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On non-consensus motions of dynamical linear multiagent systems

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Abstract. The non-consensus problems of high-order linear time-invariant dynamical homogeneous multiagent systems are studied. Based on the conditions of consensus achievement, the mechanisms that lead to non-consensus motions are analysed. Besides, a comprehensive classification of diverse types of non-consensus phases corresponding to different conditions is conducted, which is jointly depending on the self-dynamics of the agents, the interactive protocol and the graph topology. A series of numerical examples are explained to illustrate the theoretical analysis.

Keywords. Consensus; multiagent systems; interactive protocol; graph topology.

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1. Introduction

The dynamical multiagent systems, as a type of large-scale complex system, are composed of numerous autonomous or semiautonomous subsystems, being relatively equipotent and connected through the information interacting network. During the recent decades, the consensus problem of dynamical multiagent systems has attracted extensive attention in control theory. The basic idea of consensus is that a group of agents achieve an agreement over some variables of interest by local interactions.

Olfati-Saber and Murray were the first to introduce the term 'consensus' to the control theory [1]. Ren and Beard [2] relaxed the conditions in [1], and proved that a spanning tree within the communication topology is essential for a multiagent system to achieve consensus. A set-valued Lyapunov function method was developed by Moreau [3]. Until 2007, a majority of studies on consensus problem had dealt with first-order models. Since 2007, consensus problems for high-order multiagent systems have been addressed. For instance, Xiao and Wang [4] proposed a criterion based on the structure of certain high-dimensional matrices. Wang *et al* [5] endeavored to determine whether an appropriate linear high-order consensus protocol exists under a given undirected graph topology. Cai *et al* proved necessary and sufficient conditions for both swarm stability and consensus of high-order LTI (linear time-invariant) normal systems [6], nonlinear systems [7] and singular systems [8]. Li *et al* studied the robust stability problem of linear multiagent systems with observer-type interactive protocols [9], and later developed an approach [10] to realise consensus using merely the local information, without needing to know the global structure of the network topology. Xi *et al* devised a technique based on oblique decomposition of the state space [11], and addressed the guaranteed cost control for consensus [12,13], which can be regarded as a suboptimal control problem for interconnected systems. For other relevant or analogous works, one can refer to [14–20] and the references therein.

A majority of past studies mainly focussed on discovering the approaches and conditions to achieve consensus for different multiagent systems under various situations. The reasons for giving so much attention to the problem of achieving consensus are that: (1) The consensus is a specific and relatively simple case of the stability of multiagent systems, also being referred to as asymptotic swarm stability [6]; meanwhile, a conventional viewpoint in control theory is that stability is a prerequisite for systems to operate normally;

(2) some other control problems which are seemingly more complicated can be transformed into consensus under certain conditions, e.g. the formation control [21,22].

Despite the theoretical significance of consensus, one should know that it is still a particular example of the stability of multiagent systems. Actually, in many practical applications, mere consensus is sometimes restrictive. For example, in most cases, the formation control [21,22], flocking control [23], containment control [24] and some other control problems are free of consensus and cannot be transformed into equivalent consensus-based problems. Evidently, compared with consensus, non-consensus is more common in various practical applications, containing broader generality. In addition, consensus could be even harmful to the overall stability of some real systems. For instance, the state of the system will oscillate seriously when consensus of certain variables occurs in many economic systems [25], and thus consensus should instead be avoided deliberately. Hence, it is time to extend our perspective on consensus to the study of diverse non-consensus cases, which are much more complicated and challenging.

A few studies on the conditions and methods for certain particular non-consensus problems have been conducted so far, e.g. group clustering. Yu and Wang [26] studied group consensus in multiagent systems with switching topologies and communication delays. Hu *et al* addressed group consensus problem for heterogeneous systems [27] and systems with cooperation-competition networks [28]. Su *et al* investigated the pinning control for cluster synchronisation of undirected complex dynamical networks using a decentralised adaptive strategy [29] and the cluster synchronisation of coupled harmonic oscillators with multiple leaders in an undirected fixed network [30].

The study on the non-consensus problem in depth not only can lay a solid theoretical foundation for many practical applications, but can also further facilitate our deeper comprehension on the consensus problems. However, no papers exist hitherto which aim to systematically study the motions being free of consensus and reveal the mechanism of non-consensus. In this paper, the non-consensus problem of LTI multiagent systems will be addressed. Unlike many existing investigations around the non-consensus problem which are just concerned with certain particular phase of motions, e.g. group clustering, a comprehensive classification of different types of non-consensus phases along with the analysis for each corresponding condition will be provided. In fact, one might discover that the mechanism of non-consensus, which mainly depends on the graph topology, the agent dynamics and the interactions of agents, could be less complicated than imagination, with different non-consensus motions of LTI multiagent systems being classified roughly into two categories: agreement and mutual repulsion between any distinct agents. Concretely speaking, the features of non-consensus dynamics can be classified into three primary classes according to the different conditions about consensus achievement.

The main contribution of this paper is a comprehensive classification and analysis of the conditions for various types of non-consensus motions of LTI multiagent systems, illustrated by several typical numerical instances. The discussions here can deepen our understanding of the mechanisms about the dynamics of multiagent systems, both consensus and non-consensus.

The rest of this paper is organised as follows: In §2, the mathematical model of homogeneous high-order LTI multiagent systems and relevant fundamentals are introduced. Section 3 provides a comprehensive classification for the paradigms of non-consensus motions, with both empirical and theoretical analysis. Finally, §4 concludes this paper.

2. Model formulation and preliminaries

In this paper, the mathematical model of homogeneous high-order LTI multiagent systems takes the following form:

$$\begin{cases} \dot{x}_1 = Ax_1 + F\sum_{j=1}^{N} w_{1j}(x_j - x_1), \\ \dot{x}_2 = Ax_2 + F\sum_{j=1}^{N} w_{2j}(x_j - x_2), \\ \vdots \\ \dot{x}_N = Ax_N + F\sum_{j=1}^{N} w_{Nj}(x_j - x_N), \end{cases} \tag{1}$$

where $x_i \in R^d \quad (i \in \{1, 2, \ldots, N\})$ denotes the state vector of agent i; $A, \quad F \in R^{d\times d}$ represent the autonomous dynamics of each agent and the interactive protocol between the nearest agents, respectively; $w_{ij} \in R^+$ denotes the arc weight between agents i and j in the graph of the system. The adjacency matrix of G is

$$G : W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1N} \\ w_{21} & w_{22} & \cdots & w_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ w_{N1} & w_{N2} & \cdots & w_{NN} \end{bmatrix}.$$

This matrix is symmetric if and only if the graph is undirected, otherwise it is asymmetric. Weighted adjacency matrix is more general than the case with binary 0-1 element values, with the weight w_{ij} being regarded as the strength of information link between agents i and j.

LTI systems are addressed in this paper due to two main reasons. First, results about LTI systems are most fundamental and enlightening for studying various classes of systems. Second, many practical engineering systems can be described by the LTI model taking the form of (1), e.g. the multiagent supporting systems (MASS) [31].

It has been a common knowledge that consensus of a multiagent system implies joint convergence, which is formulated by the following definition:

DEFINITION 1

For system (1), if

$$\lim_{t\to\infty} \left\| x_i(t) - x_j(t) \right\| = 0$$

then agents i and j achieve an agreement. For a given vertex set $V_k \subset V$, if $\forall i, j \in V_k$, agents i and j achieve an agreement, then the multiagent system (1) achieves a consensus in V_k. If consensus is achieved in $V_1, V_2, \ldots, V_\alpha$, respectively, with

$$V_1 \cup V_2 \cup \cdots \cup V_\alpha = V,$$

then the overall system achieves a group consensus.

Based on the above mathematical model of high-order LTI multiagent systems and the definition of consensus, many studies on the conditions and approaches for achieving consensus have been conducted. The well-known consensus criterion of high-order LTI multiagent systems is fundamental for studying the non-consensus problem in this paper, which is illustrated as follows:

Lemma 1 [6]. *For multiagent system* (1) *with* $\lambda_1, \lambda_2, \ldots, \lambda_N \in C$ *as the eigenvalues of the Laplacian matrix* $L(G)$, *if* A *is not Hurwitz, the system achieves consensus iff both conditions* 1 *and* 2 *below are true:*

1. *The graph topology* G *includes a spanning tree*;
2. *All the matrices* $A - \lambda_i F$ $(\lambda_i \neq 0)$ *are Hurwitz.*

If A *is Hurwitz, then the system achieves consensus iff condition* 2 *is true.*

Remark 1. The formal definition of Laplacian matrix for undirected graphs is clear, which is simply $L = D - W$, where D denotes the degree matrix. However, it is somewhat ambiguous for directional graphs. In this paper, we inherit the definition given in [1], with D expressing the in-degrees, which is a natural generalisation from the undirected cases.

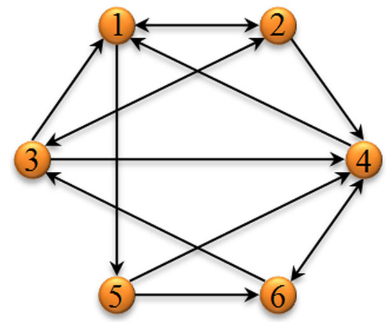

Figure 1. Graph of systems in Examples 1 and 2. Default edge weight is 1.

3. Classification and numerical simulation on non-consensus motions

According to the criterion on checking consensus of high-order LTI multiagent systems, various conditions of non-consensus motions can be classified into three primary classes. The main theme of the current section is to elaborate the different classes of motions by theoretical analyses and simulations.

3.1 *Class* 1

The situation of Class 1 of non-consensus motions of high-order LTI multiagent systems can be summarised as:

1. Some of the elements in

   $$\{A - \lambda_i F \,|\, i = 1, 2, \ldots, N; \lambda_i \neq 0\}$$

   are not Hurwitz, where $\lambda_i (i = 1, 2, \ldots, N; \lambda_i \neq 0)$ are the non-zero eigenvalues of the Laplacian matrix $L(G)$.
2. At least a spanning tree is included in the graph of system.

Example 1. Consider an LTI multiagent system with the graph containing a spanning tree, which is illustrated in figure 1.

The adjacency matrix is

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and the Laplacian matrix is

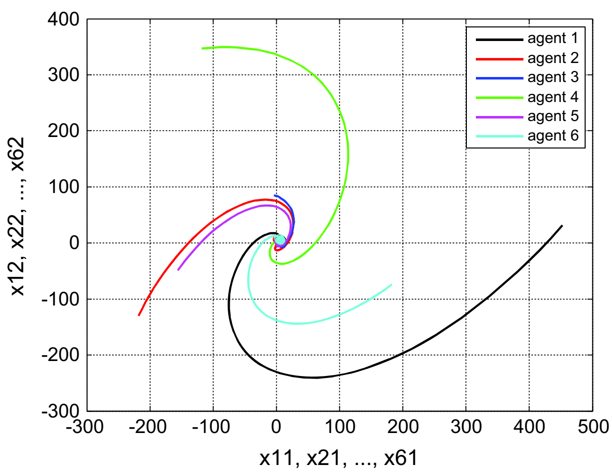

Figure 2. State trajectories of Example 1 with $t \in [0, 7]$. Thick dots denote the starting positions.

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

with the eigenvalues $\{3.96 + 0.56i, 3.96 - 0.56i, 0, 1.53 + 0.51i, 1.53 - 0.51i, 3\}$.

Suppose that the agents are LTI second-order asymptotically stable systems with

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix}.$$

For the above LTI multiagent system, the eigenvalues of state matrix A are $\{-1, -2\}$, and the eigenvalues of $A - \lambda_i F (i = 1, 2, \ldots, 6)$ are: $\{0.80 - 1.03i, 0.17 + 1.60i\}$, $\{0.80 + 1.03i, 0.17 - 1.60i\}$, $\{-1.00, -2.00\}$ $(\lambda_3 = 0)$, $\{-0.02 + 0.16i, -1.44 + 0.35i\}$, $\{-0.02 - 0.16i, -1.44 - 0.35i\}$ and $\{\pm 0.71i\}$, respectively. Evidently, some of the elements in $\{A - \lambda_i F \mid i = 1, 2, \ldots, N; \lambda_i \neq 0\}$ are not Hurwitz. The state trajectories of Example 1 are shown in figure 2.

In figure 2, one can see that each agent is mutually repulsive to others in the phase diagram such that the relative states among agents will gradually enlarge as time elapses.

Example 2. Consider an LTI multiagent system also with the graph shown in figure 1.

Suppose that the agents are LTI second-order unstable systems with

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} -0.65 & -1.65 \\ 0.07 & 0.40 \end{bmatrix}.$$

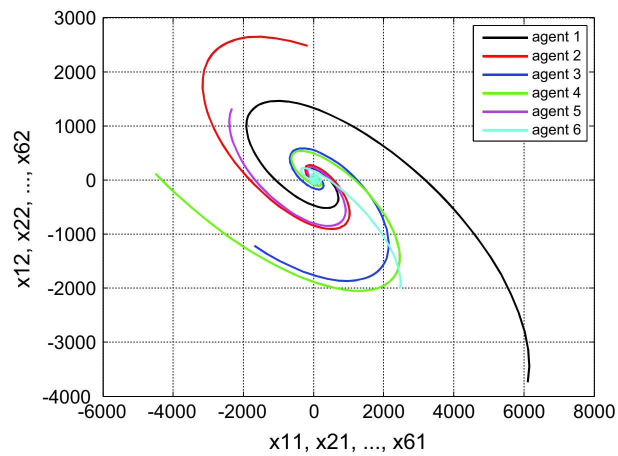

Figure 3. State trajectories of Example 2 with $t \in [0, 6.6]$. Thick dots denote the starting positions.

For the above LTI multiagent system, the eigenvalues of state matrix of each agent A are $\{0.50 \pm 1.32i\}$, and the eigenvalues of $A - \lambda_i F (i = 1, 2, \ldots, 6)$ are: $\{0.86 + 3.32i, 1.13 - 3.18i\}$, $\{0.86 - 3.32i, 1.13 + 3.18i\}$, $\{0.50 \pm 1.32i\}$ $(\lambda_3 = 0)$, $\{0.44 + 2.48i, 0.94 - 2.35i\}$, $\{0.44 - 2.48i, 0.94 + 2.35i\}$ and $\{0.88 \pm 2.97i\}$, respectively. Evidently, all the elements in $\{A - \lambda_i F \mid i = 1, 2, \ldots, N; \lambda_i \neq 0\}$ are not Hurwitz. The state trajectories of Example 2 are shown in figure 3.

In figure 3, one can see that each agent is mutually repulsive to others in the phase diagram such that the relative states between agents will gradually enlarge as time elapses.

Evidently, the state trajectories of systems in both Examples 1 and 2 are divergent, i.e. all agents in the system are mutually repulsive.

In fact, the feature of motions for Class 1 of LTI multiagent systems is complicated and indefinite, sensitively depending on the specific values of matrices A, F and L.

It is a notable feature for condition in Class 1 that quasiclustering phenomenon may appear. This subsection can be concluded by a necessary and sufficient condition for checking whether or not a clustering phenomenon will occur.

Lemma 2 [32]. *Consider the dynamical system* (1). *Suppose that the spectrum of Laplacian matrix of the directed graph with spanning tree is*

$$\{\lambda_1 = 0, \lambda_2, \ldots, \lambda_N\}$$

with the series of matrices

$$A, A - \lambda_2 F, \ldots, A - \lambda_{\alpha-1} F$$

being not Hurwitz, and

$$A - \lambda_\alpha F, A - \lambda_{\alpha+1} F, \ldots, A - \lambda_N F$$

being Hurwitz. The pair of agents i and i + 1 (or N and 1, if i = N) reaches an agreement iff the ith row ψ_i^T *of the product* $\Psi = TQ$ *possesses the configuration*:

$$\psi_i^T = \left[\underset{(1)}{*}\ \underset{(2)}{0}\ \cdots\ \underset{(\alpha-1)}{0}\ \underset{(\alpha)}{*}\ \cdots\ \underset{(N)}{*}\right],$$

where $T \in R^{N\times N}$ *represents any feasible solution of the matrix equation*

$$TL = \Phi = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \cdots\cdots & & \\ & & & 1 & -1 \\ -1 & 0 & \cdots\cdots & & 1 \end{bmatrix}.$$

$Q \in R^{N\times N}$ *represents the non-singular matrix that transforms the Laplacian matrix into the similar Jordan canonical form*:

$$Q^{-1}LQ = J = \begin{bmatrix} 0 & & & & \\ & \lambda_2 & * & & \\ & & \lambda_3 & \ddots & \\ & & & \ddots & * \\ & & & & \lambda_N \end{bmatrix}$$

and $*$ *denotes any arbitrary value.*

3.2 *Class* 2

Before introducing the condition of Class 2, some serviceable theoretical preparations should be first expounded.

Lemma 3 (*Determinant of block matrix*) [33]. *Suppose* A, B, C *and* D *are matrices of dimensions* $n\times n, n\times m, m\times n$ *and* $m\times m$, *respectively, then*

$$\det\left(\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}\right) = \det(A)\det(D) = \det\left(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\right).$$

DEFINITION 2 (independent group)

If a subgraph has spanning tree of its own and receives no information, then it is called an ‘independent group’ here.

PROPOSITION 1

The spectrum of an independent group of nth order coincides with n eigenvalues of the overall graph.

Proof. Without loss of generality, suppose that the first n vertices of the graph form an independent group, otherwise the indices can be rearranged. Because the group receives no information, the adjacency matrix can be decomposed into the following triangular configuration:

$$W = \begin{bmatrix} W_{11} & 0 \\ W_{21} & W_{22} \end{bmatrix},$$

where W_{11} represents the subgraph of the independent group and W_{22} the subgraph of the remaining vertices. Correspondingly,

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}.$$

According to Lemma 3, the eigenpolynomial of L is

$$|\lambda I_N - L| = |\lambda I_n - L_{11}|\,|\lambda I_{N-n} - L_{22}|\,.$$

Thus, the spectrum of the independent group coincides with n eigenvalues of the overall graph. □

PROPOSITION 2

The agents associated with an nth order independent group of the system achieves agreement if and only if $A - \lambda_i F$ $(i = 1, 2, \ldots, n;\ \lambda_i \neq 0)$ *are Hurwitz, where* λ_i $(i = 1, 2, \ldots, n;\ \lambda_i \neq 0)$ *are the eigenvalues of the overall Laplacian matrix of the system that correspond to the independent group. Besides, if an independent group achieves agreement, the trajectories of the agents converge to a solution of the dynamical equation* $\dot{\xi} = A\xi$.

Proof. According to Proposition 1, the spectrum of the independent group coincides with n eigenvalues of the overall graph; meanwhile, there is a single zero eigenvalue among them because the independent group includes a spanning tree of its own. Based on Lemma 1, it can be implied that the agents associated with the independent group achieve agreement if and only if $A - \lambda_i F$ $(i = 1, 2, \ldots, n;\ \lambda_i \neq 0)$ are Hurwitz.

Without loss of generality, consider the dynamics of agent 1. As the independent group receives no external information, the difference between $\dot{x}_1$ and Ax_1 is

$$\dot{x}_1 - Ax_1 = F\sum_{j=1}^{n} w_{ij}(x_j - x_1).$$

Because the agents $1 \sim n$ achieve agreement, $\lim_{t\to\infty}\|x_j - x_1\| = 0$ $(j = 1, 2, \ldots, n)$, and as a result, $\lim_{t\to\infty}\|F\sum_{j=1}^{n} w_{ij}(x_j - x_1)\| = 0$. Consequently, $\lim_{t\to\infty}\dot{x}_1 = Ax_1$. □

COROLLARY 1

If all $A - \lambda_i F$ $(i = 1, 2, \ldots, N;\ \lambda_i \neq 0)$ *in a system are Hurwitz, then the agents associated with any independent group achieve agreement.*

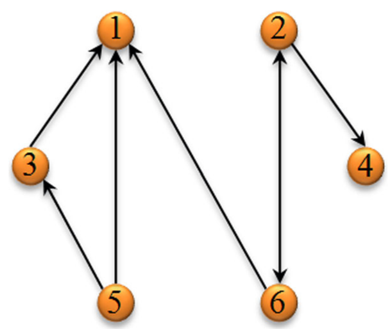

Figure 4. Graph of systems in Examples 3, 4 and 5. Default edge weight is 1.

The condition for Class 2 of non-consensus motions of high-order LTI multiagent systems is

1. Some of the elements in

   $$\{A - \lambda_i F | i = 1,\ 2,\ \ldots, N;\ \lambda_i \neq 0\}$$

   are not Hurwitz, where $\lambda_i (i = 1,\ 2, \ldots, N; \lambda_i \neq 0)$ are the non-zero eigenvalues of the Laplacian matrix $L(G)$.
2. There is no spanning tree in the graph of the system.
   In addition, this class can be classified into two types of cases with regard to whether the matrix A is Hurwitz or not.

3.2.1 *A is Hurwitz*

Example 3. Consider an LTI multiagent system with the graph having no spanning tree, which is illustrated in figure 4.

The adjacency matrix is

$$W = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the Laplacian matrix is

$$L = \begin{bmatrix} 3 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with the eigenvalues $\{3, 1, 2, 0, 1, 0\}$.

Suppose that the agents are LTI second-order asymptotically stable systems with

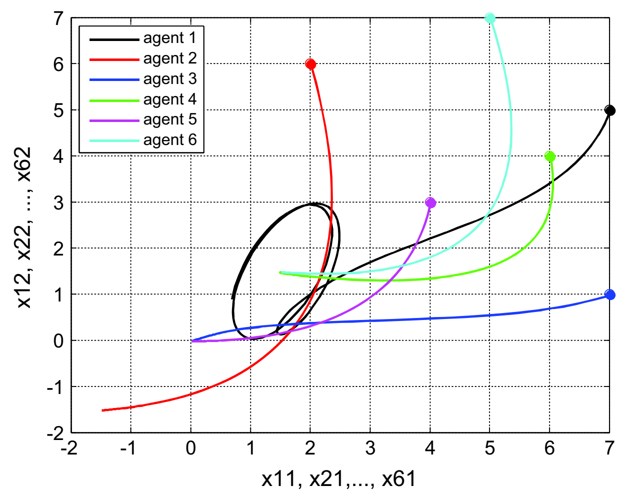

Figure 5. State trajectories of Example 3 with $t \in [0, 7]$. Thick dots denote the starting positions.

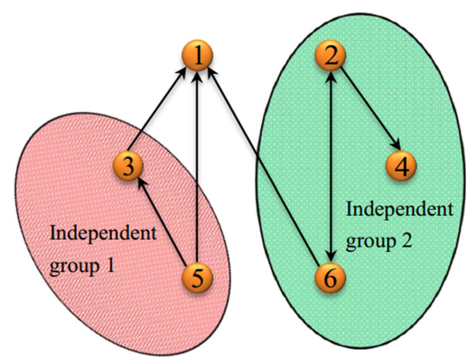

Figure 6. Two independent groups of the graph.

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix}.$$

For the above LTI multiagent system, the eigenvalues of state matrix A are $\{-1,\ -2\}$, and the eigenvalues of $A - \lambda_i F \quad (i = 1,\ 2, \ldots, 6)$ are: $\{\pm 0.71i\}$, $\{-1.71, -0.30\}$, $\{-1,\ 0\}$, $\{-1,\ -2\}(\lambda_4 = 0)$, $\{-1.71, -0.30\}$ and $\{-1,\ -2\}(\lambda_6 = 0)$, respectively. Evidently, some of the elements in $\{A - \lambda_i F |\ i = 1,\ 2,\ \ldots, N; \lambda_i \neq 0\}$ are not Hurwitz. The state trajectories of Example 3 are shown in figure 5.

In figure 5, agents 3 and 5 will gradually converge to a common state trajectory and approach the origin with the elapse of time. Likewise, agents 4 and 6 will also gradually converge to another common state trajectory. This phenomenon is referred to as group clustering. Agent 2 has its own unique state trajectory. Agent 1 will oscillate persistently.

In fact, according to Definition 2, two independent groups can be found in the graph as shown in figure 6.

The Laplacian matrices of independent groups 1 and 2 are:

$$L_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

respectively. The eigenvalues for the Laplacian matrix of independent groups 1 and 2 are $\{1,\ 0\}$ and $\{1,\ 2,\ 0\}$, respectively. The eigenvalues of $A - \lambda_i F$ in independent group 1 are $\{-1.71, -0.30\}$ and $\{-1,\ -2\}$. The eigenvalues of $A - \lambda_i F$ in independent group 2 are $\{-1.71, -0.30\}$, $\{-1,\ 0\}$ and $\{-1,\ -2\}$. Evidently, all the elements in $\{A - \lambda_i F | \lambda_i \neq 0\}$ for independent group 1 are Hurwitz, whilst some of the elements in $\{A - \lambda_i F | \lambda_i \neq 0\}$ of independent group 2 are not Hurwitz.

According to the above analysis and Proposition 1, all agents in independent group 1 will achieve agreement, with the common state trajectory converging to the origin as time elapses. In contrast, the phase of motion for independent group 2 by itself belongs to Class 1. The affiliation of agent 1 is undetermined, depending on the joint attraction from independent groups 1 and 2.

3.2.2 *A is not Hurwitz*

Example 4. Consider an LTI multiagent system with the graph having no spanning tree, which is illustrated in figure 4. The edge weight between agents 3 and 5 is adjusted to 0.3.

Correspondingly, the adjacency matrix

$$W = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the Laplacian matrix

$$L = \begin{bmatrix} 3 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0.3 & 0 & -0.3 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with the eigenvalues being $\{3,\ 1,\ 2,\ 0,\ 0.3,\ 0\}$.

Suppose that the agents are LTI second-order unstable systems with

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 7 & 5 \\ -4 & -1 \end{bmatrix}$$

For the above LTI multiagent system, the eigenvalues of state matrix A are $\{0.50 \pm 1.32i\}$, and the eigenvalues of $A - \lambda_i F (i = 1,\ 2, \ldots, 6)$ are: $\{-8.50 \pm 2.78i\}$, $\{-4.56,\ -4.44\}$, $\{-7.00,\ -4.00\}$, $\{0.50 \pm 1.32i\}(\lambda_4 = 0)$, $\{-1.34,\ 0.54\}$ and $\{0.50 \pm 1.32i\}(\lambda_6 = 0)$, respectively. Evidently, some of the elements in $\{A - \lambda_i F | i = 1,\ 2, \ldots, N;\ \lambda_i \neq 0\}$ are not Hurwitz. The state trajectories of Example 4 are shown in figure 7.

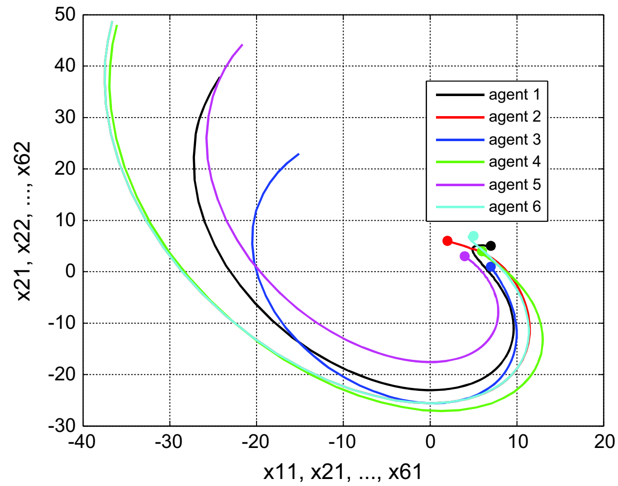

Figure 7. State trajectories of Example 4 with $t \in [0, 3.6]$. Thick dots denote the starting positions.

In figure 7, agents 2, 4 and 6 will gradually aggregate to a common state trajectory which will be away from the origin with elapse of time. This phenomenon is quasigroup clustering. Differently, agents 1, 3 and 5 have their own separate state trajectories.

Similarly, according to Definition 2, two independent groups can be found in the graph shown in figure 6.

The Laplacian matrices of independent groups 1 and 2 are:

$$L_1 = \begin{bmatrix} 0.3 & -0.3 \\ 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

respectively. The eigenvalues for the Laplacian matrices of independent groups 1 and 2 are: $\{0.3,\ 0\}$ and $\{1,\ 2,\ 0\}$, respectively. The eigenvalues of $A - \lambda_i F (i = 1,\ 2)$ for independent group 1 are $\{-1.34, 0.54\}$ and $\{0.50 \pm 1.32i\}$ $(\lambda_2 = 0)$. The eigenvalues of $A - \lambda_i F (i = 1,\ 2,\ 3)$ for independent group 2 are $\{-4.56, -4.44\}$, $\{-7.00, -4.00\}$ and $\{0.50 \pm 1.32i\}$ $(\lambda_3 = 0)$. Evidently, some of the elements in $\{A - \lambda_i F | \lambda_i \neq 0\}$ of independent group 1 are not Hurwitz, and all the elements in $\{A - \lambda_i F | \lambda_i \neq 0\}$ of independent group 2 are Hurwitz.

Based on the above analysis and Proposition 1, the feature of the motion of independent group 1 is known to be indefinite and belongs to Class 1. All agents in independent group 2 will achieve agreement and the common state trajectory will be away from the origin with time. The affiliation of agent 1 is unknown and depends on the joint attraction from both the independent groups 1 and 2.

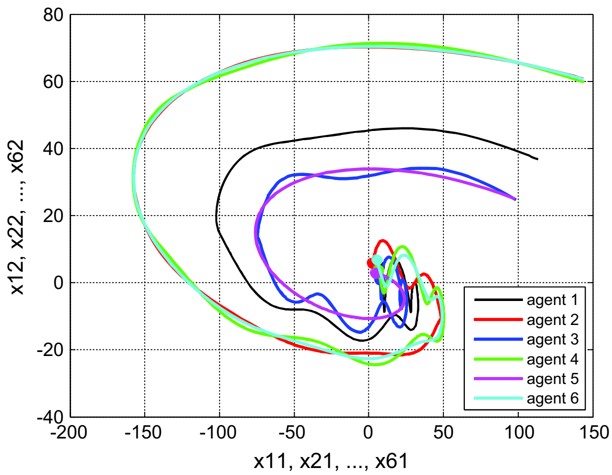

Figure 8. State trajectories of Example 5 with $t \in [0, 5]$. Thick dots denote the starting positions.

3.3 *Class* 3

The condition for the third class of non-consensus motions of high-order LTI multiagent systems is:

1. The state matrix A of each agent is not Hurwitz.
2. All the elements in

   $$\{A - \lambda_i F | i = 1, 2, \ldots, N;\ \lambda_i \neq 0\}$$

   are Hurwitz, where λ_i $(i = 1, 2, \ldots, N;\ \lambda_i \neq 0)$ are the non-zero eigenvalues of the Laplacian matrix $L(G)$.
3. There is no spanning tree in the graph of the system.

Example 5. Consider an LTI multiagent system where the graph topology has no spanning tree, which is illustrated in figure 4.

Suppose that all agents are LTI second-order unstable systems with

$$A = \begin{bmatrix} 1 & 5 \\ -0.4 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} -1.67 & 1.33 \\ 22.85 & 3.16 \end{bmatrix}.$$

For the above LTI multiagent system, the eigenvalues of state matrix of each agent A are $\{0.50 \pm 1.32i\}$, and the eigenvalues of $A - \lambda_i F$ $(i = 1, 2, \ldots, 6)$ are: $\{-1.74 \pm 3.11i\}$, $\{-0.25 \pm 8.77i\}$, $\{-0.99 \pm 8.91\}$, $\{0.50 \pm 1.32i\}$ $(\lambda_4 = 0)$, $\{-0.25 \pm 8.77i\}$ and $\{0.50 \pm 1.32i\}$ $(\lambda_6 = 0)$, respectively. Evidently, all the elements in $\{A - \lambda_i F | i = 1, 2, \ldots, N;\ \lambda_i \neq 0\}$ are Hurwitz. The state trajectories of Example 5 are shown in figure 8.

In figure 8, agents 3 and 5 will gradually aggregate to a common state trajectory. Likewise, agents 2, 4 and 6 also will gradually aggregate to another common state trajectory. This phenomenon is referred to as group clustering. Differently, agent 1 will have its own separate state trajectory.

Similarly, according to Definition 2, two independent groups can be found in the graph shown in figure 6.

According to Proposition 1, all agents in independent groups 1 and 2 will achieve agreement respectively with both the common state trajectories diverging from the origin as time elapses. The affiliation of agent 1 is unknown and depends on the joint attraction from independent groups 1 and 2 simultaneously.

Under the condition of Class 3, group clustering will be certain to occur.

4. Conclusion

The non-consensus problem of high-order homogeneous LTI multiagent systems has been investigated in this paper, mainly based on the necessary and sufficient conditions for consensus achievement. The different cases of non-consensus motions of LTI multiagent systems can be classified roughly into two basic categories: group clustering and mutual repulsion. Further, non-consensus motions can be concretely categorised into three classes, each with distinct features. For instance, the dynamical features for Class 1 are complicated and indefinite; for Class 2, the motions of independent groups are relatively simpler to anticipate; the group clustering phenomenon is certain to appear in Class 3. The theoretical conditions and typical numerical examples of each class of motion are presented.

Relevant knowledge can enrich the literature by broadening our understanding of the cause and mechanism for both consensus and non-consensus phenomena. The current research only takes the first step along the route of studying various non-consensus problems. There are several observable directions for the potential future works. First, the more explicit relationship between the topology of a graph and its Laplacian spectrum can be unveiled. Secondly, the issue on the robustness of each phase of agent motions can be addressed, with particular attention being paid to the critical conditions for the phase transitions. Thirdly, as most of the real-world networks are not constant, e.g. being state-dependent as most of the social networks are, it is meaningful to study non-consensus phenomena of systems with time-varying or even switching graph topologies. Fourth but not the last, relevant work can be extended from theoretical analysis to engineering applications, e.g. clustering of real data points based on non-consensus motions of certain dynamical systems.

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