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Combination–combination synchronisation of time-delay chaotic systems for unknown parameters with uncertainties and external disturbances

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Abstract. In this article, an adaptive control method is proposed to study the combination–combination synchronisation phenomenon of four non-identical time-delayed chaotic systems for fully unknown parameters with parametric uncertainties and external disturbances. Based on the Lyapunov–Krasovskii functional theory, an appropriate adaptive controller is constructed so that a globally and asymptotically stable synchronisation state can be established between the master and the slave systems. Unknown parameters are identified by designing suitable parameter update laws. To elaborate the presented scheme, double-delay Rossler and time-delay Chen systems are considered as the master systems and time-delay Shimizu–Morioka and time-delay modified Lorenz systems are considered as the slave systems. Numerical simulations are presented to justify the theoretical analysis.

Keywords. Combination–combination synchronisation; time-delay chaotic systems; adaptive control; Lyapunov–Krasovskii functional theory.

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1. Introduction

Synchronisation is a phenomenon in which states of two or more oscillators alter according to a particular pattern. Synchronisation has been an extremely fascinating branch of nonlinear science since the last three decades. Synchronisation and chaotic dynamics have significant roles in real-world applications such as secure communication, biology, chemical reactions, physics, information processing etc. For these reasons, study of synchronisation phenomenon has been one of the most attracting and emerging branch of nonlinear dynamics. Since 1990, with the pioneering work of Pecora and Carroll [1], different approaches such as active control [2], adaptive control [3], sliding mode control [4], feedback control [5], impulsive control [6], time-delay control [7] and different synchronisation schemes such as generalised synchronisation [8,9], lag synchronisation [10], cluster synchronisation [11] and function projective synchronisation [12], have been explored in the literature to achieve synchronisation.

Initially, synchronisation of one master system and one slave system was assumed to be a difficult task, but later, synchronisation for more than one master and one slave system such as combination [13], combination–combination [14], compound–combination [15], double compound [16], dual combination [17] synchronisation has also been achieved.

In the field of nonlinear dynamics, synchronisation of time-delay systems has its own significance, as it is not appropriate to synchronise the master and the slave systems exactly at the same time because of unavoidable delays [18]. Time-delay dynamical systems exhibit more complexity and multistable dynamics in comparison to ordinary dynamical systems. Due to complexity of time-delay choatic and hyperchaotic systems, these systems are more secure for data transmission.

Stability of time-delay differential equations has also been studied by the researchers and Lyapunov–Krasovskii and Lyapunov–Razumikhin theorems have been suggested to tackle the stability of delay differential equations. Some of the prominent features of time-delay are: memory effect, coexistence of multiple attractors and the finite signal transmission speeds [19]. In real life, several models [20,21] have been investigated which can be presented in a more accurate way by using time delay. The aforesaid reasons motivated the researchers to investigate more results on time-delay systems.

In the last few years, different kinds of problems have been tackled on delay phenomena such as synchronisation in an array of nonlinearly coupled chaotic neural networks with coupling delays [22], synchronisation of non-identical chaotic neural networks with leakage delay and mixed time-varying delays [23], cluster synchronisation in nonlinearly coupled delayed networks of non-identical dynamic systems with time-varying delays by pinning control method [24] and synchronisation of switched delay networks with interval parameters uncertainty [25] etc.

Projective synchronisation has also been studied widely for time-delay chaotic systems through different approaches [26,27]. In the last few years, projective synchronisation for identical and non-identical time-delay dynamical systems has been investigated for fully unknown parameters by Ansari and Das [28].

Despite all the above-mentioned work, a very few researches are available on time-delay systems with more than one master system and one slave system. Recently, combination synchronisation of time-delay chaotic systems has been achieved for identical systems [29] and non-identical systems [30]. The work done in this paper is an extension of the problems considered in [29,30].

Motivated by the above discussions, problem of combination–combination synchronisation for non-identical time-delay chaotic systems is investigated under the effect of parametric uncertainties and external disturbances. In this work, by employing nonlinear control method, combination–combination synchronisation is achieved based on Lyapunov–Krasovskii functional theory [31,32].

In real-life applications, parameters are not always known and the role of external disturbances cannot be avoided. Therefore, the problem considered in this paper is significant. The main features of the presented work are: (i) To the best of our knowledge, problem of combination–combination synchronisation of time-delay chaotic systems with unknown parameters, parametric uncertainties and external disturbances has not been considered till now, (ii) several existing results [28–30] can be achieved as a particular case of the presented approach, (iii) this scheme is more useful for data transmission due to the complexity of time-delay systems and more number of master and slave systems, as information signal can be dissevered with different chaotic signals.

The paper is organised as follows: In §2, problem is formulated. In §3 description of all four time-delay systems are given. Controllers are designed and computational results are presented in §4 and §5 is conclusion, where the main features of this work are highlighted.

2. Problem formulation

Let the two time-delay master systems be

$$\begin{cases} \dot{u} = f_1(u) + g_1(u)P_1 + h_1(u(t-T_1))Q_1 \\ \quad + \Delta B_1 u + D_1(t), \\ u = \theta(t), -T_1 \le t \le 0, \\ \dot{v} = f_2(v) + g_2(v)R_1 + h_2(v(t-T_2))S_1 \\ \quad + \Delta B_2 v + D_2(t), \\ v = \eta(t), -T_2 \le t \le 0. \end{cases} \tag{1}$$

Let the two time-delay slave systems be

$$\begin{cases} \dot{w} = f_3(w) + g_3(w)P_2 + h_3(w(t-T_3))Q_2 \\ \quad + \Delta B_3 w + D_3(t) + \sigma_1, \\ w = \varphi(t), -T_3 \le t \le 0, \\ \dot{x} = f_4(x) + g_4(x)R_2 + h_4(x(t-T_4))S_2 + \Delta B_4 x \\ \quad + D_4(t) + \sigma_2, \\ x = \phi(t), -T_4 \le t \le 0, \end{cases} \tag{2}$$

where $u, v, w, x \in R^n$ are the vectors of state variables of the master and the slave systems. $f_1(u), f_2(v), f_3(w), f_4(x) \in R^n, g_1(u) \in R^{n\times m_1}, h_1(u(t-T_1)) \in R^{n\times m_2}, g_2(v) \in R^{n\times m_3}, h_2(v(t-T_2)) \in R^{n\times m_4}, g_3(w) \in R^{n\times m_5}, h_3(w(t-T_3)) \in R^{n\times m_6}, g_4(x) \in R^{n\times m_7}, h_4(x(t-T_4)) \in R^{n\times m_8}$ are nonlinear functions and $P_1 \in R^{m_1}, Q_1 \in R^{m_2}, R_1 \in R^{m_3}, S_1 \in R^{m_4}, P_2 \in R^{m_5}, Q_2 \in R^{m_6}, R_2 \in R^{m_7}, S_2 \in R^{m_8}$ are parameter vectors of the master and the slave systems. Time delays for the drive and the response systems are T_1, T_2, T_3, T_4, respectively and $\sigma_1, \sigma_2 \in R^n$ are the controllers. Trajectories of the solutions in past are presented by $\theta(t), \eta(t), \varphi(t), \phi(t)$.

Here, $\Delta B_1, \Delta B_2, \Delta B_3, \Delta B_4 \in R^{n\times n}$ are parametric uncertainties with bounded norm $\|\Delta B_1\| \le \xi_1, \|\Delta B_2\| \le \xi_2, \|\Delta B_3\| \le \xi_3, \|\Delta B_4\| \le \xi_4$ such that $\xi_1, \xi_2, \xi_3, \xi_4$ are any positive numbers. Similarly, external disturbances $D_1(t), D_1(t), D_1(t), D_1(t) \in R^n$ are bounded such as $|D_1(t)| \le \rho_1, |D_2(t)| \le \rho_2, |D_3(t)| \le \rho_3, |D_4(t)| \le \rho_4$ where $\rho_1, \rho_2, \rho_3, \rho_4 > 0$.

DEFINITION 1 [14]
Response systems (2) will achieve combination–combination synchronisation with drive systems (1), if

there exist constant matrices $P, Q, R, S \in R^{n\times n}$ and $P \neq 0$ or $Q \neq 0$ such that

$$\lim_{t\to\infty} \|Px + Qw - Rv - Su\| = 0,$$

where $\|\cdot\|$ is the matrix norm.

Remark 1. The constant matrices P, Q, R, S, are called scaling matrices. For the convenience of our discussion, we assume matrices $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$, $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n)$, $R = \mathrm{diag}(r_1, r_2, \ldots, r_n)$, $S = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ and diagonal elements of these matrices are termed as scaling factors.

Theorem 1. *Response systems (2) will achieve combination–combination synchronisation with drive systems (1) for the controller defined as*

$$\begin{aligned} U = {} & -Pg_4(x)\hat{R}_2 - Ph_4(x(t-T_4))\hat{S}_2 - Qg_3(w)\hat{P}_2 \\ & -Qh_3(w(t-T_3))\hat{Q}_2 + Rg_2(v)\hat{R}_1 \\ & +Rh_2(v(t-T_2))\hat{S}_1 \\ & +Sg_1(u)\hat{P}_1 + Sh_1(u(t-T_1))\hat{Q}_1 \\ & -Pf_4(x) - Qf_3(w) \\ & +Rf_2(v) + Sf_1(u) - P\Delta B_4 x - PD_4(t) \\ & -Q\Delta B_3 w \\ & -QD_3(t) + R\Delta B_2 v + RD_2(t) \\ & +S\Delta B_1 u + SD_1(t) \\ & -(0.5+m)E(t), \end{aligned} \tag{3}$$

and parameters update laws as

$$\begin{cases} \dot{\hat{P}}_1 = \{-Sg_1(u)\}'E + \tilde{P}_1, \\ \dot{\hat{Q}}_1 = \{-Sh_1(u(t-T_1))\}'E + \tilde{Q}_1, \\ \dot{\hat{R}}_1 = \{-Rg_2(v)\}'E + \tilde{R}_1, \\ \dot{\hat{S}}_1 = \{-Rh_2(v(t-T_2))\}'E + \tilde{S}_1, \\ \dot{\hat{P}}_2 = \{Qg_3(w)\}'E + \tilde{P}_2, \\ \dot{\hat{Q}}_2 = \{Qh_3(w(t-T_3))\}'E + \tilde{Q}_2, \\ \dot{\hat{R}}_2 = \{Pg_4(x)\}'E + \tilde{R}_2, \\ \dot{\hat{S}}_2 = \{Ph_4(x(t-T_4))\}'E + \tilde{S}_2, \end{cases} \tag{4}$$

where m is any positive constant, P_1' denotes transpose of P_1, and $\hat{P}_1, \hat{Q}_1, \hat{R}_1, \hat{S}_1, \hat{P}_2, \hat{Q}_2, \hat{R}_2, \hat{S}_2$ are the estimated parameters of $P_1, Q_1, R_1, S_1, P_2, Q_2, R_2, S_2$, and $\tilde{P}_1 = P_1 - \hat{P}_1$, $\tilde{Q}_1 = Q_1 - \hat{Q}_1$, $\tilde{R}_1 = R_1 - \hat{R}_1$, $\tilde{S}_1 = S_1 - \hat{S}_1$, $\tilde{P}_2 = P_2 - \hat{P}_2$, $\tilde{Q}_2 = Q_2 - \hat{Q}_2$, $\tilde{R}_2 = R_2 - \hat{R}_2$, $\tilde{S}_2 = S_2 - \hat{S}_2$.

Proof. Suppose error is defined as

$$E = Px + Qw - Rv - Su, \tag{5}$$

where P, Q, R, S are the scaling matrices.

Then error dynamical system will become

$$\dot{E} = P\dot{x} + Q\dot{w} - R\dot{v} - S\dot{u}. \tag{6}$$

Our aim is to design the effective controller $U = P\sigma_2 + Q\sigma_1$ and laws to estimate the unknown parameters so that the condition of Lyapunov–Krasovskii functional stability is satisfied and the desired synchronisation is attained. Then

$$\begin{aligned} \dot{E} &= P\dot{x} + Q\dot{w} - R\dot{v} - S\dot{u} \\ &= P(f_4(x) + g_4(x)R_2 + h_4(x(t-T_4))S_2 \\ &\quad + \Delta B_4 x + D_4(t) \\ &\quad + \sigma_2) + Q(f_3(w) + g_3(w)P_2 \\ &\quad + h_3(w(t-T_3))Q_2 + \Delta B_3 w \\ &\quad + D_3(t) + \sigma_1) - R(f_2(v) \\ &\quad + g_2(v)R_1 + h_2(v(t-T_2))S_1 \\ &\quad + \Delta B_2 v + D_2(t)) - S(f_1(u) + g_1(u)P_1 \\ &\quad + h_1(u(t-T_1))Q_1 \\ &\quad + \Delta B_1 u + D_1(t)). \end{aligned} \tag{7}$$

By using the controller (3), the error dynamical system (7) reduces to

$$\begin{aligned} \dot{E} = {} & Pg_4(x)(R_2 - \hat{R}_2) + Ph_4(x(t-T_4))(S_2 - \hat{S}_2) \\ & + Qg_3(w)(P_2 - \hat{P}_2) \\ & + Qh_3(w(t-T_3))(Q_2 - \hat{Q}_2) \\ & - Rg_2(v)(R_1 - \hat{R}_1) - Rh_2(v(t-T_2))(S_1 - \hat{S}_1) \\ & - Sg_1(u)(P_1 - \hat{P}_1) - Sh_1(u(t-T_1))(Q_1 - \hat{Q}_1) \\ & - (0.5+m)E(t). \end{aligned} \tag{8}$$

Suppose Lyapunov–Krasovskii functional is defined as

$$\begin{aligned} K = {} & 0.5E'(t)E(t) + 0.5\int_{-T}^{0} E'(t+\rho)E(t+\rho)\mathrm{d}\rho \\ & + 0.5(\tilde{P}_1'\tilde{P}_1 + \tilde{Q}_1'\tilde{Q}_1 + \tilde{R}_1'\tilde{R}_1 + \tilde{S}_1'\tilde{S}_1 \\ & + \tilde{P}_2'\tilde{P}_2 + \tilde{Q}_2'\tilde{Q}_2 + \tilde{R}_2'\tilde{R}_2 + \tilde{S}_2'\tilde{S}_2). \end{aligned} \tag{9}$$

The derivative of K will be

$$\begin{aligned}\dot{K} &= E'(t)\dot{E}(t)\\ &\quad +0.5E'(t)E(t)-0.5E'(t-T)E(t-T)\\ &\quad +(\dot{\tilde{P}}_1'\tilde{P}_1+\dot{\tilde{Q}}_1'\tilde{Q}_1+\dot{\tilde{R}}_1'\tilde{R}_1+\dot{\tilde{S}}_1'\tilde{S}_1+\dot{\tilde{P}}_2'\tilde{P}_2\\ &\quad +\dot{\tilde{Q}}_2'\tilde{Q}_2+\dot{\tilde{R}}_2'\tilde{R}_2+\dot{\tilde{S}}_2'\tilde{S}_2)).\end{aligned} \tag{10}$$

By using (8) and (4), we get

$$\begin{aligned}\dot{K} &= E'(t)\{Pg_4(x)(R_2-\hat{R}_2)\\ &\quad +Ph_4(x(t-T_4))(S_2-\hat{S}_2)\\ &\quad +Qg_3(w)(P_2-\hat{P}_2)\\ &\quad +Qh_3(w(t-T_3))(Q_2-\hat{Q}_2)\\ &\quad -Rg_2(v)(R_1-\hat{R}_1)\\ &\quad -Rh_2(v(t-T_2))(S_1-\hat{S}_1)\\ &\quad -Sg_1(u)(P_1-\hat{P}_1)\\ &\quad -Sh_1(u(t-T_1))(Q_1-\hat{Q}_1)\\ &\quad -(0.5+m)E(t)\}\\ &\quad +0.5E'(t)E(t)-0.5E'(t-T)E(t-T)\\ &\quad -(\{-Sg_1(u)\}'E+\tilde{P}_1)'\tilde{P}_1\\ &\quad -(\{-Sh_1(u(t-T_1))\}'E+\tilde{Q}_1)'\tilde{Q}_1\\ &\quad -(\{-Rg_2(v)\}'E+\tilde{R}_1)'\tilde{R}_1\\ &\quad -(\{-Rh_2(v(t-T_2))\}'E+\tilde{S}_1)'\tilde{S}_1\\ &\quad -(\{Qg_3(w)\}'E+\tilde{P}_2)'\tilde{P}_2\\ &\quad -\{Qh_3(w(t-T_3))\}'E+\tilde{Q}_2)'\tilde{Q}_2\\ &\quad -(\{Pg_4(x)\}'E+\tilde{R}_2)'\tilde{R}_2\\ &\quad -(\{Ph_4(x(t-T_4))\}'E+\tilde{S}_2)'\tilde{S}_2.\end{aligned} \tag{11}$$

Thus, eq. (11) reduces to

$$\begin{aligned}\dot{K} &= -mE'(t)E(t)-0.5E'(t-T)E(t-T)\\ &\quad -(\tilde{P}_1'\tilde{P}_1+\tilde{Q}_1'\tilde{Q}_1+\tilde{R}_1'\tilde{R}_1+\tilde{S}_1'\tilde{S}_1+\tilde{P}_2'\tilde{P}_2\\ &\quad +\tilde{Q}_2'\tilde{Q}_2+\tilde{R}_2'\tilde{R}_2+\tilde{S}_2'\tilde{S}_2)),\end{aligned} \tag{12}$$

which is negative definite. Hence, the asymptotical and global synchronisation state will be achieved between the master and the slave systems. □

3. Systems' descriptions

Double delay Rossler system [28] is considered as the first master system

$$\begin{cases}\dot{u}_1 = -u_2-u_3+\alpha_1 u_1(t-T_1)+\mu_1 u_1(t-T_1'),\\ \dot{u}_2 = u_1+\beta_1 u_2,\\ \dot{u}_3 = \beta_1+u_1u_3-\gamma_1 u_3,\end{cases} \tag{13}$$

which shows chaotic behaviour for $\alpha_1 = 0.2, \mu_1 = 0.5, \beta_1 = 0.2, \gamma_1 = 5.7$, and time delays $T_1 = 1.0, T_1' = 2.0$.

Suppose double delay Rossler system with parametric uncertainties and external disturbances is

$$\begin{cases}\dot{u}_1 = -u_2-u_3+\alpha_1 u_1(t-T_1)+\mu_1 u_1(t-T_1')\\ \qquad -0.04u_2+0.1\cos(20t),\\ \dot{u}_2 = u_1+\beta_1 u_2+0.01u_3-0.02u_1\\ \qquad +0.2\sin(30t),\\ \dot{u}_3 = \beta_1+u_1u_3-\gamma_1 u_3+0.05u_1\\ \qquad +0.3\sin(50t),\end{cases} \tag{14}$$

where uncertainty term

$$\Delta B_1 = \begin{pmatrix} 0 & -0.04 & 0\\ -0.02 & 0 & 0.01\\ 0.05 & 0 & 0\end{pmatrix}$$

and disturbance term

$$D_1(t) = \begin{pmatrix} 0.1\cos(20t)\\ 0.2\sin(30t)\\ 0.3\sin(50t)\end{pmatrix}.$$

Figure 1a shows the chaotic attractor of the double delay Rossler system and figure 1b shows the chaotic attractor of Rossler system containing parametric uncertainties and external disturbances. Time-delay Chen system [30] is considered as the second master system which is given by the equations

$$\begin{cases}\dot{v}_1 = \alpha_2(v_2-v_1(t-T_2)),\\ \dot{v}_2 = (\gamma_2-\alpha_2)v_1(t-T_2)-v_1v_3+\gamma_2 v_2,\\ \dot{v}_3 = v_1v_2-\beta_2 v_3(t-T_2).\end{cases} \tag{15}$$

This system exhibits chaotic behaviour for $\alpha_2 = 35, \beta_2 = 3, \gamma_2 = 27$, and time delay $T_2 = 0.005$. Suppose time-delay Chen system with parametric uncertainties and external disturbances is

$$\begin{cases}\dot{v}_1 = \alpha_2(v_2-v_1(t-T_2))+0.4v_1+0.6v_2\\ \qquad +0.3\sin(10t),\\ \dot{v}_2 = (\gamma_2-\alpha_2)v_1(t-T_2)-v_1v_3\\ \qquad +\gamma_2 v_2+0.4v_3+0.4\cos(20t),\\ \dot{v}_3 = v_1v_2-\beta_2 v_3(t-T_2)+0.2v_2+0.5\sin(20t),\end{cases} \tag{16}$$

where uncertainty term

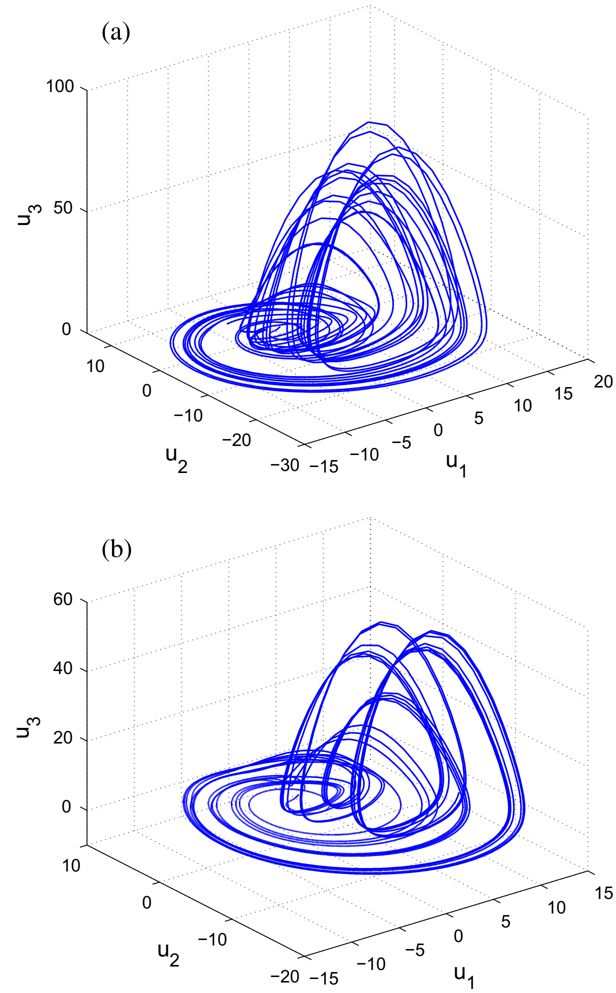

Figure 1. (**a**) Chaotic attractor of time-delay Rossler system and (**b**) chaotic attractor of time-delay Rossler system with uncertainties and disturbances.

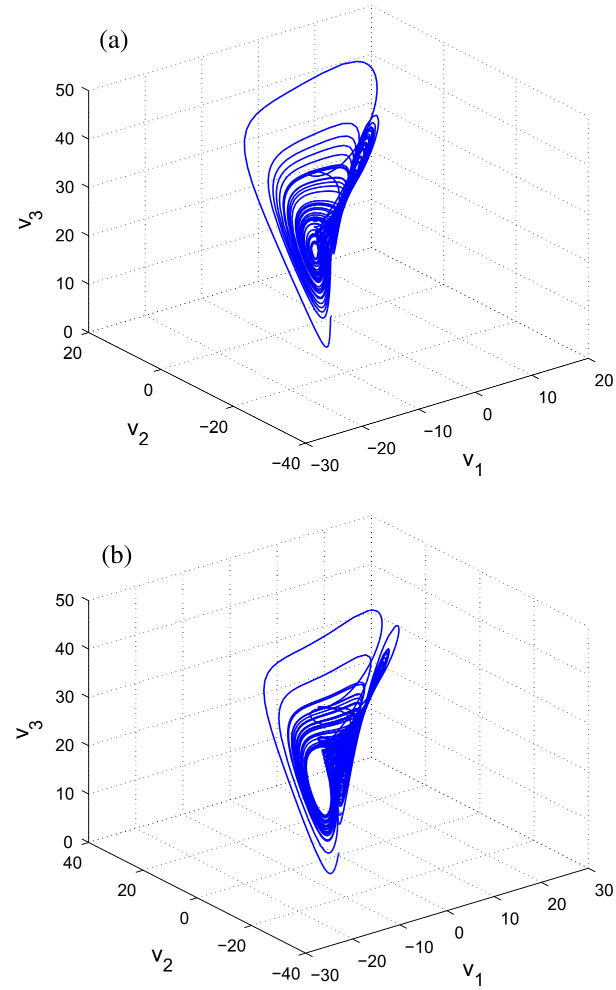

Figure 2. (**a**) Chaotic attractor of time-delay Chen system and (**b**) chaotic attractor of time-delay Chen system with uncertainties and disturbances.

$$\Delta B_2 = \begin{pmatrix} 0.4 & 0.6 & 0 \\ 0 & 0 & 0.4 \\ 0 & 0.2 & 0 \end{pmatrix}$$

and disturbance term

$$D_2(t) = \begin{pmatrix} 0.3\sin(10t) \\ 0.4\cos(20t) \\ 0.5\sin(20t) \end{pmatrix}.$$

Figure 2a shows the chaotic attractor of time-delay Chen system and figure 2b shows the chaotic attractor of Chen system containing parametric uncertainties and external disturbances.

Time-delay Shimizu–Morioka system [33] is considered as the first slave system which is given by the following equations:

$$\begin{cases} \dot{w}_1 = w_2, \\ \dot{w}_2 = w_1 - \alpha_3 w_2(t - T_3) - w_1 w_3, \\ \dot{w}_3 = -\beta_3 w_3 + w_1^2. \end{cases} \tag{17}$$

This system exhibits chaotic behaviour for the values of parameters $\alpha_3 = 0.75$, $\beta_3 = 0.45$, and time delay $T_3 = 0.15$.

Suppose time-delay Shimizu–Morioka system with parametric uncertainties and external disturbances is

$$\begin{cases} \dot{w}_1 = w_2 + 0.08w_2 + 0.3\sin(30t), \\ \dot{w}_2 = w_1 - \alpha_3 w_2(t - T_3) \\ \qquad - w_1 w_3 + 0.5w_1 - 0.02w_3 \\ \qquad - 0.4\cos(30t), \\ \dot{w}_3 = -\beta_3 w_3 + w_1^2 + 0.07w_2 + 0.2\sin(20t), \end{cases} \tag{18}$$

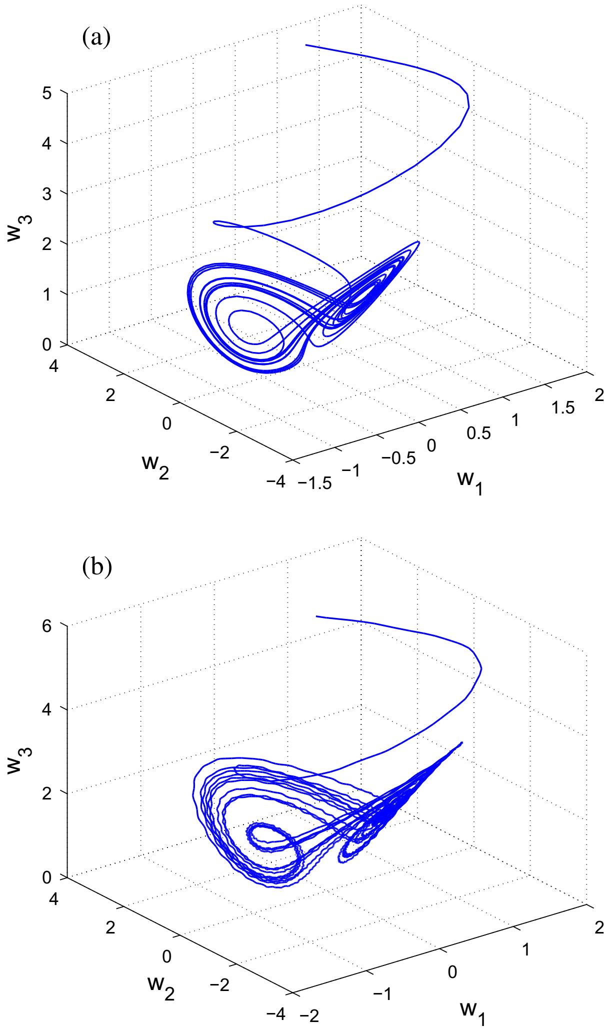

Figure 3. (**a**) Chaotic attractor of time-delay Shimizu–Morioaka system and (**b**) chaotic attractor of time-delay Shimizu–Morioka system with uncertainties and disturbances.

where uncertainty term

$$\Delta B_3 = \begin{pmatrix} 0 & 0.08 & 0 \\ 0.5 & 0 & -0.02 \\ 0 & 0.07 & 0 \end{pmatrix}$$

and disturbance term

$$D_3(t) = \begin{pmatrix} 0.3\sin(30t) \\ -0.4\cos(30t) \\ 0.2\sin(20t) \end{pmatrix}.$$

Figure 3a shows the chaotic attractor of time-delay Simizu–Morioka system and figure 3b shows the chaotic attractor of system (18) containing parametric uncertainties and external disturbances.

Modified time-delay Lorenz system [29] is considered as the second slave system which is given by the following equations:

$$\begin{cases} \dot{x}_1 = \alpha_4(x_2 - x_1), \\ \dot{x}_2 = \gamma_4 x_1 - x_1 x_3 - x_2(t - T_4), \\ \dot{x}_3 = x_1 x_2 - \beta_4 x_3(t - T_4'). \end{cases} \tag{19}$$

This system exhibits chaotic behaviour for $\alpha_4 = 0.9$, $\beta_4 = 0.1$, $\gamma_4 = 2.5$, and time delays $T_4 = 1$, $T_4' = 2$.

Suppose time-delay modified Lorenz system with parametric uncertainties and external disturbances is

$$\begin{cases} \dot{x}_1 = \alpha_4(x_2 - x_1) - 0.05x_2 \\ \qquad + 0.03x_3 + 0.3\cos(10t), \\ \dot{x}_2 = \gamma_4 x_1 - x_1 x_3 \\ \qquad - x_2(t - T_4) + 0.4x_1 + 0.2\sin(50t), \\ \dot{x}_3 = x_1 x_2 - \beta_4 x_3(t - T_4') - 0.2x_2 \\ \qquad + 0.4\cos(20t), \end{cases} \tag{20}$$

where uncertainty term

$$\Delta B_4 = \begin{pmatrix} 0 & -0.05 & 0.03 \\ 0.4 & 0 & 0 \\ 0 & -0.2 & 0 \end{pmatrix}$$

and disturbance term

$$D_4(t) = \begin{pmatrix} 0.3\cos(10t) \\ 0.2\sin(50t) \\ 0.4\cos(20t) \end{pmatrix}.$$

Figure 4a shows the chaotic attractor of time-delay modified Lorenz system and figure 4b shows the chaotic attractor of the system containing parametric uncertainties and external disturbances.

4. Illustration of synchronisation scheme and graphical results

Suppose slave systems (18) and (20) with controllers are

$$\begin{cases} \dot{w}_1 = w_2 + 0.08w_2 + 0.3\sin(30t) + \sigma_1, \\ \dot{w}_2 = w_1 - \alpha_3 w_2(t - T_3) - w_1 w_3 \\ \qquad + 0.5w_1 - 0.02w_3 \\ \qquad - 0.4\cos(30t) + \sigma_2, \\ \dot{w}_3 = -\beta_3 w_3 + w_1^2 + 0.07w_2 + 0.2\sin(20t) + \sigma_3, \end{cases} \tag{21}$$

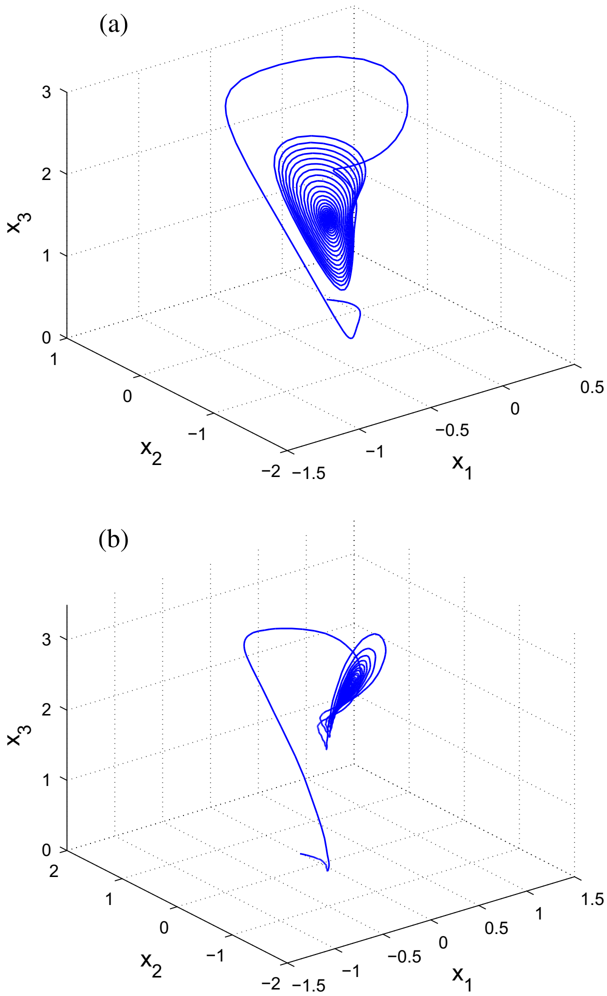

Figure 4. (**a**) Chaotic attractor of time-delay modified Lorenz system and (**b**) chaotic attractor of time-delay modified Lorenz system with uncertainties and disturbances.

and

$$\begin{cases} \dot{x}_1 = \alpha_4(x_2 - x_1) - 0.05x_2 \\ \qquad + 0.03x_3 + 0.3\cos(10t) + \sigma_1', \\ \dot{x}_2 = \gamma_4 x_1 - x_1x_3 - x_2(t - T_4) \\ \qquad + 0.4x_1 + 0.2\sin(50t) + \sigma_2', \\ \dot{x}_3 = x_1x_2 - \beta_4 x_3(t - T_4') \\ \qquad - 0.2x_2 + 0.4\cos(20t) + \sigma_3', \end{cases} \tag{22}$$

where $\sigma_1, \sigma_2, \sigma_3, \sigma_1', \sigma_2', \sigma_3'$ are the controllers.

Suppose errors are defined as

$$\begin{cases} E_1 = p_1x_1 + q_1w_1 - r_1v_1 - s_1u_1, \\ E_2 = p_2x_2 + q_2w_2 - r_2v_2 - s_2u_2, \\ E_3 = p_3x_3 + q_3w_3 - r_3v_3 - s_3u_3, \end{cases} \tag{23}$$

where $p_i, q_i, r_i, s_i, i = 1, 2, 3$ are scaling factors, and the combined controllers are

$$\begin{aligned} U_1 &= p_1\sigma_1' + q_1\sigma_1, \\ U_2 &= p_2\sigma_2' + q_2\sigma_2, \\ U_3 &= p_3\sigma_3' + q_3\sigma_3. \end{aligned} \tag{24}$$

Then, the desired synchronisation can be achieved by using the following controllers:

$$\begin{cases} U_1 = -p_1\hat{\alpha}_4(x_2 - x_1) + r_1\hat{\alpha}_2(v_2 - v_1(t - T_2)) \\ \qquad + s_1\hat{\alpha}_1u_1(t - T_1) \\ \qquad + s_1\hat{\mu}_1u_1(t - T_1') + 0.05p_1x_2 \\ \qquad - 0.03p_1x_3 - 0.3p_1\cos(10t) \\ \qquad - q_1w_2 - 0.08q_1w_2 \\ \qquad - 0.3q_1\sin(30t) + 0.4r_1v_1 \\ \qquad + 0.6r_1v_2 + 0.3r_1\sin(10t) \\ \qquad - s_1u_2 - s_1u_3 - 0.04s_1u_2 + 0.1s_1\cos(20t) \\ \qquad - (0.5 + m)E_1, \\ U_2 = -p_2\hat{\gamma}_4x_1 + q_2\hat{\alpha}_3w_2(t - T_3) \\ \qquad + r_2(\hat{\gamma}_2 - \hat{\alpha}_2)v_1(t - T_2) \\ \qquad + r_2\hat{\gamma}_2v_2 + s_2\hat{\beta}_1u_2 + p_2x_1x_3 \\ \qquad + p_2x_2(t - T_4) - 0.4p_2x_1 \\ \qquad - 0.2p_2\sin(50t) \\ \qquad - q_2w_1 + q_2w_1w_3 - 0.5q_2w_1 \\ \qquad + 0.02q_2w_3 + 0.4q_2\cos(30t) \\ \qquad - r_2v_1v_3 + 0.4r_2v_3 \\ \qquad + 0.4r_2\cos(20t) + s_2u_1 \\ \qquad + 0.01s_2u_3 - 0.02s_2u_1 \\ \qquad + 0.2s_2\sin(30t) - (0.5 + m)E_2, \\ U_3 = p_3\hat{\beta}_4x_3(t - T_4') + q_3\hat{\beta}_3w_3 - r_3\hat{\beta}_2v_3(t - T_2) \\ \qquad + s_3\hat{\beta}_1 - s_3\hat{\gamma}_1u_3 - p_3x_1x_2 \\ \qquad + 0.2p_3x_2 - 0.4p_3\cos(20t) \\ \qquad - q_3w_1^2 - 0.07q_3w_2 \\ \qquad - 0.2q_3\sin(20t) + r_3v_1v_2 \\ \qquad + 0.2r_3v_2 + 0.5r_3\sin(20t) \\ \qquad + s_3u_1u_3 + 0.05s_3u_1 \\ \qquad + 0.3s_3\sin(50t) - (0.5 + m)E_3, \end{cases} \tag{25}$$

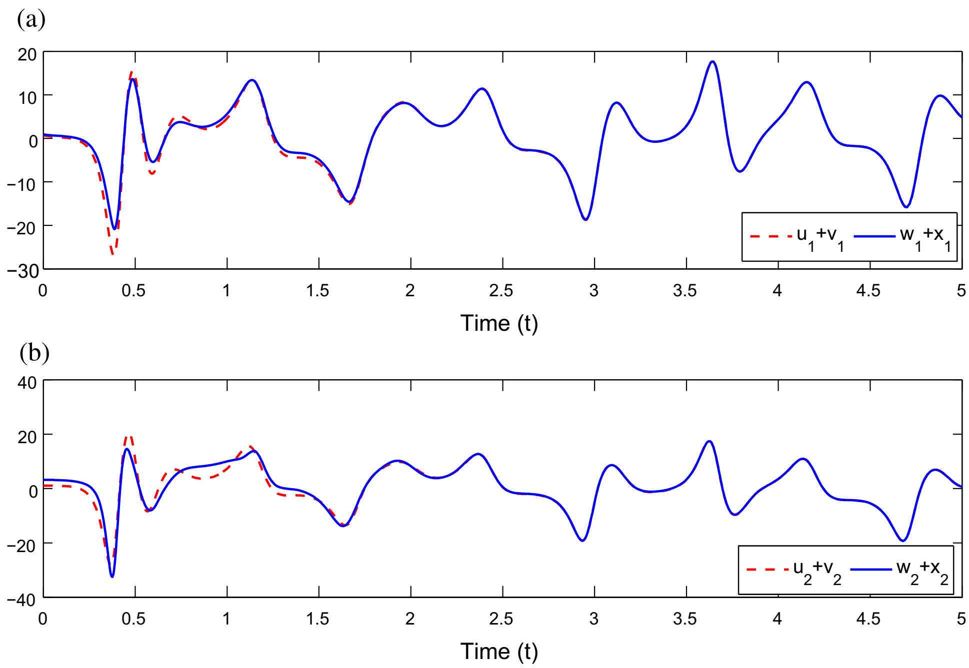

Figure 5. (**a**) Synchronisation for state variables $(u_1 + v_1, w_1 + x_1)$ and (**b**) synchronisation for state variables $(u_2 + v_2, w_2 + x_2)$.

and the parameter updating equations as

$$\begin{cases} \dot{\hat{\alpha}}_1 = -s_1 u_1(t - T_1)E_1 + \bar{\alpha}_1, \\ \dot{\hat{\mu}}_1 = -s_1 u_1(t - T_1')E_1 + \bar{\mu}_1, \\ \dot{\hat{\beta}}_1 = -s_3 E_3 - s_2 u_2 E_2 + \bar{\beta}_1, \\ \dot{\hat{\gamma}}_1 = s_3 u_3 E_3 + \bar{\gamma}_1, \\ \dot{\hat{\alpha}}_2 = -r_1(v_2 - v_1(t - T_2))E_1 \\ \qquad + r_2 v_1(t - T_2))E_2 + \bar{\alpha}_2, \\ \dot{\hat{\beta}}_2 = r_3 v_3(t - T_2)E_3 + \bar{\beta}_2, \\ \dot{\hat{\gamma}}_2 = -r_2 v_1(t - T_2)E_2 - r_2 v_2 E_2 + \bar{\gamma}_2, \\ \dot{\hat{\alpha}}_3 = -q_2 w_2(t - T_3)E_2 + \bar{\alpha}_3, \\ \dot{\hat{\beta}}_3 = -q_3 w_3 E_3 + \bar{\beta}_3, \\ \dot{\hat{\alpha}}_4 = p_1(x_2 - x_1)E_1 + \bar{\alpha}_4, \\ \dot{\hat{\beta}}_4 = -p_3 x_3(t - T_4')E_3 + \bar{\beta}_4, \\ \dot{\hat{\gamma}}_4 = p_2 x_1 E_2 + \bar{\gamma}_4, \end{cases} \tag{26}$$

where, $\bar{\alpha}_1 = (\alpha_1 - \hat{\alpha}_1), \bar{\beta}_1 = (\beta_1 - \hat{\beta}_1), \bar{\mu}_1 = (\mu_1 - \hat{\mu}_1), \bar{\gamma}_1 = (\gamma_1 - \hat{\gamma}_1), \bar{\alpha}_2 = (\alpha_2 - \hat{\alpha}_2), \bar{\beta}_2 = (\beta_2 - \hat{\beta}_2), \bar{\gamma}_2 = (\gamma_2 - \hat{\gamma}_2), \bar{\alpha}_3 = (\alpha_3 - \hat{\alpha}_3), \bar{\beta}_3 = (\beta_3 - \hat{\beta}_3), \bar{\alpha}_4 = (\alpha_4 - \hat{\alpha}_4), \bar{\beta}_4 = (\beta_4 - \hat{\beta}_4), \bar{\gamma}_4 = (\gamma_4 - \hat{\gamma}_4)$ and $\hat{\alpha}_1, \hat{\mu}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2, \hat{\alpha}_3, \hat{\beta}_3, \hat{\alpha}_4, \hat{\beta}_4, \hat{\gamma}_4$ are the estimated values of $\alpha_1, \mu_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \alpha_4, \beta_4, \gamma_4$, respectively and m is a positive constant which can be chosen arbitrarily.

It can be proved as Theorem 1.

Remark 2. If the scaling factors $p_1 = p_2 = p_3 = 0$ are chosen, then combination synchronisation will be attained by time-delay Shimizu–Morioka system (21) with delay Rossler system (14) and time-delay Chen system (16).

Remark 3. If the scaling factors $q_1 = q_2 = q_3 = 0$ are chosen, then combination synchronisation will be attained by time-delay modified Lorenz system (22) with delay Rossler system (14) and time-delay Chen system (16).

Remark 4. If scaling factors $p_1 = p_2 = p_3 = 0, q_1 = q_2 = q_3 = 1$ and $r_1 = r_2 = r_3 = 0, s_1 = s_2 = s_3$ are chosen, then projective synchronisation will be attained by time-delay Shimizu–Morioka system (21) with time delay Rossler system (14).

Remark 5. If scaling factors $p_1 = p_2 = p_3 = 1, q_1 = q_2 = q_3 = 0$ and $r_1 = r_2 = r_3 = 0, s_1 = s_2 = s_3$ are

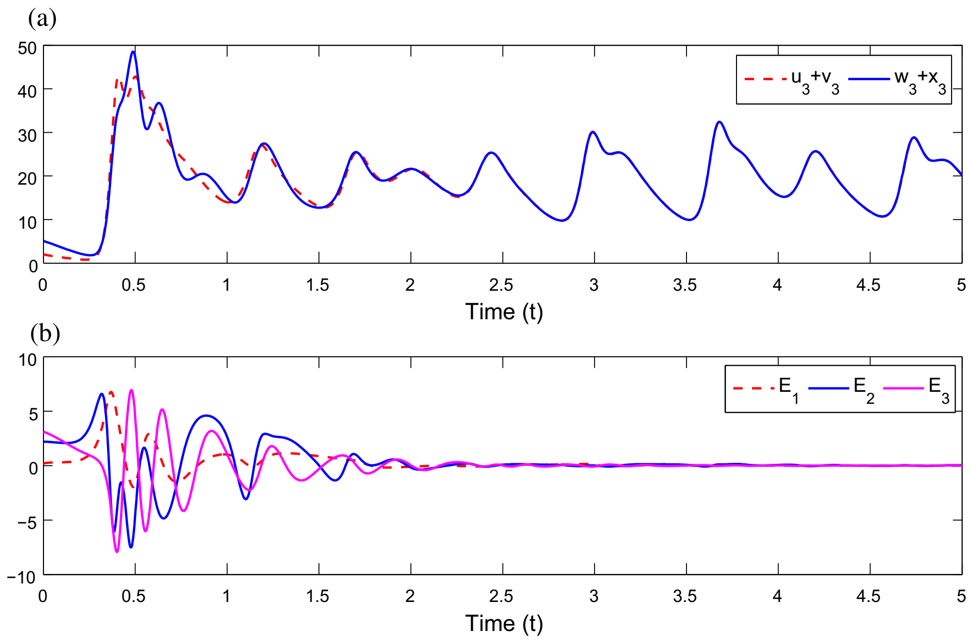

Figure 6. (**a**) Synchronisation for state variables $(u_3 + v_3, w_3 + x_3)$ and (**b**) synchronisation errors E_1, E_2, E_3 converging to zero.

chosen, then projective synchronisation will be attained by modified Lorenz system (22) with delay Rossler system (14).

Remark 6. If scaling factors $p_1 = p_2 = p_3 = 0, q_1 = q_2 = q_3 = 1$ and $r_1 = r_2 = r_3, s_1 = s_2 = s_3 = 0$ are chosen, then projective synchronisation will be attained by time-delay Shimizu–Morioka system (21) with time-delay Chen system (16).

Remark 7. If scaling factors $p_1 = p_2 = p_3 = 1, q_1 = q_2 = q_3 = 0$ and $r_1 = r_2 = r_3, s_1 = s_2 = s_3 = 0$ are chosen, then projective synchronisation will be attained by modified Lorenz system (22) with time-delay Chen system (16).

For numerical simulations, initial conditions for time-delay Rossler system and time-delay Chen system are considered as $(u_1(0), u_2(0), u_3(0)) = (0.5, 1, 1.5)$ and $(v_1(0), v_2(0), v_3(0)) = (0.2, 0, 0.5)$, respectively. Initial conditions for time-delay Shimizu–Morioka system and time-delay modified Lorenz system are considered as $(w_1(0), w_2(0), w_3(0)) = (1, 3, 5)$ and $(x_1(0), x_2(0), x_3(0)) = (-0.1, 0.2, 0.1)$, respectively. Although arbitrary values of scaling factors can be chosen, $p_1 = p_2 = p_3 = 1, q_1 = q_2 = q_3 = 1, r_1 = r_2 = r_3 = 1, s_1 = s_2 = s_3 = 1$ are taken which lead to complete synchronisation. The value of positive constant m is taken as $m = 2$. Values of parameters are taken as $\alpha_1 = 0.2, \mu_1 = 0.5, \beta_1 = 0.2, \gamma_1 = 5.7, \alpha_2 = 35, \beta_2 = 3, \gamma_2 = 27, \alpha_3 = 0.75, \beta_3 = 0.45, \alpha_4 = 0.9, \beta_4 = 0.1, \gamma_4 = 2.5$ and time delays of the master and the slave systems are $T_1 = 1, T_1{}' = 2, T_2 = 0.005, T_3 = 0.15, T_4 = 1, T_4{}' = 2$, respectively.

The initial conditions for errors are $(0.2, 2.2, 3.1)$. Figures 5a and 5b show synchronisation between the state variables $(u_1 + v_1, w_1 + x_1), (u_2 + v_2, w_2 + x_2)$ and figures 6a and 6b show synchronisation for $(u_3 + v_3, w_3 + x_3)$, and convergence of errors to zero, respectively. The estimated values of the unknown parameters $\hat{\alpha}_1, \hat{\mu}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\gamma}_2, \hat{\alpha}_3, \hat{\beta}_3, \hat{\alpha}_4, \hat{\beta}_4, \hat{\gamma}_4$ with initial conditions $(1, 0.1, 3, 1, 20, 15, 10, 1.5, 0.2, 0.1, 1.4, 0.5)$ of systems parameters converge to the true value for the parameter updating laws (26) which is shown in figures 7a and 7b and figures 8a and 8b.

5. Conclusion

In this paper, combination–combination synchronisation has been accomplished under external disturbances and parametric uncertainties for two master systems (time-delay Rossler system and time-delay Chen system) and two slave systems (time-delay Shimizu–Morioka system and time-delay modified Lorenz system) by using adaptive control method and

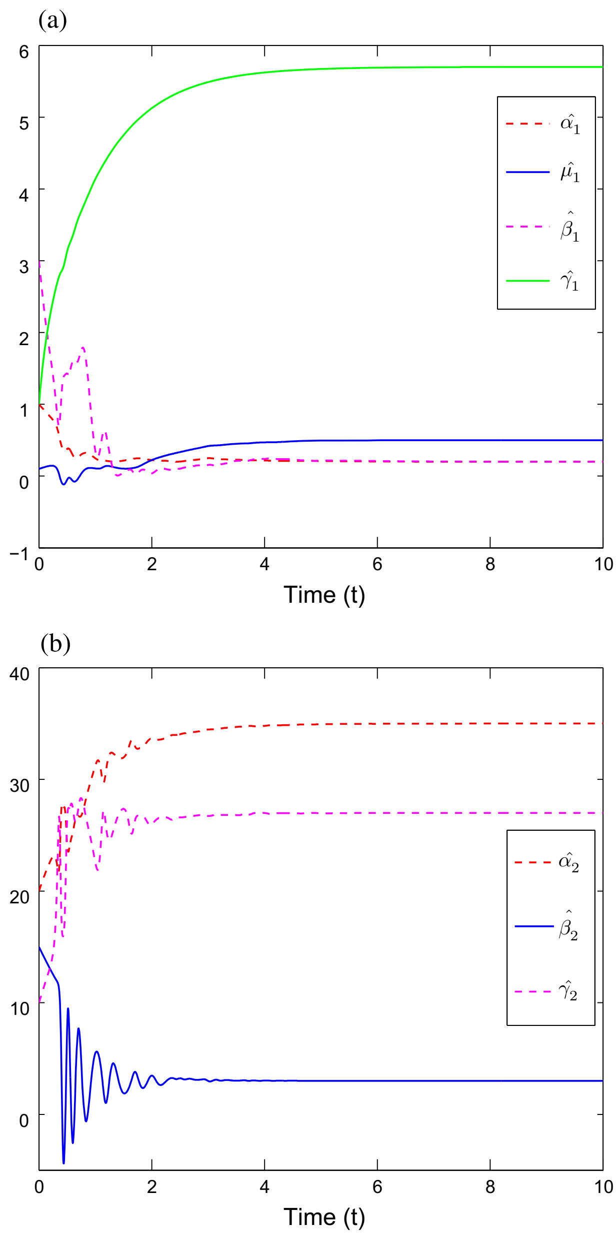

Figure 7. (**a**) Estimation of unknown parameters of double delay Rossler system and (**b**) estimation of unknown parameters of time-delay Chen system.

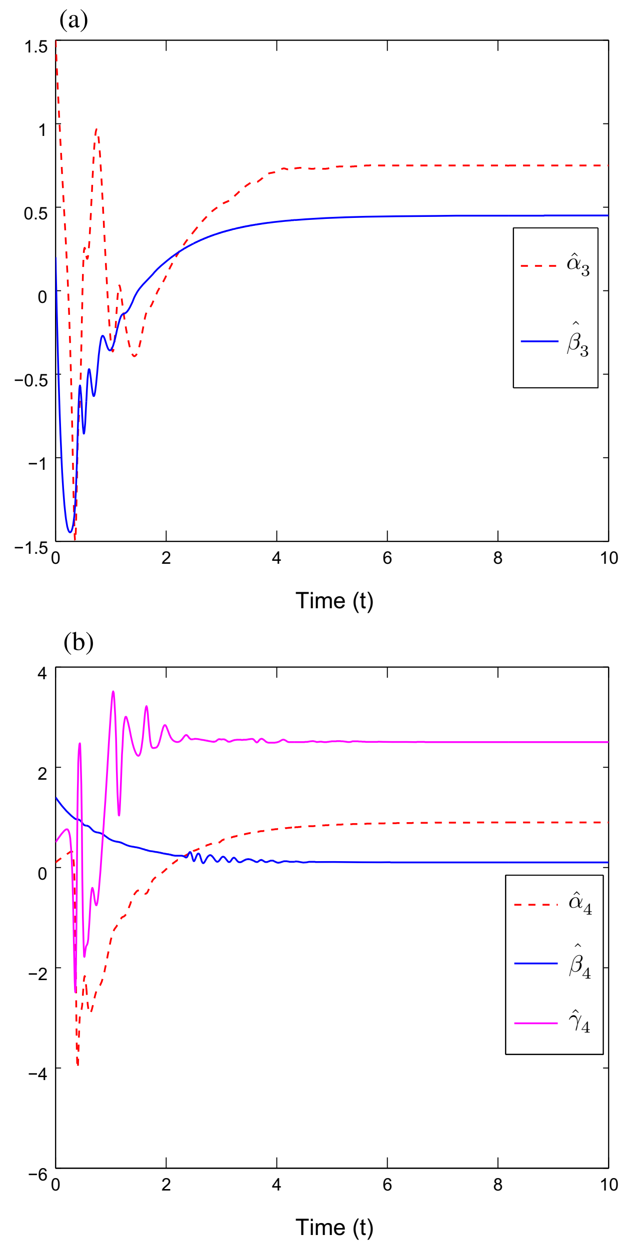

Figure 8. (**a**) Estimation of unknown parameters of Shimizu–Morioka system and (**b**) estimation of unknown parameters of time-delay modified Lorenz system.

Lyapunov–Krasovskii functional. The approach is effective for time-delay systems when parameters are not known. Computational results are in excellent agreement with theoretical results. Synchronisation of state variables, convergence of errors to zero and convergence of unknown parameters under parametric uncertainties and external disturbances are proofs that the desired synchronisation can be achieved. The problem considered in this article can also be extended to higher-dimensional systems and fractional-order systems which is a topic of further research.

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