, the above amplitude equation becomes

$$\mathrm{i}\frac{\partial}{\partial t}u - \frac{P}{2m}u_{xx} - \frac{P^2}{8m^3}u_{xxxx} + \frac{3Q}{2m}u^2\bar{u} + \frac{3PQ}{8m^3}\left(u^2\bar{u}\right)_{xx} - \frac{3PQ}{8m^3}u^2\bar{u}_{xx} + \frac{3PQ}{4m^3}u\bar{u}u_{xx} - \frac{21Q^2}{16m^3}u^3\bar{u}^2 = 0. \quad (18)$$

By applying the initial conditions of $y(x, 0)$ and $y_t(x, 0)$ to the final case of eq. (10), we have

$$u(x, 0) = \frac{1}{2}f(x) - \frac{\mathrm{i}}{2m}g(x). \quad (19)$$

As seen, eq. (18), the fourth-order cubic–quintic nonlinear Schrödinger equation, represents the amplitude equation, which can be used to study the stability behaviour of the problem. Further, its solution will be used to complete the time-dependent solution of eq. (1).

3. Stability analysis

Equation (18) is used to discuss the stability behaviour for the solution of eq. (1). To obtain the stability conditions of the above nonlinear equation, we follow the procedure adopted in [42–44]. Thus, we assume that eq. (18) varies with time only, so that it becomes

$$\mathrm{i}\frac{\partial}{\partial t}u + \frac{3Q}{2m}u^2\bar{u} - \frac{21Q^2}{16m^3}u^3\bar{u}^2 = 0. \quad (20)$$

In a steady state, the state variable does not change with time. Accordingly, the steady-state solution of (20) is given by

$$u_0 = \sqrt{\frac{8m^2}{7Q}}. \quad (21)$$

To study the stability behaviour, we suppose that the steady-state solution is modulated as

$$u(x, t) = u_0 + \xi(x, t) + \mathrm{i}\zeta(x, t), \quad (22)$$

where $\xi(x, t)$ and $\zeta(x, t)$ are the real functions. Substituting (22) into the nonlinear fourth-order Schrödinger equation (18), linearised in ξ and ζ then separating the real and imaginary parts yields

$$\zeta_t + \frac{P^2}{8m^3}\xi_{xxxx} + \frac{P}{2m}\xi_{xx} - \frac{9Qu_0^2}{2m}\xi - \frac{3PQ}{8m^3}\left(u_0^2\xi_{xx} + 2u_0\xi\right) + \frac{105Q^2}{16m^3}u_0^4\xi = 0, \quad (23)$$

$$\xi_t - \frac{P^2}{8m^3}\zeta_{xxxx} - \frac{P}{2m}\zeta_{xx} + \frac{3Q}{2m}u_0^2\zeta + \frac{3PQ}{8m^3}u_0^2\zeta_{xx} - \frac{21Q^2}{16m^3}u_0^4\zeta = 0. \quad (24)$$

These represent a system in two functions ξ and ζ. Using (21) the above system reduces to

$$\zeta_t + \frac{P^2}{8m^3}\xi_{xxxx} + \frac{P}{14m}\xi_{xx} + \left(\frac{24m}{7} - \frac{3PQ}{4m^3}u_0\right)\xi = 0, \quad (25)$$

$$\xi_t - \frac{P^2}{8m^3}\zeta_{xxxx} - \frac{P}{14m}\zeta_{xx} = 0. \quad (26)$$

For the rule of the non-trivial solution, we may write the solution of this system as

$$\xi(x,t) = \lambda\frac{P^2q^2}{8m^3}\left(q^2 - \frac{4m^2}{7P}\right)\sin(qx - \omega t), \quad (27)$$

$$\zeta(x,t) = -\lambda\omega\cos(qx - \omega t), \quad (28)$$

where λ is an arbitrary constant. Substituting (27) and (28) into (25), we find that q and ω satisfy the following dispersion relation:

$$\omega^2 = \left(\frac{P^2q}{8m^3}\right)^2\left(q^2 - \frac{4m^2}{7P}\right) \times\left(q^4 - \frac{4m^2}{7P}q^2 + \frac{8m^4}{P^2}\left(\frac{24}{7} - \frac{3PQ}{4m^4}u_0\right)\right). \quad (29)$$

It is clear from the above dispersion relation that the necessary and sufficient condition for stability is

$$\left(q^2 - \frac{4m^2}{7P}\right)\left(q^4 - \frac{4m^2}{7P}q^2 + \frac{8m^4}{P^2}\left(\frac{24}{7} - \frac{3PQ}{4m^4}u_0\right)\right) > 0. \quad (30)$$

This stability condition is satisfied for all values of q^2 whence

$$P < 0 \text{ and } Q > 0. \quad (31)$$

Conditions (31) are the stability criteria of the cubic nonlinear Klein–Gordon equation (1). These stability criteria are equivalent to those obtained before in [42] for the second-order cubic nonlinear Schrödinger equation.

4. The complete solution of the Klein–Gordon equation

The three-term approximate solution of (1) can be formulated by substituting (10), (13) and (14) into (6) and setting $\rho \to 1$ yields

$$y(x,t) = u\mathrm{e}^{\mathrm{i}mt} - \frac{Q}{8m^2}u^3\mathrm{e}^{3\mathrm{i}mt} - \frac{3Q}{64m^4}\left[2P\left(u^2u_{xx} - uu_x^2\right) - 7Qu^4\bar{u}\right] \times\mathrm{e}^{3\mathrm{i}mt} - \frac{Q^2}{64m^4}u^5\mathrm{e}^{5\mathrm{i}mt} + \text{c.c.} \quad (32)$$

The unknown amplitude function $u(x,t)$ is determined by solving eq. (18). The function $u(x,t)$ can be constructed by substituting (27) and (28) into (22), and using (21), finally, we obtain

$$u(x,t) = \sqrt{\frac{8m^2}{7Q} + \lambda\frac{P^2q^2}{8m^3}\left(q^2 - \frac{4m^2}{7P}\right)\sin(qx - \omega t)} - \mathrm{i}\lambda\omega\cos(qx - \omega t). \quad (33)$$

In this solution, the frequency of the travelling wave is given in terms of the unknown wave number q as shown by the dispersion relation (29). Therefore, the use of (33) with (32) yields the complete approximate solution of (1) containing the unknown parameter q. To obtain complete approximate solution without any unknowns, we proceed to obtain the solution of eq. (18) as follows.

4.1 *The spatial solution of the fourth-order cubic–quintic nonlinear Schrödinger equation via HPM*

As seen from eq. (18) it is without the polynomial linear variable. Therefore, for obtaining bounded oscillatory solution in the spatial form, we define the wave function $\Psi(x,t)$ through

$$u(x,t) = \Psi(x,t)\exp\left(\frac{\mathrm{i}P^2\mu}{8m^3}t\right), \quad (34)$$

where μ is any real constant. To determine the wave function $\Psi(x,t)$ we may express (34) into (18) to yield

$$\Psi_{xxxx} + \frac{4m^2}{P}\Psi_{xx} + \mu\Psi - \frac{8\mathrm{i}m^3}{P^2}\Psi_t - \frac{12m^2Q}{P^2}\Psi^2\bar{\Psi} - \frac{3Q}{P}\left(\Psi^2\bar{\Psi}\right)_{xx} + \frac{3Q}{P}\Psi^2\bar{\Psi}_{xx} - \frac{6Q}{P}\Psi\bar{\Psi}\Psi_{xx} + \frac{21Q^2}{2P^2}\Psi^3\bar{\Psi}^2 = 0. \quad (35)$$

The solution of this equation via the perturbation technique needs to define a new homotopy parameter $\delta \in [0,1]$. We apply HPM to find an approximate solution for (35). The homotopy equation has been constructed in the following form:

$$\Psi_{xxxx}+\frac{4m^2}{P}\Psi_{xx}+\mu\Psi$$
$$=\delta\left[\frac{8\mathrm{i}m^3}{P^2}\Psi_t+\frac{12m^2Q}{P^2}\Psi^2\bar{\Psi}+\frac{3Q}{P}\left(\Psi^2\bar{\Psi}\right)_{xx}\right.$$
$$\left.-\frac{3Q}{P}\Psi^2\bar{\Psi}_{xx}+\frac{6Q}{P}\Psi\bar{\Psi}\Psi_{xx}-\frac{21Q^2}{2P^2}\Psi^3\bar{\Psi}^2\right],$$
$$\Psi(x,0)=u(x,0)=\frac{1}{2}f(x)-\frac{\mathrm{i}}{2m}g(x). \tag{36}$$

The primary solution of (36) will be derived as $\delta\to 0$, which is governed by the following equation:

$$\Psi_{0xxxx}+\frac{4m^2}{P}\Psi_{0xx}+\mu\Psi_0=0. \tag{37}$$

The solution of the fourth-order partial differential equation (37) has the form

$$\Psi_0(x,t)=C_j(t)\,\mathrm{e}^{\mathrm{i}K_jx}+\bar{C}_j(t)\,\mathrm{e}^{-\mathrm{i}K_jx}, \tag{38}$$

where $C_j(t)$ and its complex conjugate are arbitrary functions. The parameters K_j, $j=1,2$, are real and given by the following characteristic equation:

$$K^4-\frac{4m^2}{P}K^2+\mu=0. \tag{39}$$

Its eigenvalues are

$$K_j^2=\frac{2m^2}{P}\pm\sqrt{\frac{4m^4}{P^2}-\mu}. \tag{40}$$

These eigenvalues are all real if $\mu<0$ or

$$\mu\le\frac{4m^4}{P^2}. \tag{41}$$

In view of the initial condition given in (36), we have

$$C_j(0)=\frac{1}{4}\left(\frac{f(x)}{\cos K_jx}-\mathrm{i}\frac{g(x)}{m\sin K_jx}\right). \tag{42}$$

Let the function $\Psi(x,t;\delta)$ expand as

$$\Psi(x,t;\delta)=C_j(t)\,\mathrm{e}^{\mathrm{i}K_jx}+\bar{C}_j(t)\,\mathrm{e}^{-\mathrm{i}K_jx}+\delta\Psi_1(x,t)$$
$$+\delta^2\Psi_2(x,t)+\cdots. \tag{43}$$

Employing (43) into (36) and equating the identical powers of the first order in δ to zero we get

$$\Psi_{1xxxx}+\frac{4m^2}{P}\Psi_{1xx}+\mu\Psi_1$$
$$=\frac{8m^3}{P^2}\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}C_j+\frac{9Q}{4m^3}(2m^2-PK_j^2)C_j^2\bar{C}_j\right.$$
$$\left.-\frac{105Q^2}{8m^3}C_j^3\bar{C}_j^2\right]\mathrm{e}^{\mathrm{i}K_jx}$$
$$+\frac{3Q}{2P^2}[4(2m^2-5PK_j^2)-35QC_j\bar{C}_j]C_j^3\mathrm{e}^{3\mathrm{i}K_jx}$$
$$-\frac{21Q^2}{2P^2}C_j^5\mathrm{e}^{5\mathrm{i}K_jx}. \tag{44}$$

Removing the source of secular terms yields

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}C_j+\frac{9Q}{4m^3}(2m^2-PK_j^2)C_j^2\bar{C}_j-\frac{105Q^2}{8m^3}C_j^3\bar{C}_j^2=0. \tag{45}$$

This is known as the cubic–quintic nonlinear Landau equation and subject to the initial condition as given by (42). The solution of eq. (45) may be given in the polar form [45] as

$$C_j(t)=\alpha_{0j}\exp\left\{\frac{3\mathrm{i}Q\alpha_{0j}^2}{8m^3}[6(2m^2-PK_j^2)\right.$$
$$\left.-35Q\alpha_{0j}^2]t+\mathrm{i}\beta_{0j}\right\}. \tag{46}$$

This solution shows that the amplitude function $C_j(t)$ has a periodic form in time, where α_{0j} and β_{0j} are two parts of the arbitrary complex constants in time. Applying the initial condition (42), we have

$$\alpha_{0j}^2=\frac{f^2(x)}{16\cos^2K_jx}+\frac{g^2(x)}{16m^2\sin^2K_jx},$$
$$\beta_{0j}=-\tan^{-1}\left(\frac{g(x)}{mf(x)}\cot K_jx\right). \tag{47}$$

The uniform solution of (44) without secular terms has the form

$$\Psi_1(x,t)=\frac{3Q}{32P(5PK_j^2-2m^2)}$$
$$\times[4(2m^2-5PK_j^2)-35Q\alpha_{0j}^2]C_j^3(t)\,\mathrm{e}^{3\mathrm{i}K_jx}$$
$$-\frac{7Q^2}{32P(13PK_j^2-2m^2)}C_j^5(t)\,\mathrm{e}^{5\mathrm{i}K_jx}. \tag{48}$$

The two-term approximate solution of eq. (36) is formulated by letting $\delta \to 1$ into (43) which leads to

$$\Psi(x,t) = 2\alpha_{0j}\cos\varphi_j(x,t) + \frac{3Q[4(2m^2-5PK_j^2)-35Q\alpha_{0j}^2]}{16P(5PK_j^2-2m^2)} \times\alpha_{0j}^3\cos 3\varphi_j(x,t) - \frac{7Q^2}{16P(13PK_j^2-2m^2)} \times\alpha_{0j}^5\cos 5\varphi_j(x,t), \tag{49}$$

where the function $\varphi_j(x,t)$ is given by

$$\varphi_j(x,t) = \frac{3Qt}{128m^3}\left(\frac{f^2(x)}{\cos^2 K_jx}+\frac{g^2(x)}{m^2\sin^2 K_jx}\right) \times\left[6(2m^2-PK_j^2) - \frac{35Q}{16}\left(\frac{f^2(x)}{\cos^2 K_jx}+\frac{g^2(x)}{m^2\sin^2 K_jx}\right)\right] + K_jx - \tan^{-1}\frac{g(x)}{mf(x)}\cot K_jx. \tag{50}$$

Substituting (49) into (32) yields the complete solution in the form

$$y(x,t) = 2\Psi\cos\eta t - \frac{Q}{16m^4}\left[4m^2\Psi^2+3P\Psi\Psi_{xx}-3P\Psi_x^2\right]\Psi\cos 3\eta t + \frac{Q^2}{32m^4}\Psi^5(21\cos 3\eta t-\cos 5\eta t), \tag{51}$$

where

$$\eta = \frac{P^2\mu}{8m^3}+m. \tag{52}$$

Solution (51) is the final approximate solution which has a periodic form in both x and t for all negative values of μ or $\mu < 4m^4/P^2$.

4.2 *Solution at the marginal state of the spatial solution*

The marginal state arises when $\mu = 4m^4/P^2$ in (39), which leads to $K_j^2 = 2m^2/P$. At this stage, we have $\eta \to \frac{3}{2}m$, and

$$\varphi_j(x,t) \to \phi(x,t) = -\frac{105Q^2\alpha_0^4}{8m^3}t + \frac{2m^2}{P}x, \tag{53}$$

$$\Psi(x,t) \to \psi(x,t) = 2\alpha_0\cos\phi(x,t) - \frac{3Q\left(32m^2+35Q\alpha_0^2\right)}{128Pm^2}\alpha_0^3\cos 3\phi(x,t) - \frac{7Q^2}{384Pm^2}\alpha_0^5\cos 5\varphi(x,t). \tag{54}$$

Accordingly, the final solution (51) will reduce to

$$y(x,t) = 2\psi\cos\left(\frac{3}{2}mt\right) - \frac{Q}{16m^4}\left[4m^2\psi^2+3P\psi\psi_{xx}-3P\psi_x^2\right] \times\psi\cos\left(\frac{9}{2}mt\right) + \frac{Q^2}{32m^4}\left(21\cos\left(\frac{9}{2}mt\right) - \cos\left(\frac{15}{2}mt\right)\right)\psi^5. \tag{55}$$

5. Nonlinear frequency approaches

In order to derive the nonlinear frequency, corresponding to the nonlinear Klein–Gordon equation (1), we proceed to find wave travelling solution. To do this, we introduce the wave variable $\theta = k\left(x - V_{\mathrm{g}}t\right)$so that

$$y(x,t) = Y(\theta), \tag{56}$$

where the localised wave solution $Y(\theta)$ travels with the group velocityV_{g} and k is the wavenumber. For this reason, we make the following changes:

$$\frac{\partial}{\partial t} = -kV_{\mathrm{g}}\frac{\mathrm{d}}{\mathrm{d}\theta},\quad \frac{\partial}{\partial x} = k\frac{\mathrm{d}}{\mathrm{d}\theta},\quad \frac{\partial^2}{\partial t^2} = k^2V_{\mathrm{g}}^2\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}$$

and

$$\frac{\partial^2}{\partial x^2} = k^2\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}. \tag{57}$$

Using (57) changes the partial differential equation (1) to the following ordinary differential equation:

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\theta^2} + \left(\frac{m^2}{k^2(V_{\mathrm{g}}^2-P)}\right)Y + \left(\frac{Q}{k^2(V_{\mathrm{g}}^2-P)}\right)Y^3 = 0,\quad Y = Y(\theta). \tag{58}$$

This is the cubic Duffing equation and can be solved using the HPM. When the travelling wave travels with velocity equivalent to the phase velocity, the wave variable θ becomes zero. Therefore, we may define the initial conditions as $Y(\theta=0) = 1$ and $Y_\theta(\theta=0) = 0$.

The homotopy statement of (58) is written as

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}\theta^2} + \Omega_0^2 Y + \varepsilon R Y^3 = 0, \quad \varepsilon \in [0, 1], \tag{59}$$

where $\varepsilon \in [0, 1]$ is the homotopy parameter and

$$\Omega_0^2 = \frac{m^2}{k^2\left(V_\mathrm{g}^2 - P\right)} \quad \text{and} \quad R = \frac{Q}{k^2\left(V_\mathrm{g}^2 - P\right)}. \tag{60}$$

For the frequency analysis, consider the expansion

$$\Omega^2(\rho) = \Omega_0^2 + \varepsilon\Omega_1 + \varepsilon^2\Omega_2 + \cdots, \tag{61}$$

where Ω_n, $n = 1, 2, \ldots$, are unknowns determined by the conditions of the uniform solutions. Expand the function $Y(\theta)$ as

$$Y(\theta; \varepsilon) = Y_0(\theta) + \varepsilon Y_1(\theta) + \varepsilon^2 Y_2(\theta) + \cdots. \tag{62}$$

Employing (61) and (62) with the homotopy equation (59) and equating the identical powers of ε to zero yields

$$\varepsilon^0 : \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} Y_0 + \Omega^2 Y_0 = 0. \tag{63}$$

$$\varepsilon^1 : \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} Y_1 + \Omega^2 Y_1 = \Omega_1 Y_0 - R Y_0^3, \tag{64}$$

$$\varepsilon^2 : \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} Y_2 + \Omega^2 Y_2 = \Omega_1 Y_1 + \Omega_2 Y_0 - 3R Y_0^2 Y_1. \tag{65}$$

Applying the initial condition to the solution of (63) yields

$$Y_0(\theta) = \cos(\Omega\theta). \tag{66}$$

The solution of (64) without secular terms is

$$Y_1(\theta) = \frac{R}{32\Omega^2}(\cos 3\Omega\theta - \cos \Omega\theta). \tag{67}$$

Elimination of secular terms yields

$$\Omega_1 = \frac{3R}{4}. \tag{68}$$

The solution of (65) without secular terms gives

$$Y_2(\theta) = \frac{R^2}{128 \times 8\Omega^4}(\cos 5\Omega\theta - \cos \Omega\theta). \tag{69}$$

Removing secular terms yields

$$\Omega_2 = -\frac{3R^2}{128\Omega^2}. \tag{70}$$

The approximate solution is

$$Y(\theta) = \left(1 - \frac{R}{32\Omega^2} - \left(\frac{R}{32\Omega^2}\right)^2\right)\cos \Omega\theta + \frac{R}{32\Omega^2}\cos 3\Omega\theta + \left(\frac{R}{32\Omega^2}\right)^2 \cos 5\Omega\theta. \tag{71}$$

The three-term approximate nonlinear frequency is

$$\Omega^4 - \left(\Omega_0^2 + \frac{3R}{4}\right)\Omega^2 + \frac{3R^2}{128} = 0. \tag{72}$$

The approximate solution (71) is an oscillator where the parameter Ω has real values. This requires the dispersion relation (72) to have real positive roots. Therefore, we have one condition

$$\Omega_0^2 + \frac{3R}{4} > 0. \tag{73}$$

This condition can be satisfied for all values of Ω_0^2 when the coefficient R is positive. It is seen from definitions (60) that $R > 0$, when

$$P < 0 \quad \text{and} \quad Q > 0. \tag{74}$$

These conditions are derived for all values of V_g^2. It is observed that (74) shows the same stability conditions as obtained before in §3. This procedure for obtaining the stability criteria is new, which depends on constructing approximate nonlinear frequency through the HPM.

6. Conclusions

In this communication, a new approach using the coupling of two HPMs is applied to the partial differential equation of the nonlinear Klein–Gordon type. The new modification of HPM, which is known as the multiple time-scale HPM, has been applied for solving nonlinear Klein–Gordon equation. In this approach, the perturbed expansion is called an outer perturbation. The homotopy approach, with three time-scales, has been applied which leads to the fourth-order cubic–quintic nonlinear Schrödinger equation as an amplitude equation. This amplitude equation is used to derive the stability conditions by means of the steady-state solution, which agrees with the same stability conditions obtained before by Nayfeh [42]. Again, by deriving the wave solution of the nonlinear Klein–Gordon equation and constructing the nonlinear frequency based on HPM, the same stability conditions are imposed. Therefore, the latter approach of the nonlinear frequency is new and can be used as an alternative tool for studying and obtaining stability conditions. To complete the approximate solution of the problem, the solution of the amplitude equation is required. Due to its complication, another perturbation technique is needed. The standard homotopy perturbation analysis with other homotopy parameters has been applied to the amplitude equation and represented as an inner perturbation. The spatial solution of the cubic–quintic fourth-order nonlinear Schrödinger equation has been obtained. In order to obtain a periodic solution in spatial form, a new auxiliary term is introduced. In this approach, the nonlinear Landau equation is obtained, for a uniform solution, which is solved in the polar form. Finally, the complete approximate solution, which is composed of the temporal solution as an outer expansion and the spatial solution as an inner expansion, for the nonlinear Klein–Gordon equation is derived. We think that this method has a great potential and can be applied to other strongly nonlinear oscillators.

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