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Trivariate analysis of two qubit symmetric separable state

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Abstract. One of the main problems of quantum information theory is developing the separability criterion which is both necessary and sufficient in nature. Positive partial transposition test (PPT) is one such criterion which is both necessary and sufficient for 2×2 and 2×3 systems but not otherwise. We express this strong PPT criterion for a system of 2-qubit symmetric states in terms of the well-known Fano statistical tensor parameters and prove that a large set of separable symmetric states are characterised by real statistical tensor parameters only. The physical importance of these states are brought out by employing trivariate representation of density matrix wherein the components of $\mathbf{J}$, namely J_x, J_y, J_z are considered to be the three variates. We prove that this set of separable states is characterised by the vanishing average expectation value of J_y and its covariance with J_x and J_z. This allows us to identify a symmetric separable state easily. We illustrate our criterion using 2-qubit system as an example.

Keywords. Separability criterion; symmetric states; angular momentum operators; statistical tensor parameters.

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1. Introduction

Entanglement is one of the striking features of quantum theory with an ever increasing number of applications [1,2] such as quantum cryptography, quantum teleportation, quantum error correction and so on. Any bipartite state ρ^{AB} is called entangled if it cannot be decomposed as $\rho^{AB} = \sum_i \lambda_i \rho_i^A \otimes \rho_i^B$ where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$ and ρ_i^A, ρ_i^B are density operators in Hilbert space H^A and H^B respectively. It is difficult to characterise an arbitrary state belonging to a Hilbert space of any dimension as separable or entangled. This separability problem is considered as NP hard [3]. However, for lower-dimensional space, there exist several results on separability criteria for both pure and mixed states [4–13] out of which PPT criterion [6] is the strongest. PPT criterion is both necessary and sufficient for the 2×2 and 2×3 systems [14]. According to this, if the partial transposed density matrix of the bipartite system with respect to either of its subsystems has positive eigenvalues, then the state is separable, otherwise it is entangled. It is worthwhile to note that in higher dimensions there are states which are called bound entangled states with positive partial transposition but are entangled [15].

In this paper, we deal with permutationally symmetric states. Recently, symmetric states correlations have been studied [16] as these classes of states are of experimental interest. With the help of PPT criterion, we provide a separability criterion for a set of permutationally symmetric states in terms of physically observable quantities, namely, the moments of angular momentum operators J_x, J_y and J_z which in turn can be expressed in terms of the well-known Pauli spin operators. We also give a set of conditions in terms of Fano statistical tensor parameters which characterise a set of separable states. Therefore, given a symmetric state, our criterion for separability is not only operationally convenient but also experimentally measurable [17].

The paper is organised as follows: In §2 we explain the Fano representation of density matrix and algebraic equivalence of spin-1 and 2-qubit symmetric systems. In §3, using the PPT criterion we develop a set of conditions in terms of Fano statistical tensor parameters that characterise a set of separable states. In §4, we use trivariate description of density matrix and express the separability conditions in terms of various moments of the angular momentum operators and Pauli spin operators. In §5 we illustrate our criterion using a 2-qubit example.

2. Fano representation of spin-j system and its correspondence with symmetric subspace

Symmetric N-qubit states are the states that remain unchanged by permutation of individual particles. That is, $\pi_{i,j}\rho^{\text{symm}}_{1,2....N} = \rho^{\text{symm}}_{1,2....N}\pi_{i,j} = \rho^{\text{symm}}_{1,2....N}$ where $\pi_{i,j}$ is called the permutation operator, $i \neq j = 1, 2, ..., N$. A general N-qubit state belongs to the Hilbert space $C^{2^{\otimes N}}$ and is represented by a density matrix of dimension $2^N \times 2^N$. An N-qubit symmetric state has one-to-one correspondence with a spin-j state where $j = N/2$. The dimension of density matrix reduces from $2^N \times 2^N$ to $(N+1) \times (N+1)$ for permutation symmetric states and this $(N+1)$-dimensional symmetric subspace can be identified with a $(2j+1)$-dimensional Hilbert space which is the carrier space of the angular momentum operator $\mathbf{J}$. We focus on such symmetric states in this article as they are of considerable experimental interest [5].

A general spin-j density matrix can be represented in terms of statistical tensor parameters [18–20] t^k_q's:

$$\rho(\vec{J}) = \frac{\text{Tr}(\rho)}{(2j+1)} \sum_{k=0}^{2j} \sum_{q=-k}^{+k} t^k_q \, \tau^{k^\dagger}_q(\vec{J}), \tag{1}$$

where $\vec{J}$ is the angular momentum operator with components J_x, J_y, J_z. The operators τ^k_q (with $\tau^0_0 = I$, the identity operator) are irreducible tensor operators of rank k in the $(2j+1)$-dimensional angular momentum space with projection q along the axis of quantisation in R^3. The elements of τ^k_q in the angular momentum basis $|jm\rangle$, $m = -j, \ldots, +j$ are given by

$$\langle jm'|\tau^k_q(\vec{J})|jm\rangle = [k]C(jkj; mqm'), \tag{2}$$

where $C(jkj; mqm')$ are the Clebsch–Gordan coefficients and $[k] = \sqrt{2k+1}$. τ^k_q satisfies the orthogonality relations

$$\text{Tr}(\tau^{k^\dagger}_q \tau^{k'}_{q'}) = (2j+1)\,\delta_{kk'}\delta_{qq'},$$

where

$$\tau^{k^\dagger}_q = (-1)^q \, \tau^k_q$$

and

$$t^k_q = \text{Tr}(\rho\, \tau^k_q) = \sum_{m=-j}^{+j} \langle jm|\rho\tau^k_q|jm\rangle.$$

As ρ is Hermitian and $\tau^{k^\dagger}_q = (-1)^q \tau^k_{-q}$, the complex conjugate of t^k_q's satisfies the condition $t^{k^*}_q = (-1)^q \, t^k_{-q}$. Furthermore, $\rho = \rho^\dagger$ and $\text{Tr}(\rho) = 1$ imply that ρ can be characterised by $n^2 - 1$ independent parameters where $n = 2j+1$ is the dimension of the Hilbert space. Under rotations, the spherical tensor parameters t^k_q transform elegantly as

$$(t^k_q)^R = \sum_{q'=-k}^{+k} D^k_{q'q}(\phi, \theta, \psi)\, t^k_{q'},$$

where $D^k_{q'q}(\phi, \theta, \psi)$ is the (q', q) element of the Wigner D matrix and (ϕ, θ, ψ) are the Euler angles.

3. PPT separable symmetric states

A spin-1 system using eq. (1) can be expanded as

$$\rho^1 = \frac{1}{3}(t^0_0\tau^{0^\dagger}_0 + t^1_1\tau^{1^\dagger}_1 + t^1_0\tau^{1^\dagger}_0 + t^1_{-1}\tau^{1\dagger}_{-1} + t^2_2\tau^{2^\dagger}_2 + t^2_1\tau^{2^\dagger}_1 + t^2_0\tau^{2^\dagger}_0 + t^2_{-1}\tau^{2\dagger}_{-1} + t^2_{-2}\tau^{2\dagger}_{-2}). \tag{3}$$

Using (2), we can find τ^k_q's of spin-1 system as

$$\tau^0_0 = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \quad \tau^1_0 = \sqrt{\frac{3}{2}} \begin{pmatrix} 1&0&0\\0&0&0\\0&0&-1 \end{pmatrix},$$

$$\tau^1_1 = -\sqrt{\frac{3}{2}} \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}, \quad \tau^1_{-1} = \sqrt{\frac{3}{2}} \begin{pmatrix} 0&0&0\\1&0&0\\0&1&0 \end{pmatrix},$$

$$\tau^2_0 = \sqrt{\frac{1}{2}} \begin{pmatrix} 1&0&0\\0&-2&0\\0&0&1 \end{pmatrix}, \quad \tau^2_1 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0&-1&0\\0&0&1\\0&0&0 \end{pmatrix},$$

$$\tau^2_{-1} = \sqrt{\frac{3}{2}} \begin{pmatrix} 0&0&0\\1&0&0\\0&-1&0 \end{pmatrix}, \quad \tau^2_2 = \sqrt{3} \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix},$$

$$\tau^2_{-2} = \sqrt{3} \begin{pmatrix} 0&0&0\\0&0&0\\1&0&0 \end{pmatrix}.$$

Therefore, ρ^1 in $|1m\rangle$ basis; $[|\uparrow\uparrow\rangle, \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}, |\downarrow\downarrow\rangle]$ is given by

$$\rho^1 = \frac{1}{3}\begin{bmatrix} 1+\sqrt{\frac{3}{2}}t_0^1+\frac{t_0^2}{\sqrt{2}} & \sqrt{\frac{3}{2}}(t_{-1}^1+t_{-1}^2) & \sqrt{3}t_{-2}^2 \\ -\sqrt{\frac{3}{2}}(t_1^1+t_1^2) & (1-\sqrt{2}t_0^2) & \sqrt{\frac{3}{2}}(t_{-1}^1-t_{-1}^2) \\ \sqrt{3}t_2^2 & -\sqrt{\frac{3}{2}}(t_1^1-t_1^2) & 1-\sqrt{\frac{3}{2}}t_0^1+\frac{t_0^2}{\sqrt{2}} \end{bmatrix}.$$

Transforming ρ^1 to computational basis, the 2-qubit density matrix

$$\rho^{AB} = U^\dagger\rho^1 U = \frac{1}{3}\begin{bmatrix} 1+\sqrt{\frac{3}{2}}t_0^1+\frac{t_0^2}{\sqrt{2}} & \frac{\sqrt{3}(t_{-1}^1+t_{-1}^2)}{2} & \frac{\sqrt{3}(t_{-1}^1+t_{-1}^2)}{2} & \sqrt{3}t_{-2}^2 \\ \frac{-\sqrt{3}(t_1^1+t_1^2)}{2} & \frac{1-\sqrt{2}t_0^2}{2} & \frac{1-\sqrt{2}t_0^2}{2} & \frac{\sqrt{3}(t_{-1}^1-t_{-1}^2)}{2} \\ \frac{-\sqrt{3}(t_1^1+t_1^2)}{2} & \frac{1-\sqrt{2}t_0^2}{2} & \frac{1-\sqrt{2}t_0^2}{2} & \frac{\sqrt{3}(t_{-1}^1-t_{-1}^2)}{2} \\ \sqrt{3}t_2^2 & \frac{-\sqrt{3}(t_1^1-t_1^2)}{2} & \frac{-\sqrt{3}(t_1^1-t_1^2)}{2} & 1-\sqrt{\frac{3}{2}}t_0^1+\frac{t_0^2}{\sqrt{2}} \end{bmatrix},$$

where U transforms $[|\uparrow\uparrow\rangle, \frac{|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle}{\sqrt{2}}, |\downarrow\downarrow\rangle, \frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}]$ to $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ and is given by

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

Now, to perform the PPT test, we consider the partially transposed ρ^{AB}

$$\rho_{PT}^{AB} = \frac{1}{3}\begin{bmatrix} 1+\sqrt{\frac{3}{2}}t_0^1+\frac{t_0^2}{\sqrt{2}} & \frac{-\sqrt{3}(t_1^1+t_1^2)}{2} & \frac{\sqrt{3}(t_{-1}^1+t_{-1}^2)}{2} & \frac{1-\sqrt{2}t_0^2}{2} \\ \frac{\sqrt{3}(t_{-1}^1+t_{-1}^2)}{2} & \frac{1-\sqrt{2}t_0^2}{2} & \sqrt{3}t_{-2}^2 & \frac{\sqrt{3}(t_{-1}^1-t_{-1}^2)}{2} \\ \frac{-\sqrt{3}(t_1^1+t_1^2)}{2} & \sqrt{3}t_2^2 & \frac{1-\sqrt{2}t_0^2}{2} & \frac{-\sqrt{3}(t_1^1-t_1^2)}{2} \\ \frac{1-\sqrt{2}t_0^2}{2} & \frac{-\sqrt{3}(t_1^1-t_1^2)}{2} & \frac{\sqrt{3}(t_{-1}^1-t_{-1}^2)}{2} & 1-\sqrt{\frac{3}{2}}t_0^1+\frac{t_0^2}{\sqrt{2}} \end{bmatrix}. \quad (4)$$

Since the eigenvalues of the above density matrix are cumbersome and difficult to analyse, one may not obtain any definite conclusion regarding the separability of the state. Therefore, we intend to devise a simple technique by observing the following: Firstly, PPT is not a symmetry preserving operation as $U\rho_{PT}^{AB}U^\dagger$ cannot be expressed as a direct sum of symmetric and antisymmetric parts. PPT preserves the symmetry only for such states which are characterised by real t_q^k's with an additional condition

$$\frac{t_0^2}{\sqrt{2}} + \sqrt{3}t_2^2 = \frac{1}{2}.$$

The resultant 3×3 symmetric matrix is certainly a density matrix representing a spin-1 system as it can be obtained by eq. (1) with real t_q^k's, $k = 0, 1, 2$, $q = -k, \ldots, k$ only. Obviously, such a density matrix has positive eigenvalues. In other words, the corresponding 4×4 matrix, ρ_{PT}^{AB}, is also positive semidefinite and hence the state is separable.

The above observations can be demonstrated explicitly in the following algebraic steps: Transforming ρ_{PT}^{AB} to symmetric subspace with basis $|\uparrow\uparrow\rangle$, $\frac{|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle}{\sqrt{2}}$, $|\downarrow\downarrow\rangle$, $\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}$, we have $U\rho_{PT}^{AB}U^\dagger$ given by

$$U\rho_{PT}^{AB}U^{\dagger} = \frac{1}{3}\begin{bmatrix} 1+\sqrt{\frac{3}{2}}t_0^1+\frac{t_0^2}{\sqrt{2}} & \frac{-\sqrt{3}(t_1^1+t_1^2)+\sqrt{3}(t_{-1}^1+t_{-1}^2)}{2\sqrt{2}} & \frac{1-\sqrt{2}t_0^2}{2} & \frac{-\sqrt{3}(t_1^1+t_1^2)-\sqrt{3}(t_{-1}^1+t_{-1}^2)}{2\sqrt{2}} \\ \frac{\sqrt{3}(t_{-1}^1+t_{-1}^2)-\sqrt{3}(t_1^1+t_1^2)}{2\sqrt{2}} & \frac{1-\sqrt{2}t_0^2+\sqrt{3}t_2^2+\sqrt{3}t_{-2}^2}{2} & \frac{\sqrt{3}(t_{-1}^1-t_{-1}^2)-\sqrt{3}(t_1^1-t_1^2)}{2\sqrt{2}} & \frac{\sqrt{3}t_2^2-\sqrt{3}t_{-2}^2}{2} \\ \frac{1-\sqrt{2}t_0^2}{2} & \frac{\sqrt{3}(t_{-1}^1-t_{-1}^2)-\sqrt{3}(t_1^1-t_1^2)}{2\sqrt{2}} & 1-\sqrt{\frac{3}{2}}t_0^1+\frac{t_0^2}{\sqrt{2}} & \frac{-\sqrt{3}(t_{-1}^1-t_{-1}^2)-\sqrt{3}(t_1^1-t_1^2)}{2\sqrt{2}} \\ \frac{\sqrt{3}(t_{-1}^1+t_{-1}^2)+\sqrt{3}(t_1^1+t_1^2)}{2\sqrt{2}} & \frac{\sqrt{3}t_{-2}^2-\sqrt{3}t_2^2}{2} & \frac{\sqrt{3}(t_{-1}^1-t_{-1}^2)+\sqrt{3}(t_1^1-t_1^2)}{2\sqrt{2}} & \frac{1-\sqrt{2}t_0^2-\sqrt{3}t_2^2-\sqrt{3}t_{-2}^2}{2} \end{bmatrix} \quad (5)$$

For the above density matrix to be totally symmetric, entries in the last column and last row should vanish. Therefore, we have the following conditions:

$$t_2^2 = t_{-2}^2 \quad (6)$$

$$t_{-1}^1 = -t_1^1 \quad (7)$$

$$t_1^2 = -t_{-1}^2 \quad (8)$$

$$\frac{t_0^2}{\sqrt{2}} + \sqrt{3}t_2^2 = \frac{1}{2}. \quad (9)$$

Observe that, with these conditions the above density matrix can be identified with a spin-1 system (ρ^1) in $|1m\rangle$ basis. As ρ^1 is a semipositive definite matrix, we know that ρ_{PT}^{AB} with real t_q^k's and $\frac{t_0^2}{\sqrt{2}} + \sqrt{3}t_2^2 = \frac{1}{2}$ will also be semipositive definite matrix because they are connected by a unitary transformation. Therefore, a symmetric state with the above condition is always separable.

Now consider a spin-*j* system which is composed of two qudits (j_1, j_2). Unitary transformation from coupled to uncoupled basis of angular momentum is given by

$$|(j_1j_2)Jm\rangle = \sum_{m_1=-j_1}^{j_1} C(j_1j_2J; m_1m_2m)\,|j_1m_1j_2m_2\rangle$$

and uncoupled to coupled basis is

$$|j_1m_1j_2m_2\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2}\sum_{m=-J}^{J} C(j_1j_2J; m_1m_2m)\,|(j_1j_2)Jm\rangle$$

Now consider,

$$\langle j_1m_1'j_2m_2'|\,\rho\,|j_1m_1j_2m_2\rangle = \sum_{J,J',m,m'} C(j_1j_2J; m_1m_2m)C(j_1j_2J'; m_1'm_2'm') \times \langle (j_1j_2)J'm'|\,\rho\,|(j_1j_2)Jm\rangle\,.$$

$J = j_1+j_2\cdots|j_1-j_2|, m = -J, \ldots, J$ and $J' = j_1+j_2\cdots|j_1-j_2|, m' = -J', \ldots, J'$. For symmetric state, $\langle (j_1j_2)J'm'|\,\rho\,|(j_1j_2)Jm\rangle \neq 0$ for $J = J' = |j_1+j_2|$ and are zero for all the other values of J and J'. Now we consider a spin-j state composed of two spins $j_1 = j_2 = j/2$. This is equivalent to bipartite two-qudit symmetric state with each of the qudit state residing in $(j+1)$-dimensional Hilbert space. Thus,

$$\left\langle \left(\frac{j}{2}\frac{j}{2}\right) jm'\rho^j \left(\frac{j}{2}\frac{j}{2}\right) jm \right\rangle$$

in uncoupled basis can be written as,

$$\left\langle \frac{j}{2}m_1'\frac{j}{2}m_2'\rho^j \middle| \frac{j}{2}m_1\frac{j}{2}m_2 \right\rangle\Big\rangle = \sum_{J,J',m,m'} C\left(\frac{j}{2}\frac{j}{2}J; m_1m_2m\right) \times C\left(\frac{j}{2}\frac{j}{2}J'; m_1'm_2'm'\right)\left\langle \left(\frac{j}{2}\frac{j}{2}\right) J'm' \middle| \rho^j \times \middle| \left(\frac{j}{2}\frac{j}{2}\right) Jm \right\rangle,$$

where $J, J' = 0, \ldots, j$.

Therefore, the density matrix elements of the partially transposed matrix can be written as

$$\left\langle \frac{j}{2}m_1\frac{j}{2}m_2' \middle| \rho^j \middle| \frac{j}{2}m_1'\frac{j}{2}m_2 \right\rangle = \sum_{J,J',m,m'} C\left(\frac{j}{2}\frac{j}{2}J; m_1'm_2m\right) \times C\left(\frac{j}{2}\frac{j}{2}J'; m_1m_2'm'\right)\left\langle \left(\frac{j}{2}\frac{j}{2}\right) J'm' \middle| \rho^j \times \middle| \left(\frac{j}{2}\frac{j}{2}\right) Jm \right\rangle.$$

Demanding the PPT operation to be symmetry preserving (permutation symmetry) and the resultant density matrix to be the same as the initial density matrix in the coupled basis, we end up with equations which dictate the conditions for t_q^k's. This has been demonstrated in eqs (6)–(9) for the case of two qubit symmetric states. Explicit calculations of these conditions will be taken up elsewhere. Obviously, these conditions cannot differentiate bound entangled states from separable states in higher dimensions.

4. Trivariate description of spin-1 density matrix

A spin-j density matrix can be visualised as a statistical distribution of three non-commuting observables [21], namely J_x, J_y and J_z, as the operators τ^k_q's in (1) are homogeneous polynomials of rank k in the angular momentum operators J_x , J_y , J_z. In particular, following the well-known Weyl construction [22,23] for τ^k_q's in terms of angular momentum operators J_x, J_y and J_z, we have

$$\tau^k_q(\vec{J}) = \mathcal{N}_{kj}\,(\vec{J}\cdot\vec{\bigtriangledown})^k\, r^k\, Y^k_q(\hat{r})\,, \tag{10}$$

where

$$\mathcal{N}_{kj} = \frac{2^k}{k!}\sqrt{\frac{4\pi(2j-k)!(2j+1)}{(2j+k+1)!}} \tag{11}$$

are the normalisation factors and $Y^k_q(\hat{r})$ are the spherical harmonics. Using (10) for a spin-1 system, we have the following relations:

For $k=1$,

$$\tau^1_0 = \sqrt{\frac{3}{2}}J_z$$
$$\tau^1_1 = \frac{-\sqrt{3}}{2}(J_x + iJ_y)$$
$$\tau^1_{-1} = \frac{\sqrt{3}}{2}(J_x - iJ_y).$$

For $k=2$,

$$\tau^2_0 = \frac{1}{\sqrt{2}}(2J_z^2 - J_x^2 - J_y^2)$$
$$\tau^2_1 = \frac{-\sqrt{3}}{2}(J_xJ_z + J_zJ_x + i(J_yJ_z + J_zJ_y))$$
$$\tau^2_{-1} = \frac{\sqrt{3}}{2}(J_xJ_z + J_zJ_x - i(J_yJ_z + J_zJ_y))$$
$$\tau^2_2 = \frac{\sqrt{3}}{2}(J_x^2 - J_y^2 + i(J_xJ_y + J_yJ_x))$$
$$\tau^2_{-2} = \frac{\sqrt{3}}{2}(J_x^2 - J_y^2 - i(J_xJ_y + J_yJ_x)).$$

As the set of PPT separable states are characterised by eqs (6)–(9) and $t^k_q = \mathrm{Tr}(\rho\tau^k_q)$, conditions for separability can be equivalently represented in terms of first- and second-order moments of the angular momentum operators as

$$\mathrm{Tr}(\rho^1 J_y) = 0$$
$$\mathrm{Cov}(J_yJ_z) = \mathrm{Tr}\left[\rho^1\frac{(J_yJ_z + J_zJ_y)}{2}\right] = 0$$
$$\mathrm{Cov}(J_yJ_x) = \mathrm{Tr}\left[\rho^1\frac{(J_yJ_x + J_xJ_y)}{2}\right] = 0$$
$$\mathrm{Tr}[\rho(3J_z^2 - J^2)] + 3\mathrm{Tr}[\rho^1(J_x^2 - J_y^2)] = 1.$$

These conditions can be written for a two-qubit system in terms of the well-known Pauli spin matrices as

$$\mathrm{Tr}[\rho^{AB}(\sigma_y \otimes I_2 + I_1 \otimes \tau_y)] = 0,$$
$$\mathrm{Tr}[\rho^{AB}(\sigma_x \otimes \tau_y + \sigma_y \otimes \tau_x)] = 0,$$
$$\mathrm{Tr}[\rho^{AB}(\sigma_z \otimes \tau_y + \sigma_y \otimes \tau_z)] = 0,$$

where $\vec{\sigma}$ and $\vec{\tau}$ are the Pauli spin matrices in the space H^A and H^B respectively.

5. Example

Consider a triaxial system which is realised when a spin-1 nucleus with non-zero quadrupole moment is exposed to a combined external magnetic dipole and electric quadrupole field found in suitable crystal lattice [24]. If the dipole field is along the z-axis of the principle axis frame of the electric quadrupole field, then the resultant system is characterised by $t^1_0 \neq 0$, $t^1_1 = t^1_{-1} = 0$, $t^2_0 \neq 0$, $t^2_2 = t^2_{-2} \neq 0$. Consider one such system of the form

$$\rho = \frac{1}{3}\begin{bmatrix} 1+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}} & 0 & \frac{1}{2}-\frac{1}{\sqrt{6}} \\ 0 & 1-\sqrt{\frac{2}{3}} & 0 \\ \frac{1}{2}-\frac{1}{\sqrt{6}} & 0 & 1-\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}} \end{bmatrix}$$

with $t^1_0 = 1$, $t^1_1 = t^1_{-1} = 0$, $t^2_0 = \frac{1}{\sqrt{3}}$, $t^2_2 = t^2_{-2} = \frac{1}{2\sqrt{3}} - \frac{1}{3\sqrt{2}}$. Observe that the above t^k_q' satisfy our conditions for separability mentioned in eqs (6)–(9). Now to verify if it is PPT separable, we transform ρ to computational basis, and we have

$$\rho^c = U^\dagger \rho U =$$
$$\frac{1}{3}\begin{bmatrix} 1+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{2}-\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{2}-\frac{1}{\sqrt{6}} & \frac{1}{2}-\frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{2}-\frac{1}{\sqrt{6}} & \frac{1}{2}-\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{2}-\frac{1}{\sqrt{6}} & 0 & 0 & 1+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}} \end{bmatrix}.$$

Observe that under partial transposition ρ^c remains the same and we get the eigenvalues to be greater than or equal to zero explicitly, $\lambda_1 = \frac{1}{18}[6+\sqrt{6}+\sqrt{69-6\sqrt{6}}$, $\lambda_2 = \frac{1}{18}[6+\sqrt{6}-\sqrt{69-6\sqrt{6}}$, $\lambda_3 = \frac{3-\sqrt{6}}{9}$ and $\lambda_4 = 0$. Hence the state is separable.

6. Conclusion

With the help of PPT criterion, we have arrived at a set of separability conditions in terms of Fano statistical tensor parameters which show that if a 2-qubit symmetric state is characterised by real Fano statistical tensor parameters with the additional condition $\frac{t_0^2}{\sqrt{2}} + \sqrt{3}t_2^2 = \frac{1}{2}$, then the state is separable. Further, using trivariate description of the density matrix, we have expressed the above separability conditions in terms of the moments of angular momentum operators which are experimentally measurable. We have shown that a set of symmetric separable states are characterised by vanishing average expectation value of J_y and its covariance with J_x and J_y.

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