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# Detecting identical entanglement pure states for two qubits

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**Abstract.** Entanglement is one of the most surprising features of composite quantum systems. Yet, challenges remain in our understanding and quantification of the entanglement. There is no unique degree of entanglement from various measures, as presented by numerous studies on quantifying entanglement. As indicated in this paper, any degree of entanglement for two qubits resulting from a particular measure can be detected in excess of one corresponding pure state. Evidently, those identical entanglement pure states can be counted as a quantitative condition to be satisfied by other proper measures. The most popular measures of pure states for two qubits are based on the same structure, as indicated in this paper. Then, the algorithm to detect the identical entanglement pure states for two qubits is proposed based on randomness distillation. Eventually, two sets of identical entanglement states are listed for two qubits.

**Keywords.** Identical entanglement; detection; randomness distillation.

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## 1. Introduction

Entanglement counts as the quintessential property in quantum mechanics distinguished itself from any classical physical theory [1]. A lot of studies strive to probe [2–4] and develop [5–7] this surprising feature taken on by the quantum system in particle physics. Entanglement, actually it is a mathematical definition [8], stems from the tensor product structure of the composite Hilbert space and linear superposition principle. It has also been shown that entanglement is an important physical resource in quantum information applications [9–11]. In the meantime, the degree of entanglement is inevitably quantified either in quantum communication or in quantum computation. Considerable number of approaches to pure states of two qubits have been proposed, as the theory of quantifying entanglement has been developing (e.g. Von Neumann entropy of entanglement, relative entropy of entanglement (REE), concurrence, negativity and geometric measure) [9,12–15]. Furthermore, quantum Fisher information, which is a practical tool to detect entanglement, has also been well studied recently [16–19]. Also, there are many studies on comparison of various entanglement measures [20–23].

To date, it may seem surprising, this theory is still not comprehensive. Especially for mixed states, the characterisation of the set of separable mixed states tends to be extremely complex [9]. Even though many approaches have been proposed to measure entanglement of bipartite pure states, the environment is complicated by various approaches based on different concepts and measurements without being apparently connected to each other. This situation is quite contrasted in quantifying the uncertainty in a probability distribution, and numerous math tools are presented to handle uncertainty and dependent relationships [24–30].

The degree of entanglement is used to measure the quantum correlation between qubits. More specifically, it depends on the correlation between the subsystems (i.e. the property of a reduced density matrix, see figure 1). Since then, many measures of entanglement have been proposed in the past few decades based on Schmidt decomposition (SD) [31–33] or decomposition in a 'magic basis' [34,35]. Particularly, the most popular measures, including entropy of entanglement, REE and concurrence, can be traced to the same structure based on the SD though they are seemingly distinguishable on the definitions. The details are presented here,

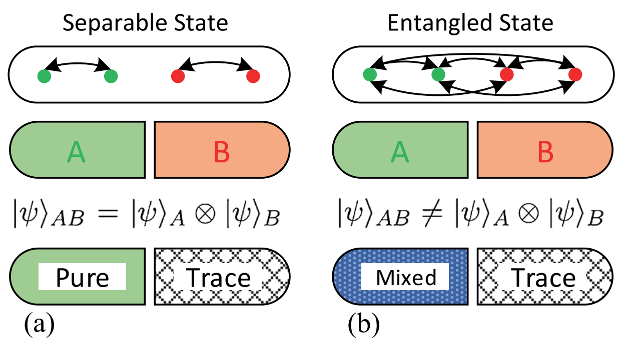


**Figure 1.** Partial trace of bipartite quantum system. There exist quantum correlations (shown as arrows) in the composite state. If the state is divided into two subsystems A and B, the state will be entangled with each other when $|\psi\rangle_{\mathrm{AB}} \neq |\psi\rangle A \otimes |\psi\rangle_{\mathrm{B}}$ **(b)**. Conversely, the state will be separable (not entangled) when $|\psi\rangle_{\mathrm{AB}} = |\psi\rangle_{\mathrm{A}} \otimes |\psi\rangle_{\mathrm{B}}$ **(a)**. The reduced density matrix of A is $\rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}(\rho_{\mathrm{AB}})$, where $\rho_{\mathrm{AB}} = |\psi\rangle_{\mathrm{AB}}\langle\psi|$ is the density matrix of the full system, and it is similar to $\rho_{\mathrm{B}}$. For a separable state, the subsystems are individually pure, that is, $\mathrm{Tr}(\rho_{\mathrm{A}}^2) = \mathrm{Tr}(\rho_{\mathrm{B}}^2) = 1$. For an entangled state, the subsystems are mixed, that is, $\mathrm{Tr}(\rho_{\mathrm{A}}^2) = \mathrm{Tr}(\rho_{\mathrm{B}}^2) < 1$.

and all these measures are demonstrated to be driven by Schmidt coefficients.

Furthermore, the Shannon entropy measuring uncertainty in probability distribution can be authenticated by the Shannon coding theorem. Yet, nothing can approve any measure to quantify the entanglement given the superficial knowledge of entanglement. Since then, the conditions for a proper measure of entanglement have been presented in [36]. The maximally entangled states are usually subjected to the highest amount of entanglement, which is also regarded as a quantitative condition [32]. In our work, we present a novel quantitative condition, i.e., the identical entanglement states should have an equal degree of entanglement for a particular measure. The key issue is how to detect those identical entanglement states.

In this paper, we show that some different measures of entanglement are actually driven by Schmidt coefficient. Then, we propose the algorithm for detecting identical entanglement state of two qubits based on randomness distillation and list some valuable results in succession. Finally, we draw summary and discussion.

## 2. The same structure

### 2.1 *Schmidt decomposition (SD)*

The SD count is a particular way to express a quantum state in quantum mechanics [37]. Consider an arbitrary bipartite quantum state

$$|\psi\rangle_{\mathrm{AB}} = \sum_{i=1}^{N}\sum_{j=1}^{M} \alpha_{ij}|i^{\mathrm{A}}\rangle \otimes |j^{\mathrm{B}}\rangle \tag{1}$$

of a composite quantum system, living in a Hilbert space $\mathcal{H}_{\mathrm{A}\otimes\mathrm{B}}^{(N+M)}$, which is divided into two smaller Hilbert spaces $\mathcal{H}_{\mathrm{A}}^{(N)}$ and $\mathcal{H}_{\mathrm{B}}^{(M)}$ of dimensions $N$ and $M$ ($N \leq M$), where $\{|i^{\mathrm{A}}\rangle\}$ and $\{|j^{\mathrm{B}}\rangle\}$ are assumed to be two complete bases of $\mathcal{H}_{\mathrm{A}}^{(N)}$ and $\mathcal{H}_{\mathrm{B}}^{(M)}$, respectively, and $\alpha_{ij}$ are probability amplitudes satisfying $\sum_{ij}\alpha_{ij}^2 = 1$. Then the state can be transformed into the form of SD:

$$|\psi\rangle_{\mathrm{AB}} = \sum_{i=1}^{N} \sqrt{\lambda_i}|\lambda_i^{\mathrm{A}}\rangle \otimes |\lambda_i^{\mathrm{B}}\rangle, \tag{2}$$

where $\{|\lambda_i^{\mathrm{A}}\rangle\}$ and $\{|\lambda_i^{\mathrm{B}}\rangle\}$ are orthonormal bases for $\mathcal{H}_{\mathrm{A}}^{(N)}$ and $\mathcal{H}_{\mathrm{B}}^{(M)}$, respectively, and $\lambda_i$ are real and non-negative coefficients satisfying $\sum_{i=1}^{N}\lambda_i = 1$, which capture all entanglement properties of a pure bipartite state [38]. Also, the Schmidt number

$$K = \frac{1}{\sum_{i=1}^{N}\lambda_i^2} \tag{3}$$

defined on the interval $[1, N]$ as the effective number of SD, which can be directly regarded as a measure of entanglement [9,39].

### 2.2 *Von Neumann entanglement entropy*

The bipartite von Neumann entanglement entropy $S$ is defined as the von Neumann entropy of either of its reduced density matrices. Consider a pure state $\rho_{\mathrm{AB}}$ of two subsystems A and B. The reduced density matrix is defined as a result of partial trace over one subsystem, either A or B, i.e. $\rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}(\rho_{\mathrm{AB}})$ and $\rho_{\mathrm{B}} = \mathrm{Tr}_{\mathrm{A}}(\rho_{\mathrm{AB}})$. Then the von Neumann entropy of entanglement is given by

$$S(\rho_{\mathrm{A}}) = S(\rho_{\mathrm{B}}) = -\sum_{i=1}^{N} a_i \ \log a_i, \tag{4}$$

where $a_i$ are the eigenvalues of the reduced density matrix. Actually, the eigenvalues of the reduced density matrix, i.e. $a_i$, are equal to the Schmidt coefficients, i.e. $\lambda_i$, mentioned above (for details see [33]).

### 2.3 *Relative entropy of entanglement*

The degree of entanglement of relative entropy, introduced by Vedal *et al* [36], is defined as (for bipartite)

$$E_{\mathrm{r}}(\rho_{\mathrm{AB}}) = \min_{\sigma_{\mathrm{AB}}\in D} S(\rho_{\mathrm{AB}}||\sigma_{\mathrm{AB}}), \tag{5}$$

where $S(\rho_{AB}||\sigma_{AB})$ is the REE of $\rho_{AB}$ and $\sigma_{AB}$

$$S(\rho_{AB}||\sigma_{AB}) = \mathrm{tr}\{\rho_{AB}(\log \rho_{AB} - \log \sigma_{AB})\}. \quad (6)$$

$D$ contains all disentangled states. It is clear from the definition that the degree of relative entropy actually is a solution of the optimisation problem, which is intractable for some entanglement [9]. Fortunately, an efficient numerical method is there to obtain this measure [40]. If a pure state can be expressed in the form

$$\rho_{AB} = \sum_{i\,j} \sqrt{p_i p_j}\, |\Phi_i \Psi_j\rangle\langle\Phi_i \Psi_j|\,, \quad (7)$$

then the REE is equal to the von Neumann reduced entropy, i.e. $E_r(\rho_{AB}) = -\sum_i p_i \log p_i$. Throw a glance at eq. (2), i.e. SD, the density matrix of $\rho_{AB}$ in eq. (7) can be directly obtained by the state vector of $|\psi\rangle_{AB}$ in eq. (2). Besides, $p_i = \lambda_i$.

### 2.4 *Concurrence*

The concurrence was introduced for pure states by Hill and Wootters [35], which is defined as $C(\rho_{AB}) = \sqrt{2(1 - \mathrm{Tr}(\rho_A^2))}$, where $\rho_{AB}$ is the full density matrix and $\rho_A$ is the reduced density matrix. There is another form for two qubits, $C|\psi\rangle_{AB} = 2\lambda_1\lambda_2$, where $\lambda_i$ are the Schmidt coefficients (for details, see [34]). Therefore, the concurrence is also based on the SD.

### 2.5 *Some others*

There are some other measures of entanglement subject to this same structure, i.e. SD (for details see [34,41–43]).

## 3. Algorithm of detecting identical entanglement states

As mentioned above, the various measures are actually based on the Schmidt coefficients, so that we proposed the algorithm to detect identical entanglement states (IES) based on this same structure. The basic idea is the sample as shown in figure 2, known as state distillation. Four steps are described as follows:

(i) Initialise the desired sequence number $N$, loop counter $i = 1$ and the Schmidt coefficients of an input state $|\psi\rangle_0$ as the benchmark for distillation, $\mathbf{S}_{\text{bench}}$.

(ii) Generate a random pure state for two qubits as shown in eq. (1). Four distinct coefficients, $\alpha_{00}, \alpha_{01}, \alpha_{10}$ and $\alpha_{11}$, correspond to different states and satisfy $\sum_{ij} \alpha_{ij}^2 = 1$.

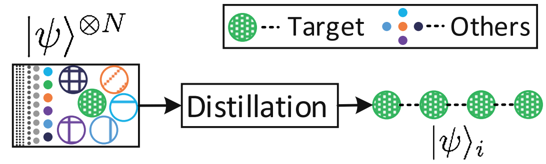


**Figure 2.** State distillation. There are a number of random pure states $|\psi\rangle^{\otimes N}$ and an input (target) state $|\psi\rangle_0$. The distillation process, by comparing with $|\psi\rangle_0$, can detect the identical entanglement states to generate the sequence of IES $|\psi\rangle_i$ from $|\psi\rangle^{\otimes N}$.

(iii) Perform the SD on the random pure state, get the Schmidt coefficients $\mathbf{S}_{\text{temp}}$ and compare with $\mathbf{S}_{\text{bench}}$. Return to (ii) if $\mathbf{S}_{\text{temp}} \neq \mathbf{S}_{\text{bench}}$. Otherwise, go to the next step.

(iv) Record the random pure state to the sequence of IES. Set $i = i + 1$, return to (ii) if $i \leq N$. Otherwise, end.

The procedure of state distillation is indicated as random-based, thus no answer is concerned with how many identical entanglement states are there, how to control the randomness efficiently or what is the underlying cause of the identical entanglement. Fortunately, at least a solution can be provided.

Two examples of distillation are presented in this paper. One refers to the distillation results for a bipartite maximal entanglement state, e.g. Bell state, as listed in table 1. The other refers to the distillation results for state of $(|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$, as listed in table 2. Degrees of different measures for different states are listed in tables 1 and 2. Apparently, degrees of negativity [44] and maximal singlet fraction [45] are also presented as validated measures. Furthermore, other validated measures, inclusive of Renyi entropy of entanglement [46], have also been performed. It is noteworthy that the equivalence of different Bell pairs is eliminated in entanglement in both cases.

Furthermore, the REE, serving as a unique way to understand entanglement, deserves more attention. Degree of REE is smaller than other measures, replicating the findings of Miranowicz and Grudka [22], as listed in table 2. The REE provides a unifying measure for all cases, and other measures of entanglement that can be given by relative entropies of an extended system, depending overall on how the information in the memory is adopted or depending on how the original system is decomposed, as stated by Henderson and Vedral [47]. Also, different decompositions shall differentiate the amounts of relative entropy of an extended system. Accordingly, the excess of degree of other measures over REE, in terms of information content, could be expounded as information of decomposition

**Table 1.** The maximal entanglement states for two qubits. The abbreviations of the corresponding measures are listed as follows: Schm.N: Schmidt number; Von-Neu.E: Von Neumann entropy of entanglement; Rela.E: Relative entropy of entanglement; Con: concurrence; Neg: negativity; MSF: maximal singlet fraction.

| State | Schm.N[a] | Von-Neu.E[a] | Rela.E[a] | Con[a] | Neg[b] | MSF[b] |
|---|---|---|---|---|---|---|
| $(\|00\rangle + \|11\rangle)/\sqrt{2}$ | 2.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| $(\|00\rangle + \|01\rangle + \|10\rangle - \|11\rangle)/2$ | 2.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| $(\|00\rangle - 2\|01\rangle + 2\|10\rangle + \|11\rangle)/\sqrt{10}$ | 2.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| $(3\|00\rangle - 1\|01\rangle + 1\|10\rangle + 3\|11\rangle)/2\sqrt{5}$ | 2.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| $(3\|00\rangle - 2\|01\rangle + 2\|10\rangle + 3\|11\rangle)/\sqrt{26}$ | 2.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| $(\|00\rangle - 7\|01\rangle + 7\|10\rangle + \|11\rangle)/10$ | 2.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

[a]This measure has been shown based on the same structure.
[b]This measure is regarded as the validated ones.

**Table 2.** A set of identical entanglement states for two qubits. The abbreviations of the corresponding measures are similar to table 1.

| State | Schm.N[a] | Von-Neu.E[a] | Rela.E[a] | Con[a] | Neg[b] | MSF[b] |
|---|---|---|---|---|---|---|
| $(\|00\rangle + \|01\rangle + \|11\rangle)/\sqrt{3}$ | 1.2857143 | 0.5500478 | 0.5500478 | 0.6666667 | 0.6666667 | 0.8333333 |
| $(2\|00\rangle + \|01\rangle + \|11\rangle)/\sqrt{6}$ | 1.2857143 | 0.5500478 | 0.5500478 | 0.6666667 | 0.6666667 | 0.8333333 |
| $(2\|00\rangle + \|01\rangle + 5\|11\rangle)/\sqrt{30}$ | 1.2857143 | 0.5500478 | 0.5500478 | 0.6666667 | 0.6666667 | 0.8333333 |
| $(\|00\rangle + 2\|01\rangle + 3\|10\rangle + 1\|11\rangle)/\sqrt{15}$ | 1.2857143 | 0.5500478 | 0.5500478 | 0.6666667 | 0.6666667 | 0.8333333 |
| $(\|00\rangle + 3\|01\rangle + 4\|10\rangle + 2\|11\rangle)/\sqrt{30}$ | 1.2857143 | 0.5500478 | 0.5500478 | 0.6666667 | 0.6666667 | 0.8333333 |

[a]This measure has been shown based on the same structure.
[b]This measure is regarded as the validated ones.

or transposition. Furthermore, it has been shown that the negativity of any state can never exceed its concurrence, i.e. $C(\rho) \geq N(\rho)$, and equality holds for pure states [21].

## 4. Summary and discussion

In this paper, we have shown that the most popular measures for pure states of two qubits are based on the Schmidt coefficients, especially including von Neumann entropy of entanglement, REE and concurrence. Based on this finding, we proposed the algorithm for detecting an identical entanglement state for two qubits based on state distillation. After that, some distillation results are listed for a bipartite maximal entanglement state and the state of $(|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$. Moreover, the proposed algorithm is examined by adopting validated measures. Additionally, it is more interesting to detect identical entanglement states for mixed states, which are still under research.

Our algorithm is random-based, and so the knowledge concerning the correlation between the identical entanglement states fails to be deepened in this paper. This topic shall arouse on-going concern with the open issue of quantifying entanglement. The construction of a physically meaningful and mathematically rigorous theory of quantifying entanglement can improve our perception of quantum correlation. By uncovering this intricate correlation between the composite quantum systems, it is a great effort to do research on identical entanglements, and we believe that the present work is a beginning in this direction.

Moreover, the powerful results of identical entanglement listed in tables 1 and 2 give a novel quantitative condition any proper measure of entanglement should satisfy, i.e. the identical entanglement states should have an equal degree of entanglement for a particular measure.

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## References

[1] E. Schrödinger, *Naturwissenschaften* **23(823)**, 807 (1935)

[2] S Abdel-Khalek, Y S El-Saman and M Abdel-Aty, *Pramana – J. Phys.* **90(1)**: 1 (2018)
[3] F Ozaydin, *Quantum pseudo-telepathy in spin systems: Magic square game under magnetic fields and Dzyaloshinskii–Moriya interaction*, arXiv preprint arXiv:1609.03881 (2016)
[4] X Zong-Cheng, L Mai-Lin, Z Ya-Ting and Y Jian-Quan, *Pramana – J. Phys.* **86(3)**, 495 (2016)
[5] N Nayak, B Ghosh and A S Majumdar, *Indian J. Phys.* **84(8)**, 1039 (2010)
[6] V Erol, F Ozaydin and A A Altintas, *Sci. Rep.* **4**, 5422 (2014)
[7] E K Bashkirov and M S Mastyugin, *Pramana – J. Phys.* **84(1)**, 127 (2015)
[8] V Vedral, *Nature* **453(7198)**, 1004 (2008)
[9] R Horodecki, P Horodecki, M Horodecki and K Horodecki, *Rev. Mod. Phys.* **81(2)**, 865 (2009)
[10] C B Dağ, W Niedenzu, Ö E Müstecaplıoğlu and G Kurizki. *Entropy* **18(7)**, 244 (2016)
[11] A K Maurya, M K Mishra and H Prakash, *Pramana – J. Phys.* **86(3)**, 515 (2016)
[12] M B Plbnio and S Virmani, *Quantum Inf. Comput.* **7(1)**, 1 (2005)
[13] C Eltschka and J Siewert, *J. Phys. A: Math. Theor.* **47(42)**, 424005 (2014)
[14] G Adesso, T R Bromley and M Cianciaruso, *J. Phys. A: Math. Theor.* **49**, 473001 (2016)
[15] S Khan and N A Khan *Pramana – J. Phys.* **87(4)**, 61 (2016)
[16] S Luo, *Lett. Math. Phys.* **53(3)**, 243 (2000)
[17] F Ozaydin, A Ali Altintas, C Yesilyurt, S Bugu and V Erol, *Acta Phys. Pol. A* **127(4)**, 1233 (2015)
[18] F Ozaydin, A A Altintas, S Bugu and C Yesilyurt, *Int. J. Theor. Phys.* **52(9)**, 2977 (2013)
[19] S Abdel-Khalek and S H A Halawani, *Pramana – J. Phys.* **85(6)**, 1089 (2015)
[20] J Eisert and M B Plenio, *J. Mod. Opt.* **46(1)**, 145 (1999)
[21] F Verstraete, K Audenaert, J Dehaene and B De Moor, *J. Phys. A: Math. Gen.* **34(47)**, 10327 (2001)
[22] A Miranowicz and A Grudka, *J. Opt. B: Quantum Semiclass. Opt.* **6(12)**, 542 (2004)
[23] D Das, R Sengupta and Aravind, *Pramana – J. Phys.* **88(6)**: 82 (2017)
[24] X Zhou, Y Hu, Y Deng, F T S Chan and A Ishizaka, *Ann. Oper. Res.* https://doi.org/10.1007/s10479-018-2769-3 (2018)
[25] R Zhang, B Ashuri and Y Deng, *Adv. Data Anal. Class.* **11(4)**, 759 (2017)
[26] H Xu and Y Deng, *IEEE Access* **6**, 11634 (2018)
[27] X Deng and Y Deng, *Soft Comput.*, published online, https://doi.org/10.1007/s00500-017-2993-9 (2018)
[28] B Kang, G Chhipi-Shrestha, Y Deng, K Hewage and R Sadiq, *Appl. Math. Comput.* **324**, 202 (2018)
[29] Q Zhang, M Li and Y Deng, *Phys. A: Stat. Mech. Appl.* **491**, 749 (2018)
[30] X Zheng and Y Deng, *Ann. Nucl. Energy* **112**, 673 (2018)
[31] J Eisert and H J Briegel, *Phys. Rev. A* **64(2)**, 022306 (2001)
[32] J Sperling and W Vogel, *Phys. Scr.* **83(4)**, 045002 (2011)
[33] P Vivo, M P Pato and G Oshanin, *Phys. Rev. E* **93(5)**, 052106 (2016)
[34] A F Abouraddy, B E A Saleh, A V Sergienko and M C Teich, *Phys. Rev. A* **64(5)**, 050101 (2001)
[35] S Hill and W K Wootters, *Phys. Rev. Lett.* **78(26)**, 5022 (1997)
[36] V Vedral, M B Plenio, M A Rippin and P L Knight, *Phys. Rev. Lett.* **78(12)**, 2275 (1997)
[37] M A Nielsen and I L Chuang, *Quantum computation and quantum information* (Cambridge University Press, Cambridge, 2010)
[38] G Sentís, C Eltschka, O Gühne, M Huber and J Siewert, *Phys. Rev. Lett.* **117(9)**, 190502 (2016)
[39] H Di Lorenzo Pires, C H Monken and M P van Exter, *Phys. Rev. A* **80(2)**, 022307 (2009)
[40] V Vedral and M B Plenio, *Phys. Rev. A* **57(3)**, 1619 (1998)
[41] A Shimony, *Ann. NY Acad. Sci.* **755(1)**, 675 (1995)
[42] M A Horne, A Shimony and A Zeilinger, *Phys. Rev. Lett.* **62(19)**, 2209 (1989)
[43] J-L Chen, L Fu, A A Ungar and X-G Zhao, *Phys. Rev. A* **65(4)**, 044303 (2002)
[44] M B Plenio, *Phys. Rev. Lett.* **95(9)**, 090503 (2005)
[45] M Horodecki, P Horodecki and R Horodecki, *Phys. Rev. A* **60(3)**, 1888 (1999)
[46] K G H Vollbrecht and M M Wolf, *J. Math. Phys.* **43(9)**, 4299 (2002)
[47] L Henderson and V Vedral, *Phys. Rev. Lett.* **84(10)**, 2263 (2000)