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Exact solutions to (2 + 1)-dimensional Chaffee–Infante equation

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Abstract. In this paper, the canonical-like transformation method and trial equation method are applied to (2+1)-dimensional Chaffee–Infante equation, and some exact solutions are obtained. In particular, a new solution in terms of elliptic functions is given.

Keywords. Chaffee–Infante equation; travelling wave solutions; canonical-like transformation method; trial equation method.

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1. Introduction

The construction of exact solutions to nonlinear differential equations is an important and difficult task. It also plays a significant role in mathematics and physics. Various powerful methods such as inverse scattering method [1], direct method [2], symmetrical method [3], the complete discrimination system for polynomial method [4–14] and so on have been proposed for obtaining approximate and exact solutions for various nonlinear equations.

In this paper, we consider (2 + 1)-dimensional Chaffee–Infante equation [15]

$$u_{xt} + (-u_{xx} + au^3 - au)_x + \sigma u_{yy} = 0, \quad (1)$$

where a and σ are arbitrary constants. Assume that a substance spreads in a region with the concentration of the diffusion as $u(x, y, z)$. Let $D(x, y, z, t)$ be the diffusion coefficient, then according to the law of diffusion, we have

$$\mathrm{d}m = -D\frac{\partial u}{\partial n}\mathrm{d}s\,\mathrm{d}t, \quad (2)$$

where m represents the amount of diffusion material and $D > 0$. According to eq. (2) and the law of conservation of mass yields

$$m = \int_{t_1}^{t_2}\left[\iint_{\Sigma} D\frac{\partial u}{\partial n}\mathrm{d}s\right]\mathrm{d}t = \iiint_{\Omega}[u(x, y, z, t_2) - u(x, y, z, t_1)]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad (3)$$

that is,

$$\frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial t}. \quad (4)$$

Assume that $D = 1$ and the influential factor is $f(u) = u^3 - u$. We get

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \lambda(u - u^3), \quad (5)$$

where parameter λ adjusts the relative balance of the diffusion term and the nonlinear term. So the (1 + 1)-dimensional Chaffee–Infante equation is

$$u_t - u_{xx} = \lambda(u - u^3), \quad (6)$$

and the derivation process of (2 + 1)-dimensional Chaffee–Infante equation is similar to it.

The (2 + 1)-dimensional Chaffee–Infante equation is a well-known reaction diffusion equation. It describes the physical process of mass transport and particle diffusion, and has been widely used in environmental science, fluid dynamics, high-energy physics, electronic science, and so on. Therefore, building exact solutions to this equation has great scientific significance and a broad application background (see [16] and references therein). In [15], the authors use exp-function method to obtain solutions to eq. (1), and by this method, we cannot get solutions of other types such as elliptic function solutions. It is also difficult to get elliptic function solutions to eq. (1) by some other methods, but canonical-like transformation method can be used to get elliptic function solutions. The canonical-like transformation method was proposed by Liu [17] to obtain exact travelling wave solutions to some diffusion-reaction equations such as Fisher equation, Burgers–KdV equation and Newell–Whitehead–Kwahara equation and so on. In this paper, we use Liu's canonical-like transformation method to find solutions to eq. (1) and get a new solution in terms of elliptic functions. In addition, we use trial equation method [18–23] to get exact solutions of other forms. Further studies for exact solutions and integrability of differential equations in mathematical physics can be found in many papers (see, for example, [24–30]).

2. Elliptic functions solutions by canonical-like transformation method

According to ref. [17], the canonical-like transformation method is as follows. We consider the following ordinary differential equation:

$$u''(\xi) - Au'(\xi) = Bu^{\alpha}(\xi) + Du(\xi). \tag{7}$$

If we take $\omega = u'$, eq. (7) becomes

$$\frac{\mathrm{d}\omega}{\mathrm{d}u} = \frac{Bu^{\alpha} + Du + A\omega}{\omega}. \tag{8}$$

In order to solve the first-order nonlinear ODE (8), we re-parametrise u and ω by a new parameter s, that is, we take the canonical-like transformations

$$u = a_{11}(s)v(s) + a_{12}(s)v'(s), \tag{9}$$

$$\omega = a_{21}(s)v(s) + a_{22}(s)v'(s), \tag{10}$$

where $a_{ij}(s)(i, j = 1, 2)$ and $v(s)$ are functions to be determined. In order to obtain the parameters, we set $a_{12} = 0$. After a lot of calculations, we get

$$a_{11} = \left\{\frac{\alpha+3}{(\alpha-1)A}\right\}^{1/2}(s-s_0)^{2/(\alpha-1)}, \tag{11}$$

$$a_{22} = \left\{\frac{(\alpha-1)A}{\alpha+3}\right\}^{1/2}(s-s_0)^{2/(\alpha-1)}, \tag{12}$$

$$a_{21} = \frac{A-h}{2}\left\{\frac{\alpha+3}{(\alpha-1)A}\right\}^{1/2}(s-s_0)^{2/(\alpha-1)}. \tag{13}$$

Then eq. (7) becomes

$$v''(s) = Mv^{\alpha}(s), \tag{14}$$

and s is determined by

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \frac{1}{\lambda(s)}, \tag{15}$$

where $M = Bh^{-(\alpha+3)/2}$, $h = \pm\sqrt{4D + A^2}$ and $\lambda(s) = h(s - s_0)$.

The general solution of eq. (14) is given by

$$\pm(s - s_0) = \int \frac{\mathrm{d}v}{\sqrt{\frac{2M}{\alpha+1}v^{\alpha+1} + c}}, \tag{16}$$

where c is an integral constant. If $c = 0$, we have

$$v(s) = \left\{\pm\frac{1-\alpha}{2}\sqrt{\frac{2M}{\alpha+1}}(s-s_0)\right\}^{2/(1-\alpha)}. \tag{17}$$

And in the special case of $c = 1$ and $\alpha = 3$, we get

$$v(s) = \left(\frac{2}{M}\right)^{1/4}\frac{a\,\mathrm{sn}(\eta(s-s_0), m) + b\,\mathrm{cn}(\eta(s-s_0), m)}{c\,\mathrm{sn}(\eta(s-s_0), m) + d\,\mathrm{cn}(\eta(s-s_0), m}, \tag{18}$$

where

$$a = 3 - 2\sqrt{2}, \quad b = 2\sqrt{2} - 1,$$
$$c = 2\sqrt{2} - 4, \quad d = \sqrt{2},$$
$$m = \frac{16 + 12\sqrt{2}}{17 + 12\sqrt{2}}, \quad \eta = \sqrt{\frac{19 + 12\sqrt{2}}{52 - 32\sqrt{2}}}.$$

So, we can get solutions of eq. (7) from the formulas (17) and (18)

$$u(\xi) = \frac{6}{B}\left\{\frac{(\alpha-1)A}{\alpha+3}\right\}^{(\alpha+2)/2}\frac{\mathrm{e}^{\frac{2A}{\alpha+3}(\xi-\xi_0)}}{\left(\mathrm{e}^{\frac{(\alpha-1)A}{\alpha+3}(\xi-\xi_0)} - s_0\right)^{\frac{2}{\alpha-1}}} \tag{19}$$

and

$$u(\xi) = \left(\frac{3}{A}\right)^{1/2}(\mathrm{e}^{h(\xi-\xi_0)} - s_0)\left(\frac{2}{M}\right)^{1/4} \times \frac{a\,\mathrm{sn}(\eta(\mathrm{e}^{h(\xi-\xi_0)} - s_0), m) + b\,\mathrm{cn}(\eta(\mathrm{e}^{h(\xi-\xi_0)} - s_0), m)}{c\,\mathrm{sn}(\eta(\mathrm{e}^{h(\xi-\xi_0)} - s_0), m) + d\,\mathrm{cn}(\eta(\mathrm{e}^{h(\xi-\xi_0)} - s_0), m)}. \tag{20}$$

Now, we use the canonical-like transformation method to obtain the solutions of eq. (1). Let $u = u(\xi)$ and $\xi = kx + ly + wt$, then eq. (1) becomes

$$(kw + \sigma l^2)u'' - k^3u''' + 3aku^2u' - aku' = 0, \tag{21}$$

and integrating it yields

$$(kw + \sigma l^2)u' - k^3u'' + aku^3 - aku = c_0. \tag{22}$$

If $c_0 = 0$, by (19) and (20), the solutions of eq. (1) can be written as

$$u(x, y, t) = \frac{\frac{6k^2}{a}\left(\frac{kw+\sigma l^2}{3k^3}\right)^{5/2}\mathrm{e}^{\frac{kw+\sigma l^2}{3k^3}(kx+ly+\omega t-\xi_0)}}{\mathrm{e}^{\frac{kw+\sigma l^2}{3k^3}(kx+ly+\omega t-\xi_0)} - s_0} \tag{23}$$

and

$$u(x,y,t)=\left(\frac{3}{A}\right)^{1/2}(\mathrm{e}^{h(kx+ly+\omega t-\xi_0)}-s_0)\left(\frac{2}{M}\right)^{1/4} \times\frac{a_1\,\mathrm{sn}(\eta(\mathrm{e}^{h(kx+ly+\omega t-\xi_0)}-s_0),m)+b_1\,\mathrm{cn}(\eta(\mathrm{e}^{h(kx+ly+\omega t-\xi_0)}-s_0),m)}{c_1\,\mathrm{sn}(\eta(\mathrm{e}^{h(kx+ly+\omega t-\xi_0)}-s_0),m)+d_1\,\mathrm{cn}(\eta(\mathrm{e}^{h(kx+ly+\omega t-\xi_0)}-s_0),m)}, \tag{24}$$

where

$$A=\frac{kw+\sigma l^2}{k^3},\quad h=\pm\frac{\sqrt{(kw+\sigma l^2)^2+4ak^4}}{k^3},$$
$$M=\frac{a}{k^2}h^{-3},\quad a_1=3-2\sqrt{2},\quad b_1=2\sqrt{2}-1,$$
$$c_1=2\sqrt{2}-4,\quad d_1=\sqrt{2},$$
$$m=\frac{16+12\sqrt{2}}{17+12\sqrt{2}},\quad \eta=\sqrt{\frac{19+12\sqrt{2}}{26-16\sqrt{2}}}.$$

Furthermore, according to ref. [17], we have

$$2(kw+\sigma^2)^2=-9ak^4, \tag{25}$$

from which we have

$$h=\frac{kw+\sigma l^2}{3k^3}.$$

Formula (24) is a new solution represented by elliptic functions with double period.

3. Other solutions by trial equation method

If $c_0\neq 0$, by taking

$$u=a_0+a_1\varphi+\frac{b_1}{\varphi} \tag{26}$$

and

$$\omega=\varphi', \tag{27}$$

where φ is a function to be determined, eq. (22) becomes

$$\left(a_1-\frac{b_1}{\varphi^2}\right)\omega\omega'-\left(Aa_1-\frac{Ab_1}{\varphi^2}\right)\omega+\frac{2b_1}{\varphi^3}\omega^2 =B\left[\left(a_0+a_1\varphi+\frac{b_1}{\varphi}\right)^3-\left(a_0+a_1\varphi+\frac{b_1}{\varphi}\right)\right] +c_0, \tag{28}$$

where

$$A=\frac{k\omega+\sigma l^2}{k^3},\quad B=\frac{a}{k^2},$$

and the prime indicates differential with respect to φ, e.g., $\omega'=\mathrm{d}\omega/\mathrm{d}\varphi$. Without loss of generality, we take the solution of eq. (28) as follows:

$$\omega=\varphi^2+b_0, \tag{29}$$

where b_0 is a constant to be determined. Substituting (29) into (28) yields a polynomial of φ. Setting all coefficients of this polynomial to zero, we get a system of algebraic equations

$$2b_1b_0^2-Bb_1^3=0, \tag{30}$$
$$Ab_1b_0-Ba_0b_1^2=0, \tag{31}$$
$$4b_1b_0-2b_1b_0-Ba_1b_1^2-Ba_0^2b_1+Bb_1=0, \tag{32}$$
$$-Aa_1b_0+Ab_1-Ba_0^3-Ba_0a_1b_1+Ba_0-c=0, \tag{33}$$
$$2a_1b_0-2b_1+2b_1-Ba_0^2a_1-Ba_1^2b_1+Ba_1=0, \tag{34}$$
$$-Aa_1-Ba_0a_1^2=0, \tag{35}$$
$$2a_1-Ba_1^3=0. \tag{36}$$

Solving this algebraic equation system, we obtain a family of values of parameters

$$a_0=-\frac{A}{\sqrt{2B}},\quad a_1=\sqrt{\frac{2}{B}},\quad b_0=-\frac{2B-A^2}{8},$$
$$b_1=-\frac{A^2-2B}{4\sqrt{2B}}. \tag{37}$$

So we have

$$\varphi=-\sqrt{\frac{2B-A^2}{8}}\tanh\left[\sqrt{\frac{2B-A^2}{8}}\xi-\xi_0\right] \tag{38}$$

and

$$\varphi=-\sqrt{\frac{2B-A^2}{8}}\coth\left[\sqrt{\frac{2B-A^2}{8}}\xi-\xi_0\right], \tag{39}$$

where ξ_0 is an arbitrary constant. Therefore, the solutions of eq. (1) are given by

$$u(x,y,t)=-\frac{kw+\sigma l^2}{k^2\sqrt{2a}} -\frac{m}{2k^2\sqrt{a}}\tanh\left[\frac{m}{2\sqrt{2}k^3}(kx+ly+wt)-\xi_0\right] +\frac{m}{2k^2\sqrt{a}}\coth\left[\frac{m}{2\sqrt{2}k^3}(kx+ly+wt)-\xi_0\right] \tag{40}$$

and

$$u(x, y, t) = -\frac{kw + \sigma l^2}{k^2\sqrt{2a}} - \frac{m}{2k^2\sqrt{a}} \coth\left[\frac{m}{2\sqrt{2}k^3}(kx + ly + wt) - \xi_0\right] + \frac{m}{2k^2\sqrt{a}} \tanh\left[\frac{m}{2\sqrt{2}k^3}(kx + ly + wt) - \xi_0\right], \quad (41)$$

where $m = \sqrt{2ak^4 - k^2\omega^2 - 2k\omega\sigma l^2 - \sigma^2 l^4}$. Expressions (40) and (41) are solutions represented by hyperbolic tangent function and hyperbolic cotangent function.

4. Conclusion

In this paper, the canonical-like transformation method and trial equation method are applied to obtain exact solutions to $(2+1)$-dimensional Chaffee–Infante equation. Among those, some new solutions are given. The results show that canonical-like transformation method and trial equation method are powerful for solving nonlinear problems arising in mathematical physics.

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