Pramana – J. Phys. (2018) 91:77
https://doi.org/10.1007/s12043-018-1649-3

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Potentials and phase shifts for nucleon–light nuclei systems

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MS received 22 July 2017; revised 10 March 2018; accepted 27 April 2018; published online 26 September 2018

Abstract. Two simple models based on the Coulomb-distorted phase function and supersymmetry-inspired factorisation methods are adapted to deal with the nucleon–light nuclei elastic scattering at low energies. The first one is associated with the derivation of a closed-form expression of the scattering phase shift for motion in Coulomb-distorted separable non-local potentials. The second one deals with the development of an energy-dependent phase equivalent local potential to the non-local one for s-wave and its subsequent generation of higher partial wave interactions through the formalism of supersymmetric quantum mechanics. The usefulness of our models is demonstrated through the computation of α–nucleon scattering phase shifts at low energies up to partial waves $\ell = 2$. Certain energy-dependent correction factors are also incorporated into energy-dependent higher partial wave potentials to achieve an excellent agreement with the standard data.

Keywords. Non-local separable potential; energy-dependent local potential; supersymmetry formalism; phase function methods for non-local and local interactions; α–nucleon systems; elastic scattering phase shifts.

PACS Nos 24.10.−i; 03.65.Nk; 11.30.Pb; 13.75.Cs

1. Introduction

In ref. [1], the result of α–nucleon elastic scattering phase shifts within the framework of the supersymmetric quantum mechanics (SQM) [2–4] has been reported where the short-range nuclear interaction is represented by a two-term nuclear Hulthén-type potential. The proton–helium interaction is a combination of a short-range interaction and an electromagnetic part. The essential features of the short-range interaction have been clarified by the microscopic theory of nucleus–nucleus interaction such as the generator coordinate method (GCM) and the resonating group method (RGM) [5–9]. In the RGM, an interaction is generated from two nucleon forces consisting of two parts – a direct part and a complicated non-local kernel. The effect of this non-locality on the wave function is similar to those of a repulsive local potential. This non-local kernel presents mathematical difficulties for the rigorous inclusion of electromagnetic interaction in the studies of proton–helium (p−α) elastic scattering.

The α-particles are tightly bound and have no low-lying excited states. Therefore, within a limited range of energies (<20 MeV) α–nucleon interaction can be regarded as a two-body problem. The low-energy α–nucleon elastic scattering is not only an interesting problem but also has some importance in the cluster model description of the nuclei. The results of the α–nucleon elastic scattering have been reported earlier in a number of publications [10–16]. In view of the tight binding of the α–particles, one can use the Schrödinger equation with Coulomb plus separable non-local potentials to compute the phase shifts for the α–nucleon scattering. The analysis of α–nucleon data with the non-local separable interaction has been performed both in momentum and representation space by a number of researchers [17–26] to achieve a good agreement with the experimental data [10–14]. However, the use of separable non-local potential for the (α−n) system has been objected [27] due to the inclusion of a bound 1 s state for helium-5 whose existence is forbidden by the Pauli exclusion principle. On the one hand, with this attractive two-body s-wave interaction, this difficulty is unavoidable but on the other hand, the standard results of the phase shifts are reproduced properly. We shall adapt here the phase function method (PFM) for the Coulomb-distorted separable non-local potential [28] to compute scattering phase shifts of

(α−p) and (α−n) systems up to partial waves $\ell = 2$. Subsequently, to avoid the ambiguity encountered with the use of separable non-local potential, we shall also construct the associated phase equivalent energy-dependent local potentials. In this context, we first follow the method of Arnold and MacKellar [29] to develop an energy-dependent local potential for the s-wave only. Furthermore, this s-wave potential and its corresponding wave function are exploited by the judicious use of the formalism of SQM [30–34] to generate higher partial wave interactions and their corresponding phase shifts are computed using the PFM [35].

Section 2 is devoted to reviewing the PFM for motion in the Coulomb plus separable non-local potential. In §3, we develop energy-dependent local potentials from non-local interactions through the supersymmetry-inspired factorisation method. In §4, we compute elastic scattering phase shifts of the α–nucleon systems and discuss our results and finally, concluding remarks are given in §5.

2. Phase function for the Coulomb-distorted separable non-local potential

The PFM is an efficient approach to compute the scattering phase shifts of the quantum mechanical problems involving local [35] and non-local interactions [30,36] and is based on the separation of radial wave function of the Schrödinger equation into an amplitude part $\alpha_\ell(k,r)$ and an oscillating part with variable phase $\delta_\ell(k,r)$. The function $\delta_\ell(k,r)$ called the phase function describes the meaning of the phase shift, at each point of the wave function for scattering by the potential truncated at a distance r. The local potential $\delta_\ell(k,r)$ satisfies the first-order nonlinear differential equation given by

$$\delta'_\ell(k,r) = -k^{-1}V(r)\big[\hat{j}_\ell(kr)\cos\delta_\ell(k,r) - \hat{\eta}_\ell(kr)\sin\delta_\ell(k,r)\big]^2, \tag{1}$$

where $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$ are the Riccati–Bessel functions. We shall follow the phase convention of Calogero [35] with the Hankel function of the first kind written as $\hat{h}^1_\ell(x) = -\hat{\eta}_\ell(x) + \mathrm{i}\hat{j}_\ell(x)$. The scattering phase shift $\delta_\ell(k)$ is achieved by solving the equation from the origin to the asymptotic region with the initial condition $\delta_\ell(k,0) = 0$. Finally, one obtains the phase shift $\delta_\ell(k) = \lim_{r\to\infty}\delta_\ell(k,r)$.

For general non-local potentials, the phase equation has a very long complicated mathematical structure [36]. In the special case of a separable potential, however, the PFM can be written in closed form without solving the phase equation. This is, however, no longer true when the Coulomb interaction is involved. The calculation of the scattering phase shifts for Coulomb plus separable potential need not start with the kinetic energy as a zero-order Hamiltonian. Instead, one should begin by constructing the states, Green's function, etc. for a model Hamiltonian that involves the Coulomb interaction. Following this root, Sett *et al* [28] constructed an expression for the phase function $\delta(k,r)$ for motion in the Coulomb plus separable potential.

The phase function for Coulomb plus rank N potential reads as

$$\begin{aligned}\tan\delta^{\mathrm{CS}}_\ell(k,r) = &-\sum_{i,j=1}^{N}\lambda^{(i)}_\ell a^{(i,j)\mathrm{CS}}_\ell(k,\beta_\ell)Y^{(j)\mathrm{CS}}_\ell(k,\beta_\ell)\\ &\times\int_0^r \mathrm{d}r'\varphi^{\mathrm{C}}_\ell(k,r')v^{(i)}_\ell(r')\\ &\times\big[k\,|\Im^{\mathrm{C}}_\ell(k)|^2\det{}_{\mathrm{N}}\,A^{\mathrm{CS}}_\ell(k)\\ &+\sum_{i,j=1}^{N}\lambda^{(i)}_\ell a^{(i,j)\mathrm{CS}}_\ell(k,\beta_\ell)Y^{(j)\mathrm{CS}}_\ell\\ &\times(k,\beta_\ell)X^{(i)\mathrm{CS}}_\ell(k,\beta_\ell)\big]^{-1},\end{aligned} \tag{2}$$

where $a^{(i,j)\mathrm{CS}}_\ell(k)$'s stand for the elements of $\det_{\mathrm{N}}A^{\mathrm{CS}}_\ell(k)$ and $a^{(i,j)\mathrm{CS}}_\ell(k,\beta_\ell)$ are the cofactors of $a^{(i,j)\mathrm{CS}}_\ell(k)$. The other quantities $\varphi^{\mathrm{C}}_\ell(k,r)$, $f^{\mathrm{C}}_\ell(k,r)$ and $\Im^{\mathrm{C}}_\ell(k)$ represent the regular, irregular solutions and Jost function of the Coulomb potential [37]; $v^{(i)}_\ell(r)$ is the form factor of the non-local separable potential and $Y^{(j)\mathrm{CS}}_\ell(k,\beta_\ell)$ and $X^{(i)\mathrm{CS}}_\ell(k,\beta_\ell)$ are the integral transforms of Coulomb regular and irregular solutions by the form factors of the separable potential. In the limit $r\to\infty$, the above equation describes the scattering phase shifts for motion in the Coulomb plus separable non-local potential and is expressed as

$$\begin{aligned}\tan\delta^{\mathrm{CS}}_\ell(k) = &-\sum_{i,j=1}^{N}\lambda^{(i)}_\ell a^{(i,j)\mathrm{CS}}_\ell(k,\beta_\ell)\\ &\times Y^{(j)\mathrm{CS}}_\ell(k,\beta_\ell)Y^{(i)\mathrm{CS}}_\ell(k,\beta_\ell)\\ &\times\Big[k\,|\Im^{\mathrm{C}}_\ell(k)|^2\det{}_{\mathrm{N}}\,A^{\mathrm{CS}}_\ell(k)\\ &+\sum_{i,j=1}^{N}\lambda^{(i)}_\ell a^{(i,j)\mathrm{CS}}_\ell(k,\beta_\ell)\\ &\times Y^{(j)\mathrm{CS}}_\ell(k,\beta_\ell)X^{(i)\mathrm{CS}}_\ell(k,\beta_\ell)\Big]^{-1}.\end{aligned} \tag{3}$$

For the nuclear part of the α–nucleon interaction, the separable potential $V_\ell(r,r')$ is considered as

$$V_\ell(r,r') = \lambda_\ell g_\ell(r)g_\ell(r') \tag{4}$$

with

$$g_\ell(r) = 2^{-\ell}(\ell!)^{-1}r^\ell \mathrm{e}^{-\beta_\ell r} \tag{5}$$

and λ_ℓ is the strength of the interaction under consideration. In view of eq. (4), eq. (3) is obtained as

$$\tan\delta_\ell^{\rm CS}(k) = -\lambda_\ell \left[Y_\ell^{\rm CS}(k,\beta_\ell)\right]^2 \times \left[k\left|\Im_\ell^{\rm C}(k)\right|^2 D_\ell^{\rm CS}(k) + \lambda_\ell\, Y_\ell^{\rm CS}(k,\beta_\ell) X_\ell^{\rm CS}(k,\beta_\ell)\right]^{-1} \tag{6}$$

with

$$Y_\ell^{\rm CS}(k,\beta_\ell) = \frac{k\,\Gamma(2\ell+2)}{2^\ell \ell!\left(\beta_\ell^2+k^2\right)^{\ell+1}} \left(\frac{\beta_\ell - {\rm i}k}{\beta_\ell + {\rm i}k}\right)^{{\rm i}\eta}, \tag{7}$$

$$X_\ell^{\rm CS}(k,\beta_\ell) = {\rm Re}\left[\frac{\Im_\ell^{\rm C}(-k){\rm e}^{\pi\eta/2}}{(4{\rm i}k)^\ell \ell!(\beta_\ell - {\rm i}k)} \frac{\Gamma(2\ell+2)}{\Gamma(\ell+2+{\rm i}\eta)} \times {}_2F_1\left(1, {\rm i}\eta-\ell; \ell+2+{\rm i}\eta; \frac{\beta_\ell+{\rm i}k}{\beta_\ell-{\rm i}k}\right)\right] \tag{8}$$

and the Fredholm determinant [38] associated with the regular/irregular boundary conditions

$$\begin{aligned} D_\ell^{\rm CS}(k) &= 1 - \lambda_\ell \int_0^\infty \int_0^r {\rm d}r\,{\rm d}r' g_\ell(r) G_\ell^{C(R)}(r,r') g_\ell(r') \\ &= 1 - \lambda_\ell \frac{\Gamma(2\ell+2)}{2^{2\ell}\ell!(\beta_\ell - {\rm i}k)(\ell+1+{\rm i}\eta)} \\ &\quad \times \left[\frac{1}{\left(\beta_\ell^2+k^2\right)^{\ell+1}} \left(\frac{\beta_\ell-{\rm i}k}{\beta_\ell+{\rm i}k}\right)^{{\rm i}\eta} \right. \\ &\quad \times {}_2F_1\left(1, {\rm i}\eta-\ell; \ell+2+{\rm i}\eta; \frac{\beta_\ell+{\rm i}k}{\beta_\ell-{\rm i}k}\right) \\ &\quad - \frac{1}{(2\beta_\ell)^{2\ell+1}(\beta_\ell-{\rm i}k)} {}_2F_1\left(1, {\rm i}\eta-\ell; \ell\right. \\ &\quad \left.\left. +2+{\rm i}\eta; \left(\frac{\beta_\ell+{\rm i}k}{\beta_\ell-{\rm i}k}\right)^2\right)\right]. \end{aligned} \tag{9}$$

In deriving the above expressions, we have used the following standard integrals and recurrence relation [39–42]:

$$\begin{aligned} &\int_0^\infty {\rm e}^{-\lambda z} z^\nu \Phi(a,c;pz) \\ &= \frac{\Gamma(\nu+1)}{\lambda^{\nu+1}} {}_2F_1(a,\nu+1;c;p/\lambda), \end{aligned} \tag{10}$$

$$\begin{aligned} &F\left(b,S;1+S+b-d;1-\frac{\mu}{a}\right) \\ &= \frac{a^S\Gamma(1+b+S-d)}{\Gamma(1+S-d)\Gamma(S)} \\ &\quad \times \int_0^\infty {\rm e}^{-ax} x^{S-1} \Psi(b,d;\mu x){\rm d}x; \\ &\quad {\rm Re}\,S > 0, 1+{\rm Re}\,S > {\rm Re}\,d \end{aligned} \tag{11}$$

and

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b} {}_2F_1(c-a,c-b;c;z). \tag{12}$$

By utilising the closed-form expression for $\tan\delta_\ell^{\rm CS}(k)$ for the Coulomb plus separable potential, we shall compute the scattering phase shifts of the α–nucleon systems for the partial waves up to $\ell = 2$.

3. Energy-dependent potential through SQM

Arnold and MacKellar [29] developed a method for the construction of a phase equivalent energy-dependent local potential to a separable non-local interaction in the coordinate space representation. This method of generating equivalent local potentials is independent of boundary conditions imposed on the solutions of the non-local equation. For $\ell = 0$, the form factor of the non-local potential considered in the previous section coincides with that of Yamaguchi potential [43]. In the recent past [33], the nucleon–nucleon phase shifts by energy-dependent two-nucleon interaction have been studied up to the partial wave $\ell = 1$ with the formalism of SQM. The present section is devoted to the development of energy-dependent potentials up to partial wave $\ell = 2$ through the supersymmetry-inspired factorisation method.

A second-order differential operator of the form

$$H_n = -\frac{\partial^2}{\partial x^2} + V_n(x) \tag{13}$$

can be factorised in terms of two first-order differential operators of the form

$$O_n^{(\pm)}(x) = \pm\frac{\rm d}{{\rm d}x} + \frac{\rm d}{{\rm d}x}\ln\psi_n^{(0)}(x) \tag{14}$$

with

$$V_n(x) = V_{n-1}(x) - \frac{{\rm d}^2}{{\rm d}x^2}\ln\psi_{n-1}^{(0)}(x); \quad n = 1,2,3,\ldots. \tag{15}$$

The quantities $O_n^{(\pm)}(x)$ are defined as the lowering and raising operators. These relations are also equally applicable for the continuous part of the spectrum. Different eigenfunctions of the Hamiltonian under consideration are related by

$$\psi_n^{(m)}(x) = \left(E_n^{(m)} - E_{n-1}^{(0)}\right)^{-1/2} O_{n-1}^- \psi_{n-1}^{(m+1)}(x), \quad m = 0,1,2,\ldots. \tag{16}$$

For positive energy states, however, the wave functions are connected by

$$\psi_n(x) = O_{n-1}^{-}\psi_{n-1}(x). \tag{17}$$

The regular and irregular solutions of the parent non-local and its phase equivalent energy-dependent local potentials [29] are related by

$$\varphi_{\mathrm{N}}(k,r) = A(k,r)\,\varphi_{\mathrm{L}}(k,r) \tag{18}$$

and

$$f_{\mathrm{N}}(k,r) = A(k,r)\,f_{\mathrm{L}}(k,r), \tag{19}$$

where $A(k,r)$ is the damping function. The regular and irregular solutions corresponding to non-local Yamaguchi [43] potential read as

$$\varphi_{\mathrm{N}}(k,r) = \frac{\sin kr}{k} + C\left[\mathrm{e}^{-\beta r} + \frac{\beta}{k}\sin kr - \cos kr\right] \tag{20}$$

and

$$f_{\mathrm{N}}(k,r) = \mathrm{e}^{\mathrm{i}kr} + \frac{\lambda(\beta+\mathrm{i}k)}{D(k)(\beta^2+k^2)^2}\mathrm{e}^{-\beta r} \tag{21}$$

with $D(k)$ being the Fredholm determinant under the regular/irregular boundary conditions [44]

$$D(k) = 1 - \frac{\lambda}{2\beta(\beta^2+k^2)} \tag{22}$$

and the other quantity

$$C = \frac{\lambda}{D(k)(\beta^2+k^2)}. \tag{23}$$

The damping function $A(k,r)$ [29,33] corresponding to Yamaguchi potential is written as

$$A(k,r) = \left(1 + k^{-1}C_{\mathrm{S}}\mathrm{e}^{-\beta r}\sin kr\right)^{1/2} \tag{24}$$

with

$$C_{\mathrm{S}} = \frac{\lambda}{(\beta^2+k^2-\lambda/2\beta)}. \tag{25}$$

Combining eqs (19), (21), (24) and (25), the bound state solution of the equivalent local Yamaguchi potential can easily be obtained by substituting $k = \mathrm{i}\alpha$. As the zeros of the Jost functions produce the bound state energies one obtains $\lambda = -2\beta(\beta+\alpha)^2$. Therefore, the behaviour of the bound state solution of equivalent local Yamaguchi potential at the origin yields

$$\Psi_{\mathrm{L}}^{(0)} = \lim_{r\to 0} f_{\mathrm{L}}^{(B)}(\alpha,r) = \frac{\mathrm{e}^{-\alpha r} - \mathrm{e}^{-\beta r}}{\left(1 + C_B r\mathrm{e}^{-\beta r}\right)^{1/2}} \tag{26}$$

with

$$C_B = \frac{\lambda}{(\beta^2-\alpha^2-\lambda/2\beta)}. \tag{27}$$

Combining eqs (14), (26) and (27), one can construct the raising operator $O^{(-)}$ as

$$O^{(-)} = -\frac{\partial}{\partial r} + \frac{(\beta\mathrm{e}^{-\beta r} - \alpha\mathrm{e}^{-\alpha r})}{(\mathrm{e}^{-\alpha r} - \mathrm{e}^{-\beta r})} - \frac{1}{2}\frac{C_B\mathrm{e}^{-\beta r}(1-\beta r)}{(1 + C_B r\,\mathrm{e}^{-\beta r})}. \tag{28}$$

Equations (17), (18), (20), (22) and (24), in conjunction with eq. (28), lead to the p-wave regular solution

$$\phi_{\mathrm{L}}^{\mathrm{p}}(k,r) = -\frac{\phi_{\mathrm{N}}'(k,r)}{A(k,r)} + \frac{\phi_{\mathrm{N}}(k,r)A'(k,r)}{(A(k,r))^2} + \frac{(\beta\mathrm{e}^{-\beta r} - \alpha\mathrm{e}^{-\alpha r})}{(\mathrm{e}^{-\alpha r} - \mathrm{e}^{-\beta r})}\frac{\phi_{\mathrm{N}}(k,r)}{A(k,r)} - \frac{1}{2}\frac{C_B\mathrm{e}^{-\beta r}(1-\beta r)}{(1 + C_B r\mathrm{e}^{-\beta r})}\frac{\phi_{\mathrm{N}}(k,r)}{A(k,r)}. \tag{29}$$

The construction of the next higher partial wave potential requires the behaviour of $\phi_{\mathrm{L}}^{\mathrm{p}}(k,r)$ near the origin. As $r \to 0$, $\phi_{\mathrm{L}}^{\mathrm{p}}(k,r)$ is obtained as

$$\phi_{\mathrm{L}}^{\mathrm{p}(0)}(k,r) = -\frac{\phi_{\mathrm{N}}'^{(0)}(k,r)}{A^{(0)}(k,r)} + \frac{\phi_{\mathrm{N}}^{(0)}(k,r)A'^{(0)}(k,r)}{(A^{(0)}(k,r))^2} + \frac{(\beta\mathrm{e}^{-\beta r} - \alpha\mathrm{e}^{-\alpha r})}{(\mathrm{e}^{-\alpha r} - \mathrm{e}^{-\beta r})}\frac{\phi_{\mathrm{N}}^{(0)}(k,r)}{A^{(0)}(k,r)} - \frac{1}{2}\frac{C_B\mathrm{e}^{-\beta r}(1-\beta r)}{(1 + C_B r\mathrm{e}^{-\beta r})}\frac{\phi_{\mathrm{N}}^{(0)}(k,r)}{A^{(0)}(k,r)} \tag{30}$$

with

$$\phi_{\mathrm{N}}^{(0)}(k,r) = r + C(\mathrm{e}^{-\beta r} + \beta r - 1), \tag{31}$$

$$\phi_{\mathrm{N}}'^{(0)}(k,r) = 1 + C\beta(1 - \mathrm{e}^{-\beta r}), \tag{32}$$

$$A^{(0)}(k,r) = 1 + C_{\mathrm{S}}r\mathrm{e}^{-\beta r} \tag{33}$$

and

$$A'^{(0)}(k,r) = C_{\mathrm{S}}(1-\beta r)\,\mathrm{e}^{-\beta r}. \tag{34}$$

From eqs (15) and (30) one can easily construct the d-wave interaction as

$$V_2(k,r) = V_1(k,r) - \frac{\partial^2}{\partial r^2}\ln\phi_{\mathrm{L}}^{p(0)}(k,r) \tag{35}$$

with

$$\frac{\partial^2}{\partial r^2}\ln\phi_{\mathrm{L}}^{p(0)}(k,r) = \frac{T_1''(k,r) + T_2''(k,r) + T_3''(k,r) + T_4''(k,r)}{T_1(k,r) + T_2(k,r) + T_3(k,r) + T_4(k,r)} - \left(\frac{T_1'(k,r) + T_2'(k,r) + T_3'(k,r) + T_4'(k,r)}{T_1(k,r) + T_2(k,r) + T_3(k,r) + T_4(k,r)}\right)^2, \tag{36}$$

Table 1. Parameters for the non-local separable potentials for various partial wave states.

States	α–neutron system		α–proton system	
	λ (MeV fm$^{-(2\ell+1)}$)	β (fm^{-1})	λ (MeV fm$^{-(2\ell+1)}$)	β (fm^{-1})
$1/2^+$	−9.995	1.2	−21.56	1.3
$1/2^-$	−25.28	1.2	−37.28	1.3
$3/2^-$	−36.5	1.2	−76.2	1.4
$3/2^+$	−27.2	1.2	−42.7	1.3
$5/2^+$	−40.2	1.2	−58.7	1.3

$$T_1(k,r) = -\frac{\phi_{\rm N}^{\prime(0)}(k,r)}{A^{(0)}(k,r)} = -\frac{1+C\beta(1-{\rm e}^{-\beta r})}{(1+C_B r\,{\rm e}^{-\beta r})^{1/2}}, \tag{37}$$

$$T_2(k,r) = \left[\frac{\phi_{\rm N}^{(0)}(k,r)}{A^{(0)}(k,r)}\right]\left[\frac{A^{\prime(0)}(k,r)}{A^{(0)}(k,r)}\right] = T_{21}(k,r)T_{22}(k,r), \tag{38}$$

$$T_3(k,r) = \left[\frac{\phi_{\rm N}^{(0)}(k,r)}{A^{(0)}(k,r)}\right]\left[\frac{(\beta{\rm e}^{-\beta r}-\alpha{\rm e}^{-\alpha r})}{({\rm e}^{-\alpha r}-{\rm e}^{-\beta r})}\right] = T_{31}(k,r)T_{32}(k,r), \tag{39}$$

$$T_4(k,r) = \left[\frac{\phi_{N}^{(0)}(k,r)}{A^{(0)}(k,r)}\right]\left[-\frac{1}{2}\frac{C_B{\rm e}^{-\beta r}(1-\beta r)}{(1+C_B r{\rm e}^{-\beta r})}\right] = T_{41}(k,r)T_{42}(k,r), \tag{40}$$

$$T_{21}(k,r) = T_{31}(k,r) = T_{41}(k,r) = \frac{C({\rm e}^{-\beta r}-1)}{(1+C_B r{\rm e}^{-\beta r})^{1/2}}, \tag{41}$$

$$T_{22}(k,r) = \frac{1}{2}\frac{C_B{\rm e}^{-\beta r}(1-\beta r)}{(1+C_B r\,{\rm e}^{-\beta r})}, \tag{42}$$

$$T_{32}(k,r) = \frac{(\beta{\rm e}^{-\beta r}-\alpha{\rm e}^{-\alpha r})}{({\rm e}^{-\alpha r}-{\rm e}^{-\beta r})} \tag{43}$$

and

$$T_{42}(k,r) = -\frac{1}{2}\frac{C_B{\rm e}^{-\beta r}(1-\beta r)}{(1+C_B r{\rm e}^{-\beta r})}. \tag{44}$$

Here $T_i'(k,r)$ $(i=1,2,3,4)$ and $T_i''(k,r)$ $(i=1,2,3,4)$ stand for the single and double derivatives of $T_i(k,r)$ $(i=1,2,3,4)$ with respect to r. The other quantity $V_1(k,r)$ in (35) refers to the p-wave potential [33] and is given by

$$V_1(k,r) = V_0(k,r) + \frac{J''(k,r)}{2J(k,r)} - \frac{1}{2}\left[\frac{J'(k,r)}{J(k,r)}\right]^2 + \frac{\beta^2{\rm e}^{-\beta r}}{({\rm e}^{-\beta r}-1)^2}, \tag{45}$$

where

$$V_0(k,r) = -\frac{J''(k,r)}{2J(k,r)} + \frac{3}{4}\left[\frac{J'(k,r)}{J(k,r)}\right]^2 + \frac{\lambda{\rm e}^{-\beta r}}{(\beta^2+k^2)J(k,r)} \times\left(1+\frac{\lambda}{2\beta(\beta^2+k^2)D(k)}\right) \times(k\sin kr+\beta\cos kr) \tag{46}$$

with

$$J(k,r) = [A(k,r)]^2\,. \tag{47}$$

With the knowledge of the energy-dependent interactions for various partial wave states one will be in a position to compute phase shifts by applying eq. (1).

4. Results and discussion

In the first phase, by applying eq. (6) together with the parameters given in table 1, we have computed elastic scattering phase shifts for s, p and d states for (α−p) and (α−n) systems and portray them in figures 1–4. Here, we have chosen to work with $\hbar^2/2m = 25.92\,{\rm MeV\,fm^2}$, $(2k\eta)^{-1} = 18.12908$ fm and $a = 20$ fm. The same for the (α−n) system is obtained by turning off the Coulomb interaction in the numerical routine for the (α−p) system. In figure 1, it is observed that our phase shift values of the $1/2^+$ state match well with values in [12–14] up to 5 MeV and then deviates slightly as energy increases. For the $1/2^-$ state, we achieve an excellent agreement with the standard data [12–14]; however, for the $3/2^-$ state our phase shifts produce a constant lower value of about 7° than those of [12,13] beyond 3.5 MeV, whereas this variation is about 5° in [14]. Although $\delta^{3/2-}$ differs numerically beyond 3.5 MeV, it traces the correct nature of the phase shifts. For $3/2^+$ and $5/2^+$ states, as shown in figure 2, our phase shift values of the (α−p) system produce correct values up to energy 6 MeV and beyond that they

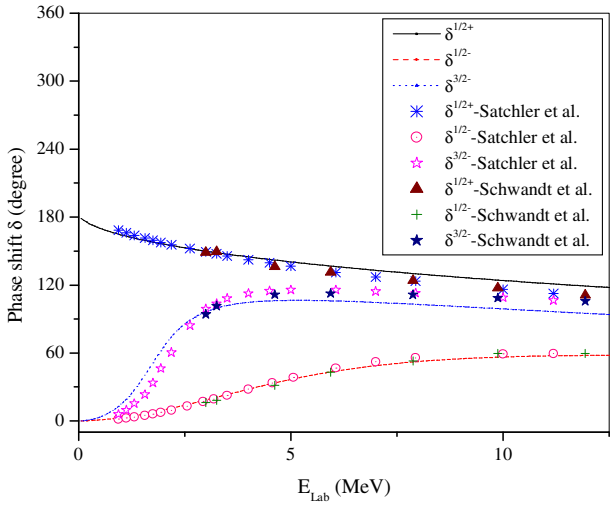

Figure 1. Phase shifts (s and p states) for the (α−p) system (Coulomb-distorted non-local potentials).

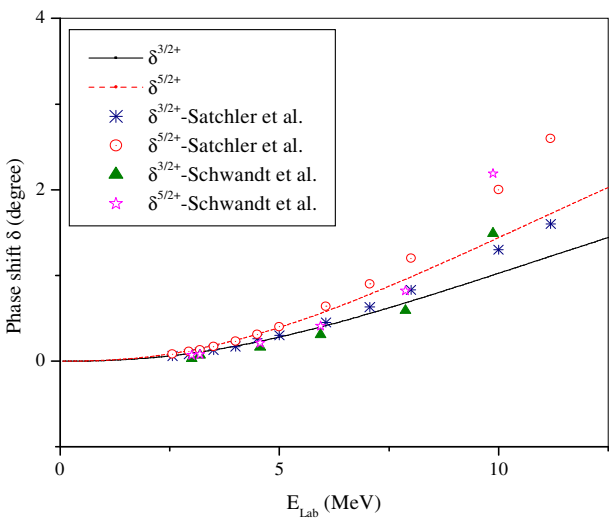

Figure 2. Phase shifts (d-wave) for the (α−p) system (Coulomb-distorted non-local potential).

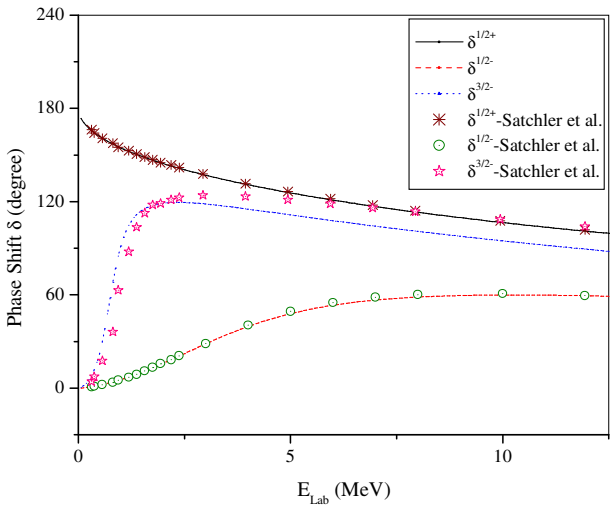

Figure 3. Phase shifts (s and p states) for the (α−n) system (non-local potentials).

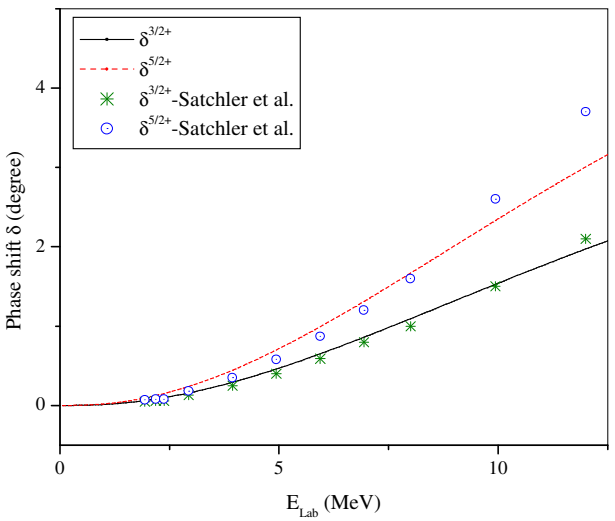

Figure 4. Phase shifts (d-state) for the (α−n) system (non-local potential).

show difference within 0.8° from the standard results [12–14] up to 12 MeV. On the other hand, our phase shift values of the $1/2^+$ and $1/2^-$ states for the (α−n) system as shown in figure 3 are in exact agreement with the standard results [12–14] and the $3/2^-$ state resembles the same state of the (α −p) system except that it differs beyond 2.5 MeV. In figure 4, the phase shifts $\delta^{3/2+}$ are in good agreement with [12–14] over the entire range, whereas $\delta^{5/2+}$ agrees well up to 8 MeV and beyond that they differ slightly (nearly 0.5°) from those of the experimental data. Ahmad *et al* [23] and Rafiqullah *et al* [25] have studied (α−p) elastic scattering with one term separable potential and found good agreement with the experimental values of Schwandt *et al* [14]. However, for non-local separable potential, their form factor differs from us by a constant factor of $2^{-\ell}(\ell!)^{-1}$.

In the second phase, we have computed phase shifts of our energy-dependent potentials for both the systems by applying the phase equation given in eq. (1) together with the parameters of the $1/2^+$ state only of table 1, respectively. Our phase shift values of the $1/2^+$ state with energy-dependent local potential for the (α−p) system shown in figure 5, reproduce fairly correct result except at very low energy (up to 1 MeV), whereas the same for the (α−n) system, represented in figure 6, is in exact agreement with [12–14]. Phase shifts of supersymmetry-generated p-wave potentials for both (α −p) and (α−n) systems, shown in figures 5 and 6, reproduce lower values of phase shifts than the standard data [12–14] except at very low energies. The numerical values of the phase shifts do not match exactly, but are able to produce the correct nature of

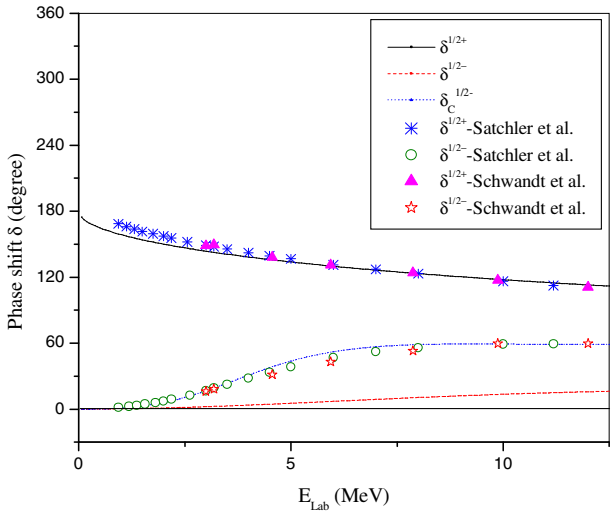

Figure 5. Phase shifts (s and p states) for the (α−p) system (energy-dependent local potentials).

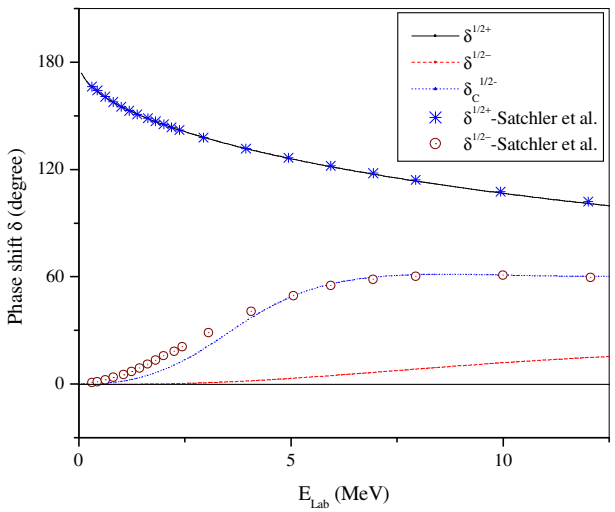

Figure 6. Phase shift (s and p states) for the (α−n) system (energy-dependent local potentials).

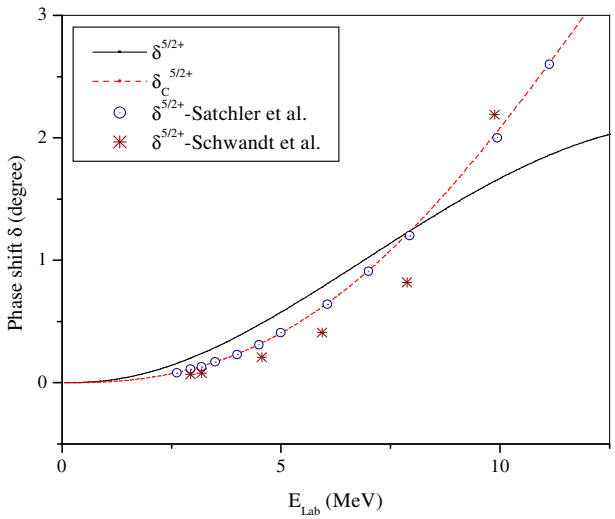

Figure 7. Phase shifts (d-state) for the (α−p) system (energy-dependent local potential).

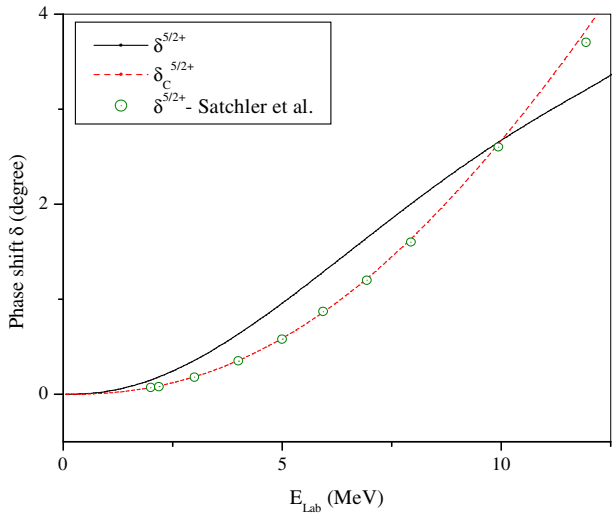

Figure 8. Phase shifts (d-state) for the (α−n) system (energy-dependent local potential).

the phase shift curve. This is due to the addition of a strong centrifugal barrier in the process of generation of p-wave potentials from their s-wave counterparts. It is well known that nuclear potentials are highly state-dependent and higher partial wave potentials cannot be generated by the simple addition of proper centrifugal barrier to its ground state one. Thus, the respective potentials of both (α−p) and (α−n) systems need certain correction to reproduce proper values of phase shifts as the p-wave phase shifts differ significantly except that at very low energies (about zero) an energy-dependent correction factor to the respective potential is incorporated which has a general form

$V^{(i)\mathrm{C}}(k,r) = (E_{\mathrm{Lab}}$ at the match point$-\gamma E_{\mathrm{Lab}}$ variable$)\,\mathrm{e}^{-S\beta r}$; γ and $S-$screening factors. By additing energy-dependent correction factors $V_{\alpha\mathrm{p}}^{(1)\mathrm{C}}(k,r) = -3k^2\mathrm{e}^{-0.17\beta r}$ and $V_{\alpha\mathrm{n}}^{(1)\mathrm{C}}(k,r) = -3k^2\mathrm{e}^{-0.19\beta r}$ with $\gamma = 2.4$ to the supersymmetry generated p-wave potential for (α−p) and (α−n) systems, we achieve close agreement in the phase shift values, hereby denoted as $\delta_{\mathrm{C}}^{1/2-}$, and are also shown in figures 5 and 6, respectively. The relation between the centre of mass energy and the laboratory energy for the systems under consideration is $E_{\mathrm{Lab}} = 1.25E_{\mathrm{cm}}$. For the d-state, the phase shifts $\delta^{5/2+}$ for (α−p) and (α−n) systems match exactly at 8 and 10 MeV, respectively, but differ slightly on either side of these energy values. Thus, correction factors in the respective interactions are indispensable about

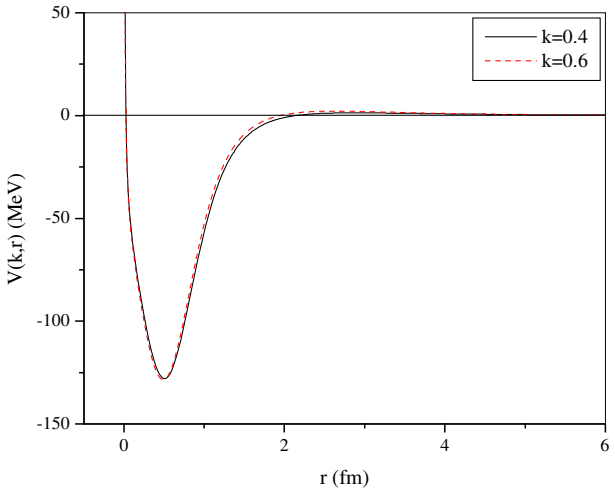

Figure 9. Energy-dependent local potential (s-state) for the (α−p) system.

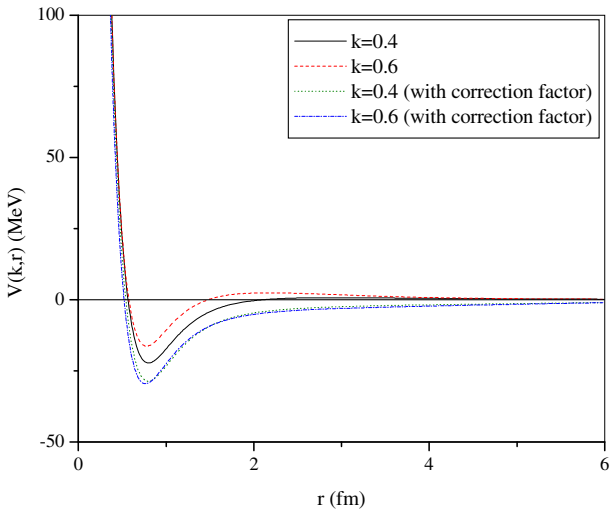

Figure 10. Energy-dependent local potential (p-state) for the (α−p) system.

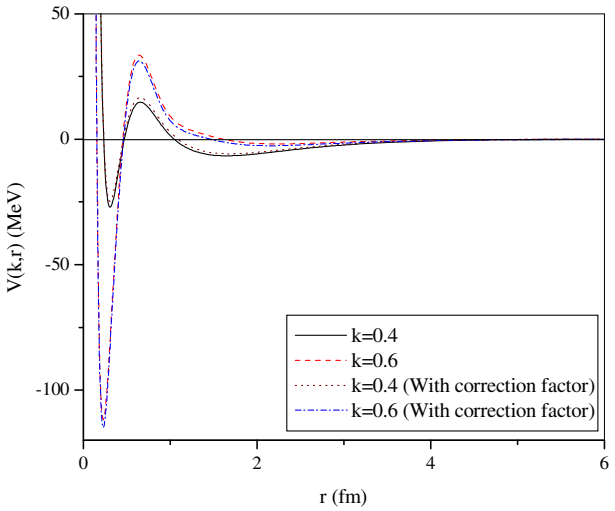

Figure 11. Energy-dependent local potential (d-state) for the (α−p) system.

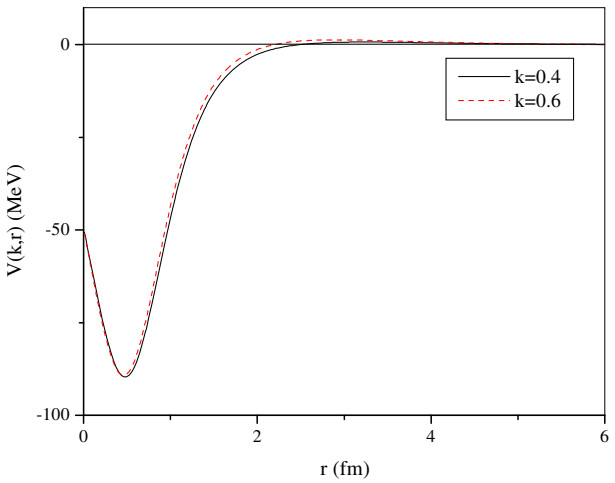

Figure 12. Energy-dependent local potential (s-state) for the (α−n) system.

these points to achieve better agreement in phase shift values. With the following energy-dependent correction factors

$$V_{\alpha p}^{(2)C}(k,r) = -1.25(k^2 - 0.2459)e^{-0.58\beta r}$$

and

$$V_{\alpha n}^{(2)C}(k,r) = -1.25(k^2 - 0.3086)e^{-0.58\beta r}$$

for $\gamma = 1.0$ in the d-wave potential, we have computed and depicted the scattering phase shifts for the $5/2^+$ state, designated as $\delta_C^{5/2+}$, in figures 7 and 8, respectively. These phase shifts are in exact agreement with the standard result [12–14].

Our energy-dependent interactions, both without and with their respective correction factors, are also shown in figures 9–14. It is observed that our energy-dependent potentials for various partial wave states contain varying degrees of repulsion. The s-wave potentials for (α−p) and (α−n) systems, which do not involve any correction factor, are shown in figures 9 and 12 for two different energy values. The s-state (α−n) potential is mostly an attractive potential, whereas the same for the (α−p) system possesses a hard core followed by a strong attractive part. The complete pictures of the p and d-state interactions are shown in figures 10, 11, 13 and 14. The supersymmetry generated p-wave potentials for both (α−n) and (α−p) systems possess repulsive cores followed by a weak attractive part compared with their s-wave counterparts. The strong repulsive cores developed in the respective p-wave interactions are due to the addition of centrifugal barriers as shown in figures 10 and 13. With the addition of correction factors

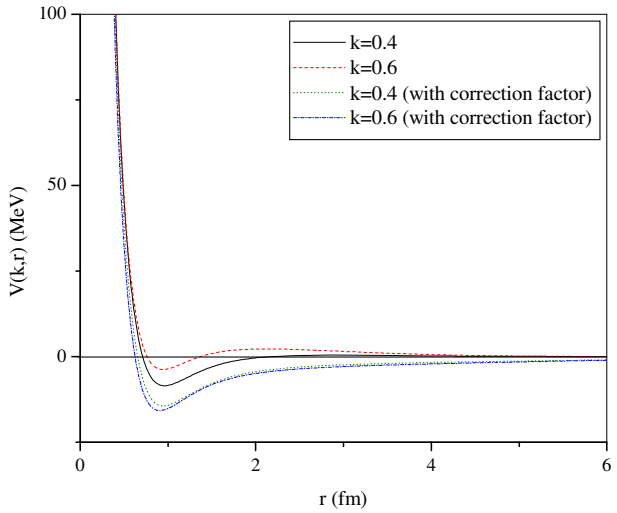

Figure 13. Energy-dependent local potential (p-state) for the (α−n) system.

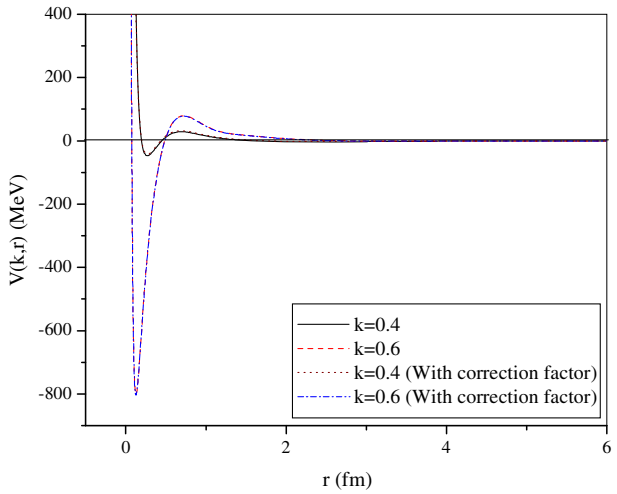

Figure 14. Energy-dependent local potential (d-state) for the (α−n) system.

to the respective potentials, which provide a fairly good agreement with experimental phase shift values [12–14], the depth of the potentials increases to fit proper data but the effective interactions for higher energy values decay slowly in the asymptotic region. This feature for the $\ell = 1$ partial wave is in agreement with that observed by Reichstein and Tang [45] in studying the effective local potential of the (α−n) system obtained from a non-local one in microscopic studies. Reichstein and Tang [45] obtained a purely attractive local potential for $\ell = 1$ at $E = 2$ MeV with very slow decrement in the asymptotic region. Our d-wave interactions obtained from SQM formalism become deeper as energy increases as shown in figures 11 and 14 for the (α−p) and (α−n) systems, respectively. The d-wave potentials with and without correction factors for both the (α−n) and (α−p) systems possess hard cores of 0.15 and 0.2 fm radii, respectively. Both potentials have a weak attractive part beyond 1 fm. The correction factors to the d-wave interactions, however, have no appreciable effect on the interaction as the phase shift values differ within 0.5° over the entire range of energies under consideration. However, too much physical significance should not be attributed to the hard cores due to the fact that these hard cores fall below the regions where the clusters are expected to overlap and at low energies, the tail parts of these potentials play an important role.

5. Conclusion

The α–nucleon system is one of the simplest and useful two-body problems in nuclear physics which can be treated as a good ground for testing various approximations in the microscopic studies of interactions between complicated systems. Such systems have been studied quite extensively by several researchers [5–26] over the last few decades and a large number of phase shift data are obtained. In contrast to our early approach to the problem [1], the present work involves the study of α–nucleon elastic scattering through two different approaches to the problem: (i) through the PFM for the Coulomb-distorted separable non-local potential and (ii) by constructing an energy-dependent phase equivalent local potential of the non-local one. Higher partial wave energy-dependent interactions have been constructed by following the formalism of SQM. Our separable non-local potential can well account the phase shift values for all the s, p and d-states except the $3/2^-$ state beyond 3.5 MeV. However, this variation in our case is within 6% only but more than 10% in the case of Ahmad *et al* [23] from the standard data [12,14]. Our results for non-local interactions are well consistent with those of Cattapan *et al* [24] and Ahmad *et al* [23]. Therefore, our phase method for the Coulomb-distorted separable potential is of immense importance for studying charged particle scattering with the rigorous inclusion of the Coulomb forces.

For our energy-dependent potential, we achieve an excellent agreement in phase shifts for the s-state. The supersymmetry generated p-wave potentials with the addition of energy-dependent correction factors for both the systems provide a good agreement in phase shifts with the standard data [12–14]. In the $5/2^+$ state also small correction factors are required to have an exact agreement with the experimental data [12–14]. The Hamiltonian hierarchy corresponds to the addition of an appropriate centrifugal potential and consequently, higher partial wave interactions are generated fairly accurately in atomic cases. The interest in this subject is

primarily due to the connection between degeneracy and symmetry. For example, the rotational invariance leading to the conservation of angular momentum implies $(2\ell + 1)$ degeneracy of the energy levels. In contrast to the atomic case, the nuclear potentials are strongly state-dependent and separate strength and range parameters are required to describe the interactions in various partial waves. In the present work, we have studied the α–nucleon elastic scattering by energy-dependent potentials in which higher partial wave interactions have been developed from the ground state through the formalism of the SQM. Thus, certain correction factors are indispensable in higher partial wave interactions to achieve good agreement with the experimental data. Recently, Dohet-Eraly and Baye [16] have applied the unitary correlation operator method for the calculation of elastic scattering phase shifts for α–nucleon systems which produce reasonably good agreement with the experimental results with the addition of some correlation factors. In the folding model approach, Chew-Lean Lee and Robson [26] also generated a nucleon–nucleus optical potential with the inclusion of spin–orbit and tensor forces which described s-wave scattering well but failed to reproduce p-states properly. In this context, they applied higher-order corrections for p-waves in the G-matrix concept. In the recent past, Mazur *et al* [46] have studied α–nucleon systems within the no-core shell model using JISP 16 and Daejeon 16 NN interactions within the formalism of the SS-HORSE approach. We have verified that our phase shift results are also in good agreement with those of Mazur *et al* [46]. As it is important to have in the literature an alternative mathematical prescription to the problem, it is our belief that the present methods to study α–nucleon scattering at low energies are of considerable interest to a wide variety of physicists. We conclude by noting that our simple minded potential models which provide a good description of the low-energy nucleon–light nuclei elastic scattering may be extended to other colliding systems involving heavier nuclei.

References

[1] U Laha and J Bhoi, *Phys. Rev. C* **91**, 034614 (2015)
[2] E Witten, *Nucl. Phys. B* **188**, 513 (1981)
[3] F Cooper and D Freedman, *Ann. Phys.* **146**, 262 (1983)
[4] C V Sukumar, *J. Phys. A* **18**, L57 (1985)
[5] H H Hackenbroich, P Heiss and Le-Chi-Niem, *Phys. Lett. B* **51**, 334 (1974)
[6] H Horiuchi, *Prog. Theor. Phys. Suppl.* **62**, 90 (1977)
[7] K Wildermuth and Y C Tang, *A unified theory of the nucleus* (Vieweg, Branschweig, 1977)
[8] D R Thompson, R E Brown, M LeMere and Y C Tang, *Phys. Rev. C* **16**, 1 (1977)
[9] Y C Tang, Microscopic description of the nuclear cluster theory, in: *Topics in nuclear physics II, lecturer notes in physics* (Springer, Berlin, 1981) Vol. 145, pp. 571–692
[10] B Hoop Jr and H H Barschall, *Nucl. Phys.* **83**, 65 (1966)
[11] L Brown, W Haeberli and W Trächslin, *Nucl. Phys. A* **90**, 339 (1967)
[12] G R Satchler, L W Owen, A Jelwin, G L Morgan and R L Walter, *Nucl. Phys. A* **112**, 1 (1968)
[13] G L Morgan and R L Walter, *Phys. Rev.* **168**, 1114 (1968)
[14] P Schwandt, T B Clegg and W Haeberli, *Nucl. Phys. A* **163**, 432 (1971)
[15] R Kamouni and D Baye, *Nucl. Phys. A* **791**, 68 (2007)
[16] J Dohet-Eraly and D Baye, *Phys. Rev. C* **84**, 014604 (2011)
[17] E Van der Spuy, *Nucl. Phys.* **1**, 381 (1956)
[18] J Gammel and R Thaler, *Phys. Rev.* **109**, 2041 (1958)
[19] A N Mitra, V S Bhasin and B S Bhakar, *Nucl. Phys.* **38**, 316 (1962)
[20] J Pigeon, J Barguil, C Fayard, G H Lamot and E El Baz, *Nucl. Phys. A* **145**, 319 (1970)
[21] S Ali, M Rahaman and D Hussain, *Phys. Rev. D* **6**, 1178 (1972)
[22] S Ali, M Rahaman and D Hussain, *Phys. Rev. C* **9**, 1657 (1974)
[23] A A Z Ahmad, S Ali, N Ferdous and M Ahmed, *Nuovo Cimento A* **30**, 385 (1975)
[24] G Cattapan, G Pisent and V Vanzani, *Nucl. Phys. A* **241**, 204 (1975)
[25] A K Rafiquallah, S Hossain, N Chowdhury, A Begum and N Ferdous, *Dacca Univ. Stud. B* **23**(**2**), 5 (1975)
[26] Chew-Lean Lee and D Robson, *Nucl. Phys. A* **379**, 11 (1982)
[27] P Swan, *Proc. R. Soc.* **228**, 10 (1955)
[28] G C Sett, U Laha and B Talukdar, *J. Phys. A* **21**, 3643 (1988)
[29] L G Arnold and A D MacKellar, *Phys. Rev. C* **3**, 1095 (1971)
[30] J Bhoi and U Laha, *J. Phys. G* **40**, 045107 (2013)
[31] U Laha and J Bhoi, *Pramana – J. Phys.* **81**, 959 (2013)
[32] J Bhoi, U Laha and K C Panda, *Pramana – J. Phys.* **82**, 859 (2014)
[33] U Laha and J Bhoi, *Int. J. Mod. Phys. E* **23**, 1450039 (2014)
[34] U Laha and J Bhoi, *Pramana – J. Phys.* **84**, 555 (2015)
[35] F Calogero, *Variable phase approach to potential scattering* (Academic, New York, 1967)
[36] B Talukdar, D Chatterjee and P Banarjee, *J. Phys. G* **3**, 813 (1977)
[37] R G Newton, *Scattering theory of waves and particles*, 2nd edn (Springer-Verlag, New York, 1982)
[38] U Laha and B Talukdar, *Pramana – J. Phys.* **36**, 289 (1991)
[39] L J Slater, *Confluent hypergeometric functions* (Cambridge University Press, New York, 1960)
[40] A Erdeyli, *Higher transcendental functions* (McGraw-Hill, New York, 1953) Vol. 1

[41] W Magnus and F Oberhettinger, *Formulas and theorems for the special functions of mathematical physics* (Chelsea, New York, 1949)
[42] H Buchholz, *The confluent hypergeometric function* (Springer, New York, 1969)
[43] Y Yamaguchi, *Phys. Rev.* **95**, 1628 (1954)
[44] D K Ghosh, S Saha, K Niyogi and B Talukdar, *Czech. J. Phys. B* **33**, 528 (1983)
[45] I Reichstein and Y C Tang, *Nucl. Phys. A* **158**, 529 (1970)
[46] A I Mazur, A M Shirokov, I A Mazur, E A Mazur, Y Kim, I J Shin, L D Blokhintsev and J P Vary, *Proceeding of the International Conference on Nuclear Theory in the Supercomputing Era-2016* (Khabarovsk, Russia, 19–23 September 2016)