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A reliable analytical algorithm for space–time fractional cubic isothermal autocatalytic chemical system

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Abstract. In this paper, we present an algorithm by using the homotopy analysis method (HAM), Adomian decomposition method (ADM) and variational iteration method (VIM) to find the approximate solutions of the space–time fractional cubic isothermal autocatalytic chemical system (STFCIACS). The HAM, ADM and VIM approximate solutions are evaluated and compared by using the computation program Mathematica and excellent results are obtained.

Keywords. Homotopy analysis method; Adomain decomposition method; variational iteration method; space–time fractional cubic isothermal autocatalytic chemical system.

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1. Introduction

Many physics phenomena, such as biology, chemistry, ecology, etc., can be described by reaction–diffusion systems. Merkin *et al* [1] considered the reaction–diffusion travelling waves arising in a coupled system involving simple isothermal autocatalysis kinetics. They assumed that reactions take place in two separate and parallel regions, with, in region I, the reaction is given by quadratic autocatalysis

$$A + B \rightarrow 2B (\text{rate } k_1 ab) \tag{1}$$

together with a linear decay step

$$B \rightarrow C (\text{rate } k_2 b), \tag{2}$$

where a and b are the concentrations of reactant A and autocatalyst B, k_i $(i = 1, 2)$ are the rate constants and C is some inert product of the reaction. The reaction in region II is given by the quadratic autocatalytic step (1) only. The two regions were assumed to be coupled via a linear diffusive interchange of the autocatalytic species B. In this study, we consider a system similar to region I, but with cubic autocatalysis

$$A + 2B \rightarrow 3B (\text{rate } k_3 ab^2) \tag{3}$$

together with a linear decay step

$$B \rightarrow C (\text{rate } k_4 b). \tag{4}$$

This leads to the system of eqs (5)–(8). The following problem on $0 \le x \le L, L > 0$ and $t > 0$ for the dimensionless concentrations (α_1, β_1) in region I and (α_2, β_2) in region II of species A and B is considered:

$$\frac{\partial \alpha_1}{\partial t} = \frac{\partial^2 \alpha_1}{\partial x^2} - \alpha_1 \beta_1^2, \tag{5}$$

$$\frac{\partial \beta_1}{\partial t} = \frac{\partial^2 \beta_1}{\partial x^2} + \alpha_1 \beta_1^2 - k\beta_1 + \gamma(\beta_2 - \beta_1), \tag{6}$$

$$\frac{\partial \alpha_2}{\partial t} = \frac{\partial^2 \alpha_2}{\partial x^2} - \alpha_2 \beta_2^2, \tag{7}$$

$$\frac{\partial \beta_2}{\partial t} = \frac{\partial^2 \beta_2}{\partial x^2} + \alpha_2 \beta_2^2 + \gamma(\beta_1 - \beta_2) \tag{8}$$

with the boundary conditions

$$\alpha_i(0, t) = \alpha_i(L, t) = 1,$$
$$\beta_i(0, t) = \beta_i(L, t) = 0, \quad (i = 1, 2) \tag{9}$$

and the initial conditions

$$\alpha_1(x, 0) = 1 - \sum_{n=1}^{\infty} a_1 \sin\left(\frac{\pi n}{2}\right) \times \cos(0.5\mu_n(L - 2x)), \tag{10}$$

$$\beta_1(x, 0) = \sum_{n=1}^{\infty} b_1 \sin\left(\frac{\pi n}{2}\right)$$

$$\times\ \cos(0.5\mu_n(L-2x))\,, \tag{11}$$

$$\alpha_2(x,0) = 1 - \sum_{n=1}^{\infty} a_2 \sin\left(\frac{\pi n}{2}\right) \times\ \cos(0.5\mu_n(L-2x))\,, \tag{12}$$

$$\beta_2(x,0) = \sum_{n=1}^{\infty} b_2 \sin\left(\frac{\pi n}{2}\right) \times\ \cos(0.5\mu_n(L-2x))\,, \tag{13}$$

where $\mu_n = (n\pi)/L$. The dimensionless constants k and γ represent the strength of the autocatalyst decay and the coupling between the two regions, respectively.

Recently, the fractional diffusion–reaction equations have been studied by several authors [2]. In this paper, we consider space–time fractional cubic isothermal autocatalytic chemical system (STFCIACS) of the form

$$\frac{\partial^\alpha \alpha_1}{\partial t^\alpha} = \frac{\partial^{2\beta}\alpha_1}{\partial x^{2\beta}} - \alpha_1\beta_1^2, \tag{14}$$

$$\frac{\partial^\alpha \beta_1}{\partial t^\alpha} = \frac{\partial^{2\beta}\beta_1}{\partial x^{2\beta}} + \alpha_1\beta_1^2 - k\beta_1 + \gamma(\beta_2-\beta_1), \tag{15}$$

$$\frac{\partial^\alpha \alpha_2}{\partial t^\alpha} = \frac{\partial^{2\beta}\alpha_2}{\partial x^{2\beta}} - \alpha_2\beta_2^2, \tag{16}$$

$$\frac{\partial^\alpha \beta_2}{\partial t^\alpha} = \frac{\partial^{2\beta}\beta_2}{\partial x^{2\beta}} + \alpha_2\beta_2^2 + \gamma(\beta_1-\beta_2), 0 \le \alpha, \beta \le 1. \tag{17}$$

Numerical methods for STFCIACS are quite limited. In this work, we utilise the homotopy analysis method (HAM), Adomian decomposition method (ADM) and variational iteration method (VIM) for finding the analytical approximate solution of STFCIACS.

The HAM was introduced by Lio [3–7]. The HAM is a powerful analytical method for solving the linear and nonlinear fractional differential equations.

This method has attracted the interest of many researchers and has been developed to solve many linear and nonlinear equations. El-Tawil and Huseen [8,9] have suggested an extension of the HAM known as q-homotopy analysis method (q-HAM) to discuss the nonlinear mathematical models. This developed method was then merged with standard integral transform operators to study the nonlinear equations appearing in science and engineering [10–13].

Adomian proposed a new method called the ADM for evaluating the solutions of the nonlinear equations [14–16]. Several authors have investigated the convergence of Adomain's method [17–19].

Recently, it has been proved that the ADM is a very effective method and it could be applied successfully for many problems such as systems of ordinary and partial differential equations and also integral equations [20–26].

The principles of the VIM and their applicability for various kinds of differential equations are given in [22–32]. The aim of this paper is to obtain the approximate analytic solutions of the STFCIACS by HAM, ADM and VIM, and to determine the accuracy of these methods in solving STFCIACS. Also, we will make some comparisons between these methods through finding the approximate solutions. These methods were applied on a variety of fractional time, fractional space, fractional space–time reaction–diffusion equations [33–46].

More recently, Caputo and Fabrizio [47] suggested a new fractional derivative based on the exponential decay law, which is a generalised power law function [48–53]. Abdon Atangana and Dumitru Baleanu introduced a fractional derivative with non-local kernel based on the Mittag–Leffler function (this function is, of course, the more generalised exponential function) and described the complex physical problems that follow the power and exponential decay law at the same time [54–70].

The present paper is organised as follows: §2–5 are devoted to the basic idea of both the fractional calculus and the basic idea of standard HAM, ADM and VIM, respectively. Sections 6–8 are devoted to applying the HAM, ADM and VIM on STFCIACS, respectively. Section 9 is devoted to the numerical results. In the last section, conclusion is presented.

2. Fractional calculus

Here, we give some basic definitions and properties of fractional calculus theory [71–74].

DEFINITION 2.1

If $f(t) \in L_1(a,b)$, the set of all integrable functions and $\alpha > 0$, then the Riemann–Liouville fractional integral of order α, denoted by J_{a+}^{α}, is defined by

$$J_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau. \tag{18}$$

DEFINITION 2.2

For $\alpha > 0$, the Caputo fractional derivative of order α, denoted by ${}^{C}D_{a+}^{\alpha}$, is defined by

$${}^{C}D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} D^n f(\tau)\,\mathrm{d}\tau, \tag{19}$$

where n is such that $n-1 < \alpha < n$ and $D = \mathrm{d}/\mathrm{d}\tau$.

If α is an integer, then this derivative takes the ordinary derivative

$$^{C}D^{\alpha}_{a+} = D^{\alpha}, \quad \alpha = 1, 2, 3, \ldots. \tag{20}$$

Finally, the Caputo fractional derivative on the whole space R is defined by

DEFINITION 2.3

For $\alpha > 0$, the Caputo fractional derivative of order α on the whole space, denoted by $^{C}D^{\alpha}_{a+}$, is defined by

$$^{C}D^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(n-\alpha)} \times \int_{-\infty}^{x} (x-\xi)^{n-\alpha-1} D^{n} f(\xi)\, \mathrm{d}\xi. \tag{21}$$

3. Basic idea of HAM

The principles of the HAM and their applicability for various types of differential equations are given in [3,4,75,76]. Also, new results were obtained in [77–84] using the HAM. For convenience, we shall present a review of the HAM [4]. To describe the basic idea of the standard HAM, we consider the nonlinear differential equation

$$\mathcal{N}[u(t)] = 0, \quad t \geq 0, \tag{22}$$

where $\mathcal{N}$ is a nonlinear differential operator and $u(t)$ is an unknown function. Liao [3] constructed the so-called zeroth-order deformation equation:

$$(1-q)\mathcal{L}[\phi(t;q) - u_0(t)] = qhH(t)\mathcal{N}[\phi(t;q)], \tag{23}$$

where $q \in [0, 1]$ is an embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $\phi(t;q)$ is an unknown function and $u_0(t)$ is an initial guess for $u(t)$, which satisfies the initial conditions. It should be emphasised that one has great freedom in choosing the initial guess $y_0(t)$, $\mathcal{L}$, h and $H(t)$. Obviously, when $q = 0$ and $q = 1$, the following relations hold, respectively:

$$\phi(t;0) = u_0(t), \quad \phi(t;1) = u_1(t).$$

Expanding $\phi(t;q)$ in Taylor series with respect to q, one has

$$\phi(t;q) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) q^m, \tag{24}$$

where

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t;q)}{\partial q^m}\bigg|_{q=0}.$$

The auxiliary parameter h, auxiliary function $H(t)$, initial approximation $u_0(t)$ and auxiliary linear operator $\mathcal{L}$ are selected such that the series (24) converges at $q = 1$, and one has

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t). \tag{25}$$

We can deduce the governing equation from the zero-order deformation equation by defining the vector

$$\vec{u}_n = \{u_0(t), u_1(t), u_2(t), \ldots, u_n(t)\}.$$

Differentiating (23) m times with respect to q, and then setting $q = 0$ and dividing by $m!$, we obtain, using (3), the so-called mth-order deformation equation:

$$\mathcal{L}[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t) R_m(\vec{u}_{m-1}(t)), \quad m = 1, 2, 3, \ldots, n, \tag{26}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\mathcal{N}[\phi(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{27}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1. \\ 1, & m > 1. \end{cases}$$

More detailed analysis of HAM and its modified version with various applications could be found in [13,85–91].

4. Basic idea of the ADM

In this section, we present the basic idea of the ADM [92] by considering the following nonlinear partial differential equation:

$$L(u(x,t)) + R(u(x,t)) + N(u(x,t)) = 0, \tag{28}$$

$$u(x,0) = f(x), \tag{29}$$

where L is the highest-order derivative, which is assumed to be invertible, R is the remaining linear operator and N represents a nonlinear operator. Now, applying the inverse operator L^{-1} to both sides of (28), we obtain

$$u(x,t) = f(x) - L^{-1}(R(u(x,t)) + N(u(x,t)). \tag{30}$$

Let

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) \tag{31}$$

and

$$N(u) = \sum_{m=0}^{\infty} A_m, \tag{32}$$

where A_m are Adomin polynomials which depend upon u. In view of eqs (31) and (32), eq. (30) takes the form

$$\sum_{m=0}^{\infty} u_m(x,t) = f(x) - L^{-1}(R(u(x,t)) + \sum_{m=0}^{\infty} A_m(u(x,t)). \quad (33)$$

We set

$$u_0(x,t) = f(x), \quad (34)$$

$$u_{m+1}(x,t) = -L^{-1}\left(R(u(x,t)) + \sum_{m=0}^{\infty} A_m(u(x,t)\right), \quad m = 0, 1, \ldots, \quad (35)$$

where

$$A_m(u(x,t)) = \left[\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} N\left(\sum_{i=0}^{\infty} u_i(x,t)\lambda^i\right)\right]_{\lambda=0}. \quad (36)$$

Hence, (34)–(36) lead to the following recurrence relations:

$$u_0(x,0) = f(x),$$
$$u_{m+1}(x,t) = -L^{-1}\left(R(u(x,t)) + A_m(u(x,t))\right). \quad (37)$$

The solution $u(x,t)$ can be approximated by the truncated series

$$\phi_k(x,t) = \sum_{m=0}^{k-1} u_m(x,t), \quad \lim_{k\to\infty} \phi_k = u(x,t).$$

5. Basic ideas of the VIM

In order to introduce the VIM, let us consider the following differential equation:

$$Lu(x,t) + Nu(x,t) = g(x,t), \quad (38)$$

where L is a linear operator, N is a nonlinear operator and $g(x,t)$ is a source term. According to the VIM, we construct the correction functional in the t direction as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left(Lu_n(x,\tau) + N\tilde{u}_n(x,t) - g(x,\tau)\right) \mathrm{d}\tau, \quad (39)$$

where λ is a general Lagrangian multiplier [27–29], which can be determined optimally through the variational theory. The subscript n denotes the nth-order approximation, whereas it is considered to be a restricted variation [27–29], i.e. $\delta\tilde{u}_n(x,t) = 0$.

6. HAM solutions of STFCIACS

In this section, we apply the HAM on STFCIACS. The HAM is based on a kind of continuous mapping

$$\alpha_1(x,t) \to \phi_1(x,t;q), \quad \beta_1(x,t) \to \psi_1(x,t;q),$$
$$\alpha_2(x,t) \to \phi_2(x,t;q), \quad \beta_2(x,t) \to \psi_2(x,t;q),$$

such that, as the embedding parameter q increases from 0 to 1, $\phi_1(x,t;q)$, $\psi_1(x,t;q)$, $\phi_2(x,t;q)$, $\psi_2(x,t;q)$ vary from the initial approximation to the exact solution. We define the nonlinear operators

$$\mathcal{N}_1(\phi_1(x,t;q)) = \phi_{1,t}(x,t;q) - \phi_{1,xx}(x,t;q) + \phi_1(x,t;q)\psi_1^2(x,t;q),$$
$$\mathcal{N}_2(\psi_1(x,t;q)) = \psi_{1,t}(x,t;q) - \psi_{1,xx}(x,t;q) + k\psi_1(x,t;q) - \gamma(\psi_2(x,t;q) - \psi_1(x,t;q)) - \phi_1(x,t;q) \times \psi_1^2(x,t;q),$$
$$\mathcal{N}_3(\phi_2(x,t;q)) = \phi_{2,t}(x,t;q) - \phi_{2,xx}(x,t;q) + \phi_2(x,t;q)\psi_2^2(x,t;q),$$
$$\mathcal{N}_4(\psi_2(x,t;q)) = \psi_{2,t}(x,t;q) - \psi_{2,xx}(x,t;q) - \gamma(\psi_1(x,t;q) - \psi_2(x,t;q)) - \phi_2(x,t;q)\psi_2^2(x,t;q).$$

Now, we construct a set of equations, using the embedding parameter q:

$$(1-q)\mathcal{L}_1(\phi_1(x,t;q) - \alpha_{1,0}(x,t)) = qhH(x,t)\mathcal{N}_1(\phi_1(x,t;q)),$$
$$(1-q)\mathcal{L}_2(\psi_1(x,t;q) - \beta_{1,0}(x,t)) = qhH(x,t)\mathcal{M}_1(\psi_1(x,t;q)),$$
$$(1-q)\mathcal{L}_3(\phi_2(x,t;q) - \alpha_{2,0}(x,t)) = qhH(x,t)\mathcal{N}_2(\phi_2(x,t;q)),$$
$$(1-q)\mathcal{L}_4(\psi_2(x,t;q) - \beta_{2,0}(x,t)) = qhH(x,t)\mathcal{M}_2(\psi_2(x,t;q)),$$

with the initial conditions

$$\phi_1(x,0;q) = \alpha_{1,0}(x,0),$$
$$\psi_1(x,0;q) = \beta_{1,0}(x,0),$$
$$\phi_2(x,0;q) = \alpha_{2,0}(x,0),$$
$$\psi_2(x,0;q) = \beta_{2,0}(x,0),$$

where $h \neq 0$ and $H(x,t) \neq 0$ are the auxiliary parameter and auxiliary function, respectively. We expand $\phi_1(x,t;q)$, $\psi_1(x,t;q)$, $\phi_2(x,t;q)$ and $\psi_2(x,t;q)$ in a Taylor series with respect to q and obtain

$$\phi_1(x,t;q) = \alpha_{1,0}(x,t) + \sum_{m=1}^{\infty} \alpha_{1,m}(x,t) q^m, \tag{40}$$

$$\psi_1(x,t;q) = \beta_{1,0}(x,t) + \sum_{m=1}^{\infty} \beta_{1,m}(x,t) q^m, \tag{41}$$

$$\phi_2(x,t;q) = \alpha_{2,0}(x,t) + \sum_{m=1}^{\infty} \alpha_{2,m}(x,t) q^m, \tag{42}$$

$$\psi_2(x,t;q) = \beta_{2,0}(x,t) + \sum_{m=1}^{\infty} \beta_{2,m}(x,t) q^m, \tag{43}$$

where

$$\alpha_{1,m}(x,t) = \frac{1}{m!}\frac{\partial^m \phi_1(x,t;q)}{\partial q^m}\bigg|_{q=0},$$
$$\beta_{1,m}(x,t) = \frac{1}{m!}\frac{\partial^m \psi_1(x,t;q)}{\partial q^m}\bigg|_{q=0},$$
$$\alpha_{2,m}(x,t) = \frac{1}{m!}\frac{\partial^m \phi_2(x,t;q)}{\partial q^m}\bigg|_{q=0},$$
$$\beta_{2,m}(x,t) = \frac{1}{m!}\frac{\partial^m \psi_2(x,t;q)}{\partial q^m}\bigg|_{q=0}.$$

If $q = 1$ in (40)–(43), the series become

$$\alpha_1(x,t) = \alpha_{1,0}(x,t) + \sum_{m=1}^{\infty} \alpha_{1,m}(x,t),$$
$$\beta_1(x,t) = \beta_{1,0}(x,t) + \sum_{m=1}^{\infty} \beta_{1,m}(x,t),$$
$$\alpha_2(x,t) = \alpha_{2,0}(x,t) + \sum_{m=1}^{\infty} \alpha_{2,m}(x,t),$$
$$\beta_2(x,t) = \beta_{2,0}(x,t) + \sum_{m=1}^{\infty} \beta_{2,m}(x,t).$$

Now, we construct the mth-order deformation equation from (26) to (27) as follows:

$$\mathcal{L}_1(\alpha_{1,m}(x,t) - \mathcal{X}_m \alpha_{1,m-1}(x,t)) = hH(x,t) R_1((\vec{\alpha}_{1,m-1}, \vec{\beta}_{1,m-1})),$$
$$\mathcal{L}_2(\beta_{1,m}(x,t) - \mathcal{X}_m \beta_{1,m-1}(x,t)) = hH(x,t) R_2((\vec{\alpha}_{1,m-1}, \vec{\beta}_{1,m-1})),$$
$$\mathcal{L}_3(\alpha_{2,m}(x,t) - \mathcal{X}_m \alpha_{2,m-1}(x,t)) = hH(x,t) R_3((\vec{\alpha}_{2,m-1}, \vec{\beta}_{2,m-1})),$$
$$\mathcal{L}_4(\beta_{2,m}(x,t) - \mathcal{X}_m \beta_{2,m-1}(x,t)) = hH(x,t) R_4((\vec{\alpha}_{2,m-1}, \vec{\beta}_{2,m-1}))$$

with the initial conditions $\alpha_{1,m}(x,0) = 0,\ \beta_{1,m}(x,0) = 0,\ \alpha_{2,m}(x,0) = 0,\ \beta_{2,m}(x,0) = 0, m > 1$, where

$$R_1((\vec{\alpha}_{1,m-1}, \vec{\beta}_{1,m-1})) = \frac{\partial^\alpha \alpha_{1,m-1}}{\partial t^\alpha} - \frac{\partial^{2\beta} \alpha_{1,m-1}}{\partial x^{2\beta}} + \sum_{i=0}^{m-1}\sum_{j=0}^{i} \alpha_{1,m-1-i}\beta_{1,j}\beta_{1,i-j},$$

$$R_2((\vec{\alpha}_{1,m-1}, \vec{\beta}_{1,m-1})) = \frac{\partial^\alpha \beta_{1,m-1}}{\partial t^\alpha} - \frac{\partial^{2\beta} \beta_{1,m-1}}{\partial x^{2\beta}} + k\beta_{1,m-1} - \gamma(\beta_{2,m-1} - \beta_{1,m-1}) - \sum_{i=0}^{m-1}\sum_{j=0}^{i} \alpha_{1,m-1-i}\beta_{1,j}\beta_{1,i-j},$$

$$R_3((\vec{\alpha}_{2,m-1}, \vec{\beta}_{2,m-1})) = \frac{\partial^\alpha \alpha_{2,m-1}}{\partial t^\alpha} - \frac{\partial^{2\beta} \alpha_{2,m-1}}{\partial x^{2\beta}} + \sum_{i=0}^{m-1}\sum_{j=0}^{i} \alpha_{2,m-1-i}\beta_{2,j}\beta_{2,i-j},$$

$$R_4((\vec{\alpha}_{2,m-1}, \vec{\beta}_{2,m-1})) = \frac{\partial^\alpha \beta_{2,m-1}}{\partial t^\alpha} - \frac{\partial^{2\beta} \beta_{2,m-1}}{\partial x^{2\beta}} - \gamma(\beta_{1,m-1} - \beta_{2,m-1}) - \sum_{i=0}^{m-1}\sum_{j=0}^{i} \alpha_{2,m-1-i}\beta_{2,j}\beta_{2,i-j}.$$

If we take $\mathcal{L}_i = \mathrm{d}^\alpha/\mathrm{d}t^\alpha$ $(i = 1,2,3,4)$ then the right inverse of $\mathcal{L}_i = \mathrm{d}^\alpha/\mathrm{d}t^\alpha$ will be J_t^α

$$\alpha_{1,m} = \mathcal{X}_m \alpha_{1,m-1} + hJ_t^\alpha\left(\frac{\partial^\alpha \alpha_{1,m-1}}{\partial t^\alpha} - \frac{\partial^{2\beta} \alpha_{1,m-1}}{\partial x^{2\beta}} + \sum_{i=0}^{m-1}\sum_{j=0}^{i} \alpha_{1,m-1-i}\beta_{1,j}\beta_{1,i-j}\right), \tag{44}$$

$$\beta_{1,m} = \mathcal{X}_m \beta_{1,m-1} + hJ_t^\alpha\left(\frac{\partial^\alpha \beta_{1,m-1}}{\partial t^\alpha} - \frac{\partial^{2\beta} \beta_{1,m-1}}{\partial x^{2\beta}} + k\beta_{1,m-1}\right) + hJ_t^\alpha\left(-\gamma(\beta_{2,m-1} - \beta_{1,m-1}) - \sum_{i=0}^{m-1}\sum_{j=0}^{i} \alpha_{1,m-1-i}\beta_{1,j}\beta_{1,i-j}\right), \tag{45}$$

$$\alpha_{2,m} = \mathcal{X}_m\alpha_{2,m-1} + hJ_t^\alpha\left(\frac{\partial^\alpha\alpha_{2,m-1}}{\partial t^\alpha} - \frac{\partial^{2\beta}\alpha_{2,m-1}}{\partial x^{2\beta}} + \sum_{i=0}^{m-1}\sum_{j=0}^{i}\alpha_{2,m-1-i}\beta_{2,j}\beta_{2,i-j}\right), \quad (46)$$

$$\beta_{2,m} = \mathcal{X}_m\beta_{2,m-1} + hJ_t^\alpha\left(\frac{\partial^\alpha\beta_{2,m-1}}{\partial t^\alpha} - \frac{\partial^{2\beta}\beta_{2,m-1}}{\partial x^{2\beta}}\right) + hJ_t^\alpha\left(-\gamma(\beta_{1,m-1} - \beta_{2,m-1}) - \sum_{i=0}^{m-1}\sum_{j=0}^{i}\alpha_{2,m-1-i}\beta_{2,j}\beta_{2,i-j}\right). \quad (47)$$

We choose the initial approximation

$$\alpha_{1,0}(x,t) = \alpha_{1,0}(x,0), \quad \beta_{1,0}(x,t) = \beta_{1,0}(x,0), \quad (48)$$

$$\alpha_{2,0}(x,t) = \alpha_{2,0}(x,0), \quad \beta_{2,0}(x,t) = \beta_{2,0}(x,0). \quad (49)$$

For $m = 1$, we obtain the first approximation as follows:

$$\alpha_{1,1} = hJ_t^\alpha\left(\frac{\partial^\alpha\alpha_{1,0}}{\partial t^\alpha} - \frac{\partial^{2\beta}\alpha_{1,0}}{\partial x^{2\beta}} + \alpha_{1,0}\beta_{1,0}^2\right), \quad (50)$$

$$\beta_{1,1} = hJ_t^\alpha\left(\frac{\partial^\alpha\beta_{1,0}}{\partial t^\alpha} - \frac{\partial^{2\beta}\beta_{1,0}}{\partial x^{2\beta}} + k\beta_{1,0} - \gamma(\beta_{2,0} - \beta_{1,0}) - \alpha_{1,0}\beta_{1,0}^2\right), \quad (51)$$

$$\alpha_{2,1} = hJ_t^\alpha\left(\frac{\partial^\alpha\alpha_{2,0}}{\partial t^\alpha} - \frac{\partial^{2\beta}\alpha_{2,0}}{\partial x^{2\beta}} + \alpha_{2,0}\beta_{2,0}^2\right), \quad (52)$$

$$\beta_{2,1} = hJ_t^\alpha\left(\frac{\partial^\alpha\beta_{2,0}}{\partial t^\alpha} - \frac{\partial^{2\beta}\beta_{2,0}}{\partial x^{2\beta}} - \gamma(\beta_{1,0} - \beta_{2,0}) - \alpha_{2,0}\beta_{2,0}^2\right). \quad (53)$$

7. ADM solutions of STFCIACS

In this section, we apply the ADM to evaluate the approximate solutions of (5)–(8). If we insert J^α on both sides of (14)–(17), we obtain

$$\alpha_1(x,t) = \alpha_1(x,0) + J_t^\alpha\left(\frac{\partial^{2\beta}\alpha_1}{\partial x^{2\beta}} - \alpha_1\beta_1^2\right), \quad (54)$$

$$\beta_1(x,t) = \beta_1(x,0) + J_t^\alpha\left(\frac{\partial^{2\beta}\beta_1}{\partial x^{2\beta}} + \alpha_1\beta_1^2 - k\beta_1 + \gamma(\beta_2 - \beta_1)\right), \quad (55)$$

$$\alpha_2(x,t) = \alpha_2(x,0) + J_t^\alpha\left(\frac{\partial^{2\beta}\alpha_2}{\partial x^{2\beta}} - \alpha_2\beta_2^2\right), \quad (56)$$

$$\beta_2(x,t) = \beta_2(x,0) + J_t^\alpha\left(\frac{\partial^{2\beta}\beta_2}{\partial x^{2\beta}} + \alpha_2\beta_2^2 + \gamma(\beta_1 - \beta_2)\right), \quad (57)$$

where

$$J_t^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(x,\tau)\,\mathrm{d}\tau, \quad (58)$$

$$D_x^\beta u(x,t) = \frac{\partial^\beta u}{\partial x^\beta} = \begin{cases} \dfrac{1}{\Gamma(2-\beta)}\displaystyle\int_{-\infty}^{x}\frac{\partial^2 u(\xi,\tau)}{\partial\xi^2}\frac{\mathrm{d}\xi}{(x-\xi)^{\beta-1}}, & 1<\beta<2,\\ \dfrac{\partial^2 u(x,t)}{\partial x^2}, & \beta = 2. \end{cases} \quad (59)$$

Now the ADM solutions and nonlinear functions $N_1(\alpha_1, \beta_1)$ and $N_2(\alpha_2, \beta_2)$ can be presented as an infinite series

$$\alpha_1(x,t) = \alpha_{1,0}(x,t) + \sum_{m=1}^{\infty}\alpha_{1,m}(x,t), \quad (60)$$

$$\beta_1(x,t) = \beta_{1,0}(x,t) + \sum_{m=1}^{\infty}\beta_{1,m}(x,t), \quad (61)$$

$$\alpha_2(x,t) = \alpha_{2,0}(x,t) + \sum_{m=1}^{\infty}\alpha_{2,m}(x,t), \quad (62)$$

$$\beta_2(x,t) = \beta_{2,0}(x,t) + \sum_{m=1}^{\infty}\beta_{2,m}(x,t) \quad (63)$$

and

$$N_1(\alpha_1,\beta_1) = \alpha_1\beta_1^2 = \sum_{m=0}^{\infty}A_m, \quad (64)$$

$$N_2(\alpha_2,\beta_2) = \alpha_2\beta_2^2 = \sum_{m=0}^{\infty}B_m, \quad (65)$$

where

$$A_m = \frac{1}{m!}\left[\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}N_1(\alpha_1,\beta_1)\right]_{\lambda=0}, \quad (66)$$

$$B_m = \frac{1}{m!}\left[\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}N_2(\alpha_2,\beta_2)\right]_{\lambda=0}. \quad (67)$$

A_m are called the Adomian polynomials and the components $\alpha_{1,m}(x,t)$ and $\beta_{1,m}(x,t)$ of the solutions $\alpha_1(x,t)$ and $\beta_1(x,t)$ will be determined by the following recurrence relations:

$$\alpha_{1,0}=\alpha_1(x,0),$$
$$\alpha_{1,m+1}=J_t^{\alpha}\left(\frac{\partial^{2\beta}\alpha_{1,m}}{\partial x^{2\beta}}-A_m\right), \quad (68)$$
$$\beta_{1,0}=\beta_1(x,0),$$
$$\beta_{1,m+1}=J_t^{\alpha}\left(\frac{\partial^{2\beta}\beta_{1,m}}{\partial x^{2\beta}}-k\beta_{1,m}+\gamma(\beta_{2,m}-\beta_{1,m})+A_m\right). \quad (69)$$

B_m are called the Adomian polynomials and the components $\alpha_{2,m}(x,t)$ and $\beta_{2,m}(x,t)$ of the solutions $\alpha_2(x,t)$ and $\beta_2(x,t)$ will be determined by the following recurrence relations:

$$\alpha_{2,0}=\alpha_2(x,0),$$
$$\alpha_{2,m+1}=J_t^{\alpha}\left(\frac{\partial^{2\beta}\alpha_{2,m}}{\partial x^{2\beta}}-B_m\right), \quad (70)$$
$$\beta_{2,0}=\beta_2(x,0),$$
$$\beta_{2,m+1}=J_t^{\alpha}\left(\frac{\partial^{2\beta}\beta_{2,m}}{\partial x^{2\beta}}+\gamma(\beta_{1,m}-\beta_{2,m})+B_m\right). \quad (71)$$

In view of (36) and using Mathematica software, we evaluate the Adomian polynomials A_m and B_m. They are as follows:

$$A_0=\alpha_{1,0}\beta_{1,0}^2,$$
$$A_1=\alpha_{1,1}\beta_{1,0}^2+2\alpha_{1,0}\beta_{1,0}\beta_{1,1},$$
$$A_2=\alpha_{1,2}\beta_{1,0}^2+2\alpha_{1,1}\beta_{1,0}\beta_{1,1}+\frac{1}{2}\alpha_{1,0}(2\beta_{1,1}^2+4\beta_{1,0}\beta_{1,2}), \quad (72)$$
$$B_0=\alpha_{2,0}\beta_{2,0}^2,$$
$$B_1=\alpha_{2,1}\beta_{2,0}^2+2\alpha_{2,0}\beta_{2,0}\beta_{2,1},$$
$$B_2=\alpha_{2,2}\beta_{2,0}^2+2\alpha_{2,1}\beta_{2,0}\beta_{2,1}+\frac{1}{2}\alpha_{2,0}(2\beta_{2,1}^2+4\beta_{2,0}\beta_{2,2}). \quad (73)$$

In the first iteration, we have

$$\alpha_{1,1}=J_t^{\alpha}\left(\frac{\partial^{2\beta}\alpha_{1,0}}{\partial x^{2\beta}}-A_0\right), \quad (74)$$

$$\beta_{1,1}=J_t^{\alpha}\left(\frac{2\partial^{\beta}\beta_{1,0}}{\partial x^{2\beta}}-k\beta_{1,0}+\gamma(\beta_{2,m}-\beta_{1,0})+A_0\right), \quad (75)$$

$$\alpha_{2,1}=J_t^{\alpha}\left(\frac{\partial^{2\beta}\alpha_{2,0}}{\partial x^{2\beta}}-B_0\right), \quad (76)$$

$$\beta_{2,1}=J_t^{\alpha}\left(\frac{2\partial^{\beta}\beta_{2,0}}{\partial x^{2\beta}}+\gamma(\beta_{1,0}-\beta_{2,0})+B_0\right). \quad (77)$$

The components $\alpha_{1,2},\ldots,\beta_{1,2},\ldots,\alpha_{2,2},\ldots,\beta_{2,2},\ldots$ will be computed as well as used, but for brevity are not listed. The general form of the approximations $\alpha_1,\beta_1,\alpha_2,\beta_2$ are given by (60)–(67), i.e.

$$\alpha_1=\alpha_{1,0}+\alpha_{1,1}+\alpha_{1,2}+\cdots, \quad (78)$$
$$\beta_1=\beta_{1,0}+\beta_{1,1}+\beta_{1,2}+\cdots, \quad (79)$$
$$\alpha_2=\alpha_{2,0}+\alpha_{2,1}+\alpha_{2,2}+\cdots, \quad (80)$$
$$\beta_2=\beta_{2,0}+\beta_{2,1}+\beta_{2,2}+\cdots. \quad (81)$$

8. VIM solutions of STFCIACS

In this section, we apply the VIM to evaluate the approximate solutions of (14)–(17). We can approximate the correction formula of (14)–(17) as follows:

$$\alpha_{1,n+1}(x,t)=\alpha_{1,n}(x,t)+\int_0^t\mu_1(\tau)\left(\frac{\partial}{\partial\tau}\alpha_{1,n}(x,\tau)-\frac{\partial^2}{\partial x^2}\tilde{\alpha}_{1,n}(x,\tau)+\tilde{\alpha}_{1,n}(x,\tau)\tilde{\beta}_{1,n}^2(x,\tau)\right)\mathrm{d}\tau, \quad (82)$$

$$\beta_{1,n+1}(x,t)=\beta_{1,n}(x,t)+\int_0^t\mu_2(\tau)\left(\frac{\partial}{\partial\tau}\beta_{1,n}(x,\tau)-\frac{\partial^2}{\partial x^2}\tilde{\beta}_{1,n}(x,\tau)-\tilde{\alpha}_{1,n}(x,\tau)\tilde{\beta}_{1,n}^2(x,\tau)\,\mathrm{d}\tau+k\tilde{\beta}_{1,n}(x,\tau)+\gamma(\tilde{\beta}_{1,n}(x,t)-\tilde{\beta}_{2,n}(x,\tau))\right)\mathrm{d}\tau, \quad (83)$$

$$\alpha_{2,n+1}(x,t)=\alpha_{2,n}(x,t)+\int_0^t\mu_3(\tau)\left(\frac{\partial}{\partial\tau}\alpha_{2,n}(x,\tau)-\frac{\partial^2}{\partial x^2}\tilde{\alpha}_{2,n}(x,\tau)+\tilde{\alpha}_{2,n}(x,\tau)\tilde{\beta}_{2,n}^2(x,\tau)\right)\mathrm{d}\tau, \quad (84)$$

$$\beta_{2,n+1}(x,t)=\beta_{2,n}(x,t)+\int_0^t\mu_4(\tau)\left(\frac{\partial}{\partial\tau}\beta_{2,n}(x,\tau)-\frac{\partial^2}{\partial x^2}\tilde{\beta}_{2,n}(x,\tau)-\tilde{\alpha}_{2,n}(x,\tau)\tilde{\beta}_{2,n}^2(x,\tau)+\gamma(\tilde{\beta}_{2,n}(x,t)-\tilde{\beta}_{1,n}(x,\tau))\right)\mathrm{d}\tau, \quad (85)$$

where $\tilde{\alpha}_{1,n}(x,\tau)$, $\tilde{\beta}_{1,n}(x,\tau)$, $\tilde{\alpha}_{2,n}(x,\tau)$ and $\tilde{\beta}_{2,n}(x,\tau)$ denote the restrictive variation, i.e.,

$$\delta\tilde{\alpha}_{1,n}(x,\tau)=0,\ \delta\tilde{\beta}_{1,n}(x,\tau)=0,$$

$$\delta\tilde{\alpha}_{2,n}(x,\tau)=0,\ \delta\tilde{\beta}_{2,n}(x,\tau)=0.$$

Thus, we have

$$\begin{aligned}\delta\alpha_{1,n+1}(x,t)&=\delta\alpha_{1,n}(x,t)\\&\quad+\int_0^t\delta\mu_1(\tau)\bigg(\frac{\partial}{\partial\tau}\alpha_{1,n}(x,\tau)\\&\quad-\frac{\partial^2}{\partial x^2}\tilde{\alpha}_{1,n}(x,\tau)+\tilde{\alpha}_{1,n}(x,\tau)\tilde{\beta}^2_{1,n}(x,\tau)\bigg)\,\mathrm{d}\tau\\&=\delta\alpha_{1,n}(x,t)+\int_0^t\delta\mu_1(\tau)\bigg(\frac{\partial}{\partial\tau}\alpha_{1,n}(x,\tau)\bigg)\,\mathrm{d}\tau,\end{aligned}\tag{86}$$

$$\begin{aligned}\delta\beta_{1,n+1}(x,t)&=\delta\beta_{1,n}(x,t)\\&\quad+\int_0^t\delta\mu_2(\tau)\bigg(\frac{\partial}{\partial\tau}\beta_{1,n}(x,\tau)\\&\quad-\frac{\partial^2}{\partial x^2}\tilde{\beta}_{1,n}(x,\tau)-\tilde{\alpha}_{1,n}(x,\tau)\tilde{\beta}^2_{1,n}(x,\tau)\,\mathrm{d}\tau\\&\quad+k\tilde{\beta}_{1,n}(x,\tau)+\gamma(\tilde{\beta}_{1,n}(x,t)-\tilde{\beta}_{2,n}(x,\tau))\bigg)\,\mathrm{d}\tau\\&=\delta\beta_{1,n}(x,t)+\int_0^t\delta\mu_2(\tau)\bigg(\frac{\partial}{\partial\tau}\beta_{1,n}(x,\tau)\bigg)\,\mathrm{d}\tau,\end{aligned}\tag{87}$$

$$\begin{aligned}\delta\alpha_{2,n+1}(x,t)&=\delta\alpha_{2,n}(x,t)\\&\quad+\int_0^t\delta\mu_3(\tau)\bigg(\frac{\partial}{\partial\tau}\alpha_{2,n}(x,\tau)\\&\quad-\frac{\partial^2}{\partial x^2}\tilde{\alpha}_{2,n}(x,\tau)+\tilde{\alpha}_{2,n}(x,\tau)\tilde{\beta}^2_{2,n}(x,\tau)\bigg)\,\mathrm{d}\tau\\&=\delta\alpha_{2,n}(x,t)+\int_0^t\delta\mu_3(\tau)\frac{\partial}{\partial\tau}\alpha_{2,n}(x,\tau)\,\mathrm{d}\tau,\end{aligned}\tag{88}$$

$$\begin{aligned}\delta\beta_{2,n+1}(x,t)&=\delta\beta_{2,n}(x,t)\\&\quad+\int_0^t\delta\mu_4(\tau)\bigg(\frac{\partial}{\partial\tau}\beta_{2,n}(x,\tau)-\frac{\partial^2}{\partial x^2}\tilde{\beta}_{2,n}(x,\tau)\\&\quad-\tilde{\alpha}_{2,n}(x,\tau)\tilde{\beta}^2_{2,n}(x,\tau)\\&\quad+\gamma(\tilde{\beta}_{2,n}(x,t)-\tilde{\beta}_{1,n}(x,\tau))\bigg)\,\mathrm{d}\tau\\&=\delta\beta_{2,n}(x,t)+\int_0^t\delta\mu_4(\tau)\frac{\partial}{\partial\tau}\beta_{2,n}(x,\tau)\,\mathrm{d}\tau.\end{aligned}\tag{89}$$

Integrating by parts, we obtain the Lagrange multipliers: $\mu_1(\tau)=\mu_2(\tau)=\mu_3(\tau)=\mu_4(\tau)=-1$. As a consequence, we obtain the following iterations formulae:

$$\begin{aligned}\alpha_{1,n+1}(x,t)&=\alpha_{1,n}(x,t)-\int_0^t\bigg(\frac{\partial^\alpha}{\partial\tau^\alpha}\alpha_{1,n}(x,\tau)\\&\quad-\frac{\partial^{2\beta}}{\partial x^{2\beta}}\alpha_{1,n}(x,\tau)+\alpha_{1,n}(x,\tau)\beta^2_{1,n}(x,\tau)\bigg)\,\mathrm{d}\tau,\end{aligned}\tag{90}$$

$$\begin{aligned}\beta_{1,n+1}(x,t)&=\beta_{1,n}(x,t)-\int_0^t\bigg(\frac{\partial^\alpha}{\partial\tau^\alpha}\beta_{1,n}(x,\tau)\\&\quad-\frac{\partial^{2\beta}}{\partial x^{2\beta}}\beta_{1,n}(x,\tau)-\alpha_{1,n}(x,\tau)\beta^2_{1,n}(x,\tau)\\&\quad+k\beta_{1,n}(x,\tau)+\gamma(\beta_{1,n}(x,\tau)-\beta_{2,n}(x,\tau))\bigg)\,\mathrm{d}\tau,\end{aligned}\tag{91}$$

$$\begin{aligned}\alpha_{2,n+1}(x,t)&=\alpha_{2,n}(x,t)-\int_0^t\bigg(\frac{\partial^\alpha}{\partial\tau^\alpha}\alpha_{2,n}(x,\tau)\\&\quad-\frac{\partial^{2\beta}}{\partial x^{2\beta}}\alpha_{2,n}(x,\tau)+\alpha_{2,n}(x,\tau)\beta^2_{2,n}(x,\tau)\bigg)\,\mathrm{d}\tau,\end{aligned}\tag{92}$$

$$\begin{aligned}\beta_{2,n+1}(x,t)&=\beta_{2,n}(x,t)\\&\quad-\int_0^t\bigg(\frac{\partial^\alpha}{\partial\tau^\alpha}\beta_{2,n}(x,\tau)-\frac{\partial^{2\beta}}{\partial x^{2\beta}}\beta_{2,n}(x,\tau)\\&\quad-\alpha_{2,n}(x,\tau)\beta^2_{2,n}(x,\tau)\\&\quad+\gamma(\beta_{2,n}(x,\tau)-\beta_{1,n}(x,\tau))\bigg)\,\mathrm{d}\tau.\end{aligned}\tag{93}$$

9. Numerical results

In this section, we apply the HAM, ADM and VIM to evaluate the approximate solutions of (14)–(17). First, we apply the HAM on (14)–(17) and then evaluate the average residual error of HAM. The initial and first approximations are

$$\alpha_{1,0}(x,t)=\alpha_1(x,0),\tag{94}$$

$$\begin{aligned}\alpha_{1,1}(x,t)&=\frac{ht^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^\infty a_1 2^{2\beta}\Big(\frac{\mu_n}{2}\Big)^{2\beta}\\&\quad\times\cos\Big(\frac{1}{2}(L-2x)\mu_n-\pi\beta\Big)\sin\Big(\frac{n\pi}{2}\Big)\\&\quad-\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^\infty a_1\cos\Big((L-2x)\frac{\mu_n}{2}\Big)\sin\Big(\frac{n\pi}{2}\Big)\\&\quad\times\bigg(\sum_{m=1}^\infty b_1\cos\Big((L-2x)\frac{\mu_m}{2}\Big)\sin\Big(\frac{m\pi}{2}\Big)\bigg)^2,\end{aligned}\tag{95}$$

$$\beta_{1,0}(x,t)=\beta_1(x,0),\tag{96}$$

$$\beta_{1,1}(x,t) = \frac{ht^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} b_1 2^{2\beta}\left(\frac{\mu_n}{2}\right)^{2\beta} \times \cos\left(\frac{1}{2}(L-2x)\mu_n - \pi\beta\right)\sin\left(\frac{n\pi}{2}\right) - k\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} b_1 \cos\left[(L-2x)\frac{\mu_n}{2}\right]\sin\left(\frac{n\pi}{2}\right) + \frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_1 \cos\left((L-2x)\frac{\mu_n}{2}\right)\sin\left(\frac{n\pi}{2}\right) \times \left(\sum_{m=1}^{\infty} b_1 \cos\left((L-2x)\frac{\mu_m}{2}\right)\sin\left(\frac{m\pi}{2}\right)\right)^2 + \gamma\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty}(b_2-b_1) \times \cos\left[(L-2x)\frac{\mu_n}{2}\right]\sin\left(\frac{n\pi}{2}\right), \tag{97}$$

$$\alpha_{2,0}(x,t) = \alpha_2(x,0), \tag{98}$$

$$\alpha_{2,1}(x,t) = \frac{ht^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_2 2^{2\beta}\left(\frac{\mu_n}{2}\right)^{2\beta} \times \cos\left(\frac{1}{2}(L-2x)\mu_n - \pi\beta\right)\sin\left(\frac{n\pi}{2}\right) - \frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_2 \cos\left((L-2x)\frac{\mu_n}{2}\right)\sin\left(\frac{n\pi}{2}\right) \times \left(\sum_{m=1}^{\infty} b_2 \cos\left((L-2x)\frac{\mu_m}{2}\right)\sin\left(\frac{m\pi}{2}\right)\right)^2, \tag{99}$$

$$\beta_{2,0}(x,t) = \beta_2(x,0), \tag{100}$$

$$\beta_{2,1}(x,t) = \frac{ht^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} b_2 2^{2\beta}\left(\frac{\mu_n}{2}\right)^{2\beta} \times \cos\left(\frac{1}{2}(L-2x)\mu_n - \pi\beta\right)\sin\left(\frac{n\pi}{2}\right) + \frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_2 \cos\left((L-2x)\frac{\mu_n}{2}\right)\sin\left(\frac{n\pi}{2}\right) \times \left(\sum_{m=1}^{\infty} b_2 \cos\left((L-2x)\frac{\mu_m}{2}\right)\sin\left(\frac{m\pi}{2}\right)\right)^2 + \gamma\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty}(b_1-b_2) \times \cos\left[(L-2x)\frac{\mu_n}{2}\right]\sin\left(\frac{n\pi}{2}\right). \tag{101}$$

In this method, we evaluate the optimal values of the convergence-control parameters by the minimum of the averaged residual errors [7,76,77,93–96]

$$E_{\alpha_1}(h) = \frac{1}{NM}\sum_{s=0}^{N}\sum_{j=0}^{M} \times \left[\mathcal{N}_1\left(\sum_{k=0}^{m}\alpha_{1,k}\left(\frac{100s}{N},\frac{30j}{M}\right)\right)\right]^2, \tag{102}$$

$$E_{\beta_1}(h) = \frac{1}{NM}\sum_{s=0}^{N}\sum_{j=0}^{M} \times \left[\mathcal{N}_2\left(\sum_{k=0}^{m}\beta_{1,k}\left(\frac{100s}{N},\frac{30j}{M}\right)\right)\right]^2, \tag{103}$$

$$E_{\alpha_2}(h) = \frac{1}{NM}\sum_{s=0}^{N}\sum_{j=0}^{M} \times \left[\mathcal{N}_3\left(\sum_{k=0}^{m}\alpha_{2,k}\left(\frac{100s}{N},\frac{30j}{M}\right)\right)\right]^2, \tag{104}$$

$$E_{\beta_2}(h) = \frac{1}{NM}\sum_{s=0}^{N}\sum_{j=0}^{M} \times \left[\mathcal{N}_4\left(\sum_{k=0}^{m}\beta_{2,k}\left(\frac{100s}{N},\frac{30j}{M}\right)\right)\right]^2, \tag{105}$$

corresponding to the following nonlinear algebraic equations:

$$\frac{\mathrm{d}E_{\alpha_1}(h)}{\mathrm{d}h} = 0, \tag{106}$$

$$\frac{\mathrm{d}E_{\beta_1}(h)}{\mathrm{d}h} = 0, \tag{107}$$

$$\frac{\mathrm{d}E_{\alpha_2}(h)}{\mathrm{d}h} = 0, \tag{108}$$

$$\frac{\mathrm{d}E_{\beta_2}(h)}{\mathrm{d}h} = 0. \tag{109}$$

We represent $E_{\alpha_1}(h)$, $E_{\beta_1}(h)$, $E_{\alpha_2}(h)$ and $E_{\beta_2}(h)$ in figures 1–4 and in tables 1–4. Figures 1–4 and tables 1–4 show $E_{\alpha_1}(h)$, $E_{\beta_1}(h)$, $E_{\alpha_2}(h)$ and $E_{\beta_2}(h)$ for the three terms of the HAM solutions. We set $N = 100$ and $M = 30$ in (102)–(105) with $k = 0.1, \gamma = 0.2, L = 100, a_1 = 0.001, a_2 = 0.002, b_1 = 0.001, b_2 = 0.002$. We use the command FinMinimum with Mathematica to obtain optimal values of h.

Now, if we apply the recurrence relations (68)–(71) and their initial conditions (10)–(13), we obtain the following ADM first approximations:

$$\alpha_{1,0}(x,t) = \alpha_1(x,0), \tag{110}$$

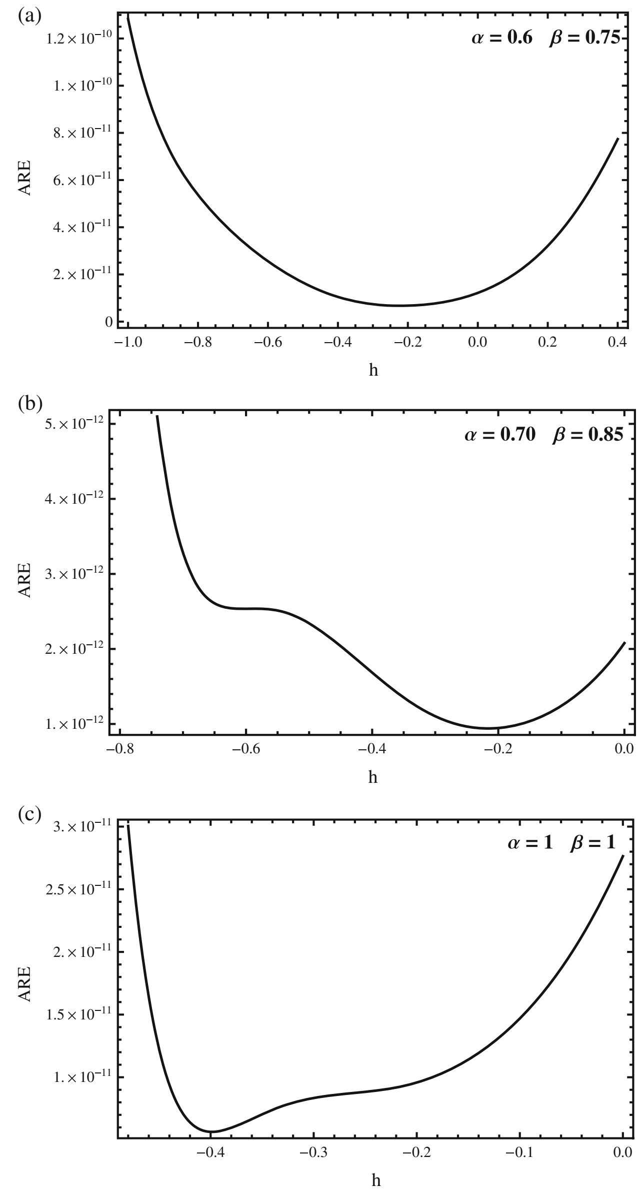

Figure 1. The averaged residual errors of $\alpha_1(x,t)$ at the three terms of the HAM solutions for $k = 0.1, \gamma = 0.2, L = 100, a_1 = 0.001, a_2 = 0.002, b_1 = 0.001$ and $b_2 = 0.002$. (**a**) $\alpha = 0.6, \beta = 0.75$, (**b**) $\alpha = 0.70, \beta = 0.85$ and (**c**) $\alpha = 1.0, \beta = 1.0$.

$$\alpha_{1,1}(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_1 2^{2\beta}\Big(\frac{\mu_n}{2}\Big)^{2\beta} \times \cos\Big(\frac{1}{2}(L-2x)\mu_n - \pi\beta\Big)\sin\Big(\frac{n\pi}{2}\Big) - \frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_1 \cos\Big((L-2x)\frac{\mu_n}{2}\Big)\sin\Big(\frac{n\pi}{2}\Big) \times \left(\sum_{m=1}^{\infty} b_1 \cos\Big((L-2x)\frac{\mu_m}{2}\Big)\sin\Big(\frac{m\pi}{2}\Big)\right)^2, \quad (111)$$

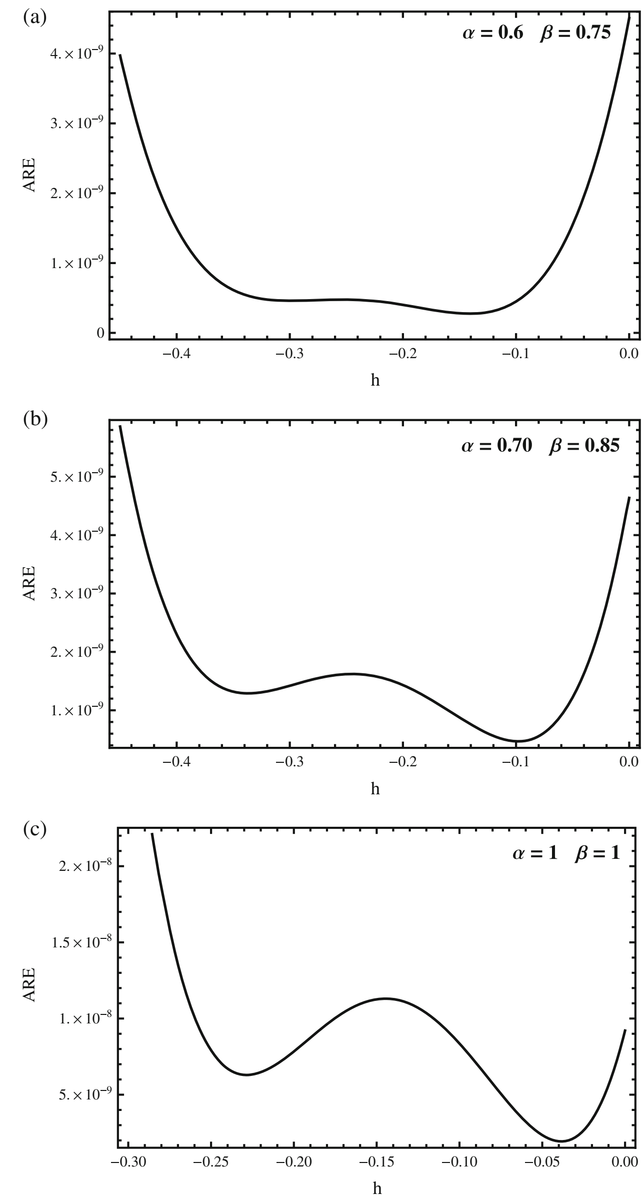

Figure 2. The averaged residual errors of $\beta_1(x,t)$ at the three terms of the HAM solutions for $k = 0.1, \gamma = 0.2, L = 100, a_1 = 0.001, a_2 = 0.002, b_1 = 0.001$ and $b_2 = 0.002$. (**a**) $\alpha = 0.6, \beta = 0.75$, (**b**) $\alpha = 0.70, \beta = 0.85$ and (**c**) $\alpha = 1.0, \beta = 1.0$.

$$\beta_{1,0}(x,t) = \beta_1(x,0), \quad (112)$$

$$\beta_{1,1}(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} b_1 2^{2\beta}\Big(\frac{\mu_n}{2}\Big)^{2\beta} \times \cos\Big(\frac{1}{2}(L-2x)\mu_n - \pi\beta\Big)\sin\Big(\frac{n\pi}{2}\Big) - k\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} b_1 \cos\Big[(L-2x)\frac{\mu_n}{2}\Big]\sin\Big(\frac{n\pi}{2}\Big)$$

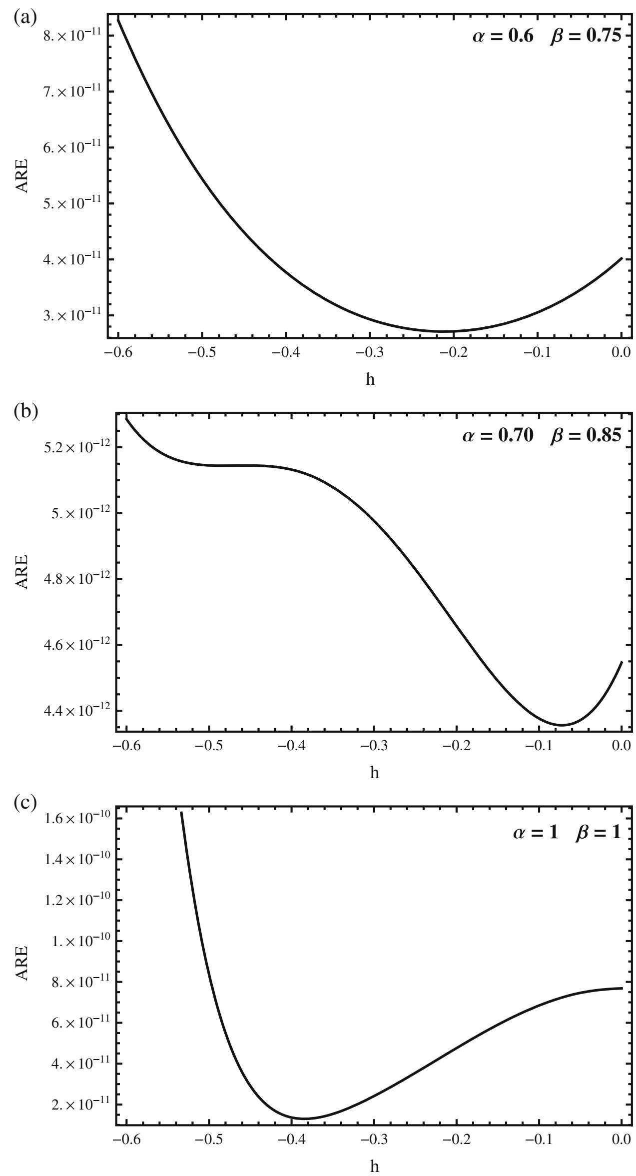

Figure 3. The averaged residual errors for $\alpha_2(x,t)$ at the three terms of the HAM solutions of (14) and (15) for $k = 0.1, \gamma = 0.2, L = 100, a_1 = 0.001, a_2 = 0.002, b_1 = 0.001$ and $b_2 = 0.002$. (**a**) $\alpha = 0.6$, $\beta = 0.75$, (**b**) $\alpha = 0.70$, $\beta = 0.85$ and (**c**) $\alpha = 1.0$, $\beta = 1.0$.

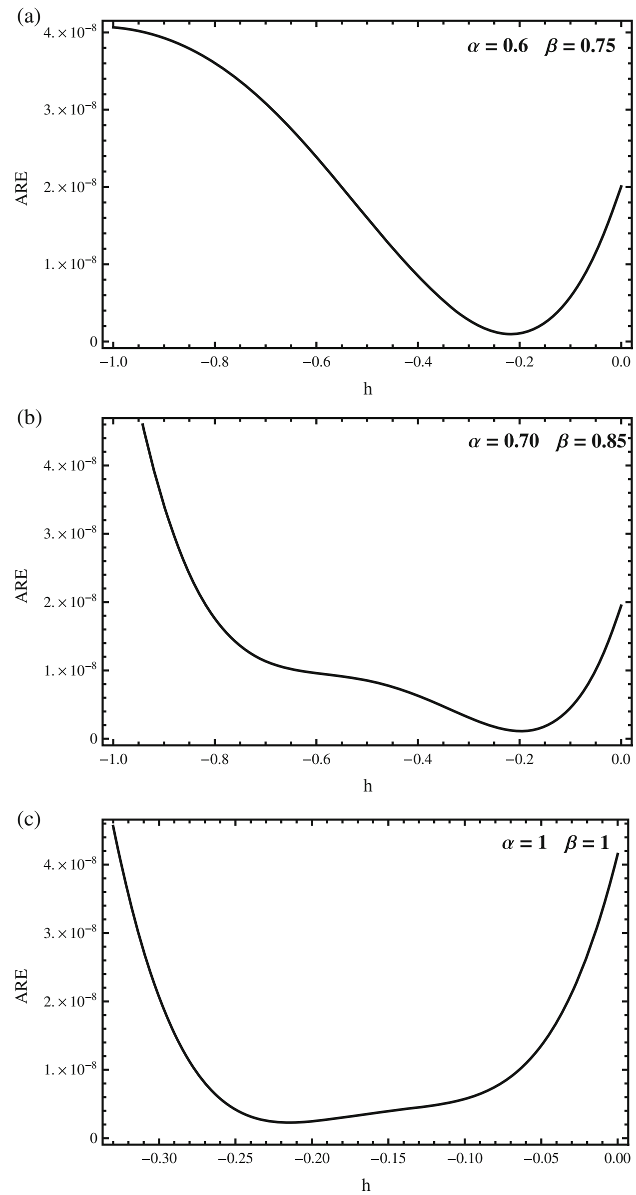

Figure 4. The averaged residual errors for $\beta_2(x,t)$ at the three terms of the HAM solutions of (14) and (15) for $k = 0.1, \gamma = 0.2, L = 100, a_1 = 0.001, a_2 = 0.002, b_1 = 0.001$ and $b_2 = 0.002$. (**a**) $\alpha = 0.6$, $\beta = 0.75$, (**b**) $\alpha = 0.70$, $\beta = 0.85$ and (**c**) $\alpha = 1.0$, $\beta = 1.0$.

$$+\frac{t^{\alpha}}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_1 \cos\left((L-2x)\frac{\mu_n}{2}\right)\sin\left(\frac{n\pi}{2}\right)$$
$$\times\left(\sum_{m=1}^{\infty} b_1 \cos\left((L-2x)\frac{\mu_m}{2}\right)\sin\left(\frac{m\pi}{2}\right)\right)^2$$
$$+\gamma\frac{t^{\alpha}}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty}(b_2-b_1)$$
$$\times\cos\left[(L-2x)\frac{\mu_n}{2}\right]\sin\left(\frac{n\pi}{2}\right), \tag{113}$$

$$\alpha_{2,0}(x,t) = \alpha_2(x,0), \tag{114}$$
$$\alpha_{2,1}(x,t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_2 2^{2\beta}\left(\frac{\mu_n}{2}\right)^{2\beta}$$
$$\times\cos\left(\frac{1}{2}(L-2x)\mu_n-\pi\beta\right)\sin\left(\frac{n\pi}{2}\right)$$
$$-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_2\cos\left((L-2x)\frac{\mu_n}{2}\right)\sin\left(\frac{n\pi}{2}\right)$$

Table 1. Optimal values of h for HAM solutions of $\alpha_1(x,t)$ at $k=0.1, \gamma=0.2, L=100, a_1=0.001, a_2=0.002, b_1=0.001$ and $b_2=0.002$.

(α,β)	Optimal value of h	Minimum of $E_{\alpha_1}(h)$
(0.6, 0.75)	−0.204096	6.70405×10^{-12}
(0.7, 0.85)	−0.214805	9.40459×10^{-13}
(1, 1)	−0.411014	5.86521×10^{-12}

Table 2. Optimal values of h for HAM solutions of $\beta_1(x,t)$ at $k=0.1, \gamma=0.2, L=100, a_1=0.001, a_2=0.002, b_1=0.001$ and $b_2=0.002$.

(α,β)	Optimal value of h	Minimum of $E_{\beta_1}(h)$
(0.6, 0.75)	−0.146486	2.76656×10^{-10}
(0.7, 0.85)	−0.18557	1.29672×10^{-9}
(1, 1)	−0.0299487	2.21299×10^{-9}

Table 3. Optimal values of h for HAM solutions of $\alpha_2(x,t)$ at $k=0.1, \gamma=0.2, L=100, a_1=0.001, a_2=0.002, b_1=0.001$ and $b_2=0.002$.

(α,β)	Optimal value of h	Minimum of $E_{\alpha_2}(h)$
(0.6, 0.75)	−0.226033	2.71645×10^{-11}
(0.7, 0.85)	−0.0273559	4.42390×10^{-12}
(1, 1)	−0.379343	1.30557×10^{-11}

Table 4. Optimal values of h for HAM solutions of $\beta_2(x,t)$ at $k=0.1, \gamma=0.2, L=100, a_1=0.001, a_2=0.002, b_1=0.001$ and $b_2=0.002$.

(α,β)	Optimal value of h	Minimum of $E_{\beta_2}(h)$
(0.6, 0.75)	−0.217057	9.56283×10^{-10}
(0.7, 0.85)	−0.213114	9.40459×10^{-13}
(1, 1)	−0.217548	1.19018×10^{-9}

$$\times\left(\sum_{m=1}^{\infty} b_2\cos\left((L-2x)\frac{\mu_m}{2}\right)\sin\left(\frac{m\pi}{2}\right)\right)^2, \quad (115)$$

$$\beta_{2,0}(x,t)=\beta_2(x,0), \quad (116)$$

$$\beta_{2,1}(x,t)=\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} b_2 2^{2\beta}\left(\frac{\mu_n}{2}\right)^{2\beta}$$
$$\times\cos\left(\frac{1}{2}(L-2x)\mu_n-\pi\beta\right)\sin\left(\frac{n\pi}{2}\right)$$
$$+\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_2\cos\left((L-2x)\frac{\mu_n}{2}\right)\sin\left(\frac{n\pi}{2}\right)$$
$$\times\left(\sum_{m=1}^{\infty} b_2\cos\left((L-2x)\frac{\mu_m}{2}\right)\sin\left(\frac{m\pi}{2}\right)\right)^2$$
$$+\gamma\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty}(b_1-b_2)$$
$$\times\cos\left[(L-2x)\frac{\mu_n}{2}\right]\sin\left(\frac{n\pi}{2}\right). \quad (117)$$

Finally, we apply the VIM to solve (14)–(17). By taking the same initial values as for ADM, we obtain the first approximation as follows:

$$\alpha_{1,1}(x,t)=\alpha_{1,0}(x,t)+\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_1 2^{2\beta}\left(\frac{\mu_n}{2}\right)^{2\beta}$$
$$\times\cos\left(\frac{1}{2}(L-2x)\mu_n-\pi\beta\right)\sin\left(\frac{n\pi}{2}\right)$$
$$-\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_1\cos\left((L-2x)\frac{\mu_n}{2}\right)\sin\left(\frac{n\pi}{2}\right)$$
$$\times\left(\sum_{m=1}^{\infty} b_1\cos\left((L-2x)\frac{\mu_m}{2}\right)\sin\left(\frac{m\pi}{2}\right)\right)^2, \quad (118)$$

$$\beta_{1,1}(x,t)=\beta_{1,0}(x,t)+\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} b_1 2^{2\beta}\left(\frac{\mu_n}{2}\right)^{2\beta}$$
$$\times\cos\left(\frac{1}{2}(L-2x)\mu_n-\pi\beta\right)\sin\left(\frac{n\pi}{2}\right)$$
$$-\frac{kt^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} b_1\cos\left[(L-2x)\frac{\mu_n}{2}\right]\sin\left(\frac{n\pi}{2}\right)$$
$$+\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_1\cos\left((L-2x)\frac{\mu_n}{2}\right)\sin\left(\frac{n\pi}{2}\right)$$
$$\times\left(\sum_{m=1}^{\infty} b_1\cos\left((L-2x)\frac{\mu_m}{2}\right)\sin\left(\frac{m\pi}{2}\right)\right)^2$$
$$+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty}(b_2-b_1)$$
$$\times\cos\left[(L-2x)\frac{\mu_n}{2}\right]\sin\left(\frac{n\pi}{2}\right), \quad (119)$$

$$\alpha_{2,1}(x,t)=\alpha_{2,0}(x,t)+\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_2 2^{2\beta}\left(\frac{\mu_n}{2}\right)^{2\beta}$$
$$\times\cos\left(\frac{1}{2}(L-2x)\mu_n-\pi\beta\right)\sin\left(\frac{n\pi}{2}\right)$$
$$-\frac{t^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} a_2\cos\left((L-2x)\frac{\mu_n}{2}\right)\sin\left(\frac{n\pi}{2}\right)$$

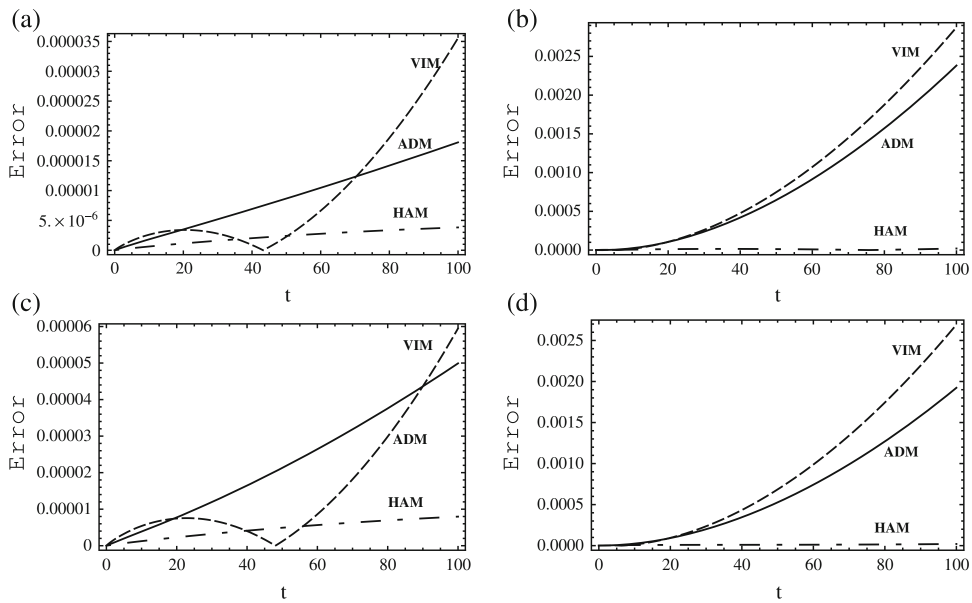

Figure 5. The absolute error of HAM, ADM and VIM for (14)–(17) with the numerical method in Mathematica at $x = 0.5$ with $\alpha = 0.9, \beta = 0.9, k = 0.1, \gamma = 0.2, L = 100, a_1 = 0.001, a_2 = 0.002, b_1 = 0.001$ and $b_2 = 0.002$. (**a**) $\alpha_1(x,t)$, (**b**) $\beta_1(x,t)$, (**c**) $\alpha_2(x,t)$ and (**d**) $\beta_2(x,t)$.

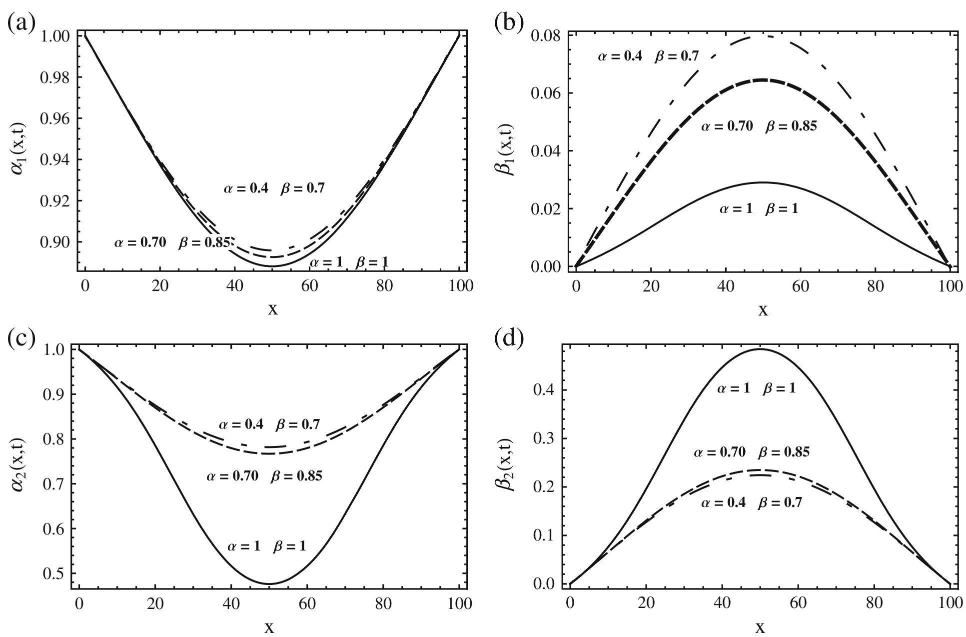

Figure 6. The plot of HAM for (14)–(17) at $t = 5$ with $k = 0.1, \gamma = 0.2, L = 100, a_1 = 0.001, a_2 = 0.002, b_1 = 0.001$ and $b_2 = 0.002$. (**a**) $\alpha_1(x,t)$, (**b**) $\beta_1(x,t)$, (**c**) $\alpha_2(x,t)$ and (**d**) $\beta_2(x,t)$.

$$\times \left(\sum_{m=1}^{\infty} b_2 \cos\left((L-2x)\frac{\mu_m}{2}\right) \sin\left(\frac{m\pi}{2}\right)\right)^2, \quad (120)$$

$$\beta_{2,1}(x,t) = \beta_{2,0}(x,t) + \frac{t^{\alpha}}{\Gamma(1+\alpha)} \sum_{n=1}^{\infty} b_2 2^{2\beta} \left(\frac{\mu_n}{2}\right)^{2\beta}$$

$$\times \cos\left(\frac{1}{2}(L-2x)\mu_n - \pi\beta\right) \sin\left(\frac{n\pi}{2}\right)$$

$$- \frac{t^{\alpha}}{\Gamma(1+\alpha)} \sum_{n=1}^{\infty} a_2 \cos\left((L-2x)\frac{\mu_n}{2}\right) \sin\left(\frac{n\pi}{2}\right)$$

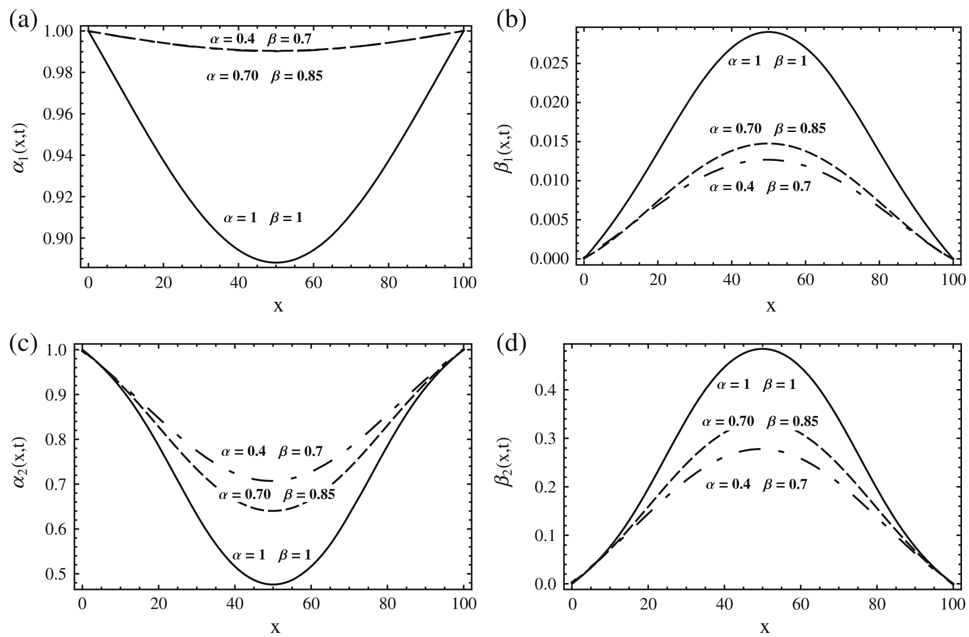

Figure 7. The plot of ADM for (14)–(17) at $t = 5$ with $k = 0.1, \gamma = 0.2, L = 100, a_1 = 0.001, a_2 = 0.002, b_1 = 0.001$ and $b_2 = 0.002$. (**a**) $\alpha_1(x,t)$, (**b**) $\beta_1(x,t)$, (**c**) $\alpha_2(x,t)$ and (**d**) $\beta_2(x,t)$

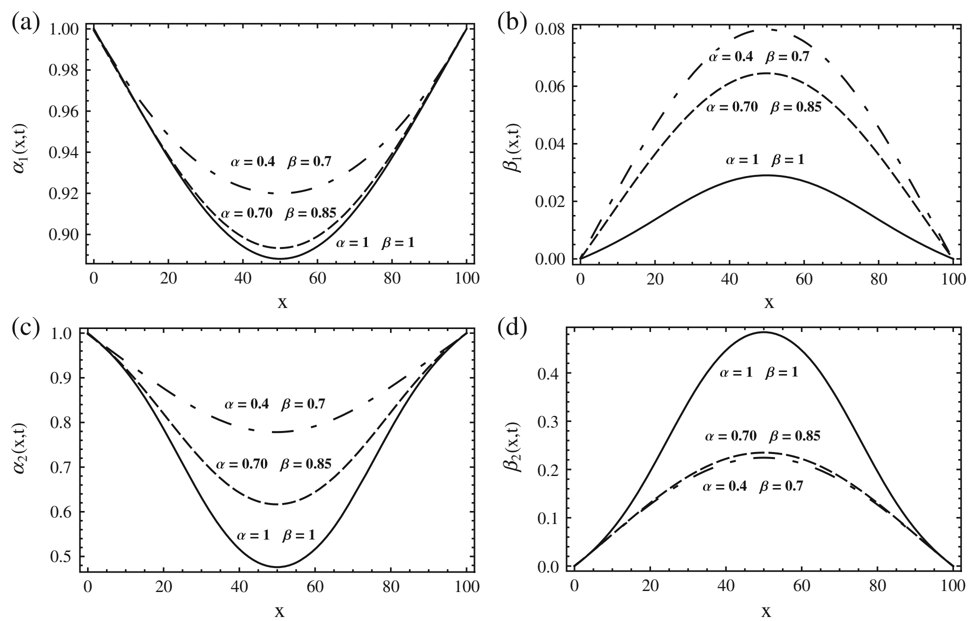

Figure 8. The plot of VIM for (14)–(17) at $t = 5$ with $k = 0.1, \gamma = 0.2, L = 100, a_1 = 0.001, a_2 = 0.002, b_1 = 0.001$ and $b_2 = 0.002$. (**a**) $\alpha_1(x,t)$, (**b**) $\beta_1(x,t)$, (**c**) $\alpha_2(x,t)$ and (**d**) $\beta_2(x,t)$

$$\times \left(\sum_{m=1}^{\infty} b_2 \cos\Big((L-2x)\frac{\mu_m}{2}\Big) \sin\Big(\frac{m\pi}{2}\Big)\right)^2 + \frac{\gamma t^{\alpha}}{\Gamma(1+\alpha)} \sum_{n=1}^{\infty} (b_1 - b_2) \times \cos\Big[(L-2x)\frac{\mu_n}{2}\Big] \sin\Big(\frac{n\pi}{2}\Big). \quad (121)$$

After substituting the initial values of $\alpha_{1,0}(x,t)$, $\beta_{1,0}(x,t)$, $\alpha_{2,0}(x,t)$ and $\beta_{2,0}(x,t)$ into (118)–(121), we obtain the first approximation of the VIM, which is the same as the two terms of the HAM and ADM for (14)–(17). A comparison of the numerical, HAM, ADM and VIM solutions is shown in figure 5 for $h = -0.16, \gamma = 0.1, k = 0.01, a_1 = 0.1, a_2 = 0.2, b_1 = 001, b_2 = 0.002, \alpha = 0.9$ and $\beta = 0.9$.

Figure 5 shows the comparison of the three terms of HAM and ADM solutions and the second approximation of VIM with the numerical solutions using the command NDsolve of MATHEMATICA 9. It can be seen from figure 5 that the absolute error obtained by HAM is better than ADM and VIM. We note that the two terms of HAM and ADM and the first approximation by VIM are identical. So their errors are of the same order. In order to get a small error, more terms need to be considered for HAM, ADM and high approximation for VIM solutions. Therefore, HAM, ADM and VIM are efficient and accurate methods that can be used to provide approximate analytical solutions of partial differential equations. Figures 6–8 show the behaviour of three terms of HAM and ADM solutions and the second approximation of VIM for (14)–(17) of different values of α and β. These figures show the effect of α and β on the concentrations of the reactant α_i and the autocatalyst β_i. We notice that from these figures the approximate solutions symmetry is at about $x = L/2$. Also, as $\alpha \rightarrow 1$ and $\beta \rightarrow 1$, the approximate solutions approach the solutions of (5)–(8). Finally as α_i and β_i decrease the concentrations of the reactant and autocatalyst, respectively, decay to the steady state, i.e. one for the reactant and zero for the autocatalyst, respectively.

10. Conclusion

In this paper, HAM, ADM and VIM have been efficiently applied to obtain approximate solutions of STFCIACS. The two terms of HAM and ADM solutions and the first approximation by VIM are identical. So, we have computed the three terms of HAM, ADM and the second approximation of VIM. We observed that HAM is better than ADM and VIM solutions. This fact is also clear when we compared them with numerical results by Mathematica. The agreement with the numerical solutions is very good. Besides that, the results demonstrate that HAM, ADM and VIM are accurate for solving space–time fractional STFCIACS. By increasing the number of iterations one can reach any desired accuracy. In this work, we compute optimal values of h for STFCIACS. In this paper, we used Mathematica 9 for all calculations.

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