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Fractional Klein–Gordon equation composed of Jumarie fractional derivative and its interpretation by a smoothness parameter

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Abstract. Klein–Gordon equation is one of the basic steps towards relativistic quantum mechanics. In this paper, we have formulated fractional Klein–Gordon equation via Jumarie fractional derivative and found two types of solutions. Zero-mass solution satisfies photon criteria and non-zero mass satisfies general theory of relativity. Further, we have developed rest mass condition which leads us to the concept of hidden wave. Classical Klein–Gordon equation fails to explain a chargeless system as well as a single-particle system. Using the fractional Klein–Gordon equation, we can overcome the problem. The fractional Klein–Gordon equation also leads to the smoothness parameter which is the measurement of the bumpiness of space. Here, by using this smoothness parameter, we have defined and interpreted the various cases.

Keywords. Jumarie fractional derivative; Mittag–Leffler function; fractional Schrödinger equation; fractional wave function.

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1. Introduction

Klein–Gordon equation is the gateway to relativistic quantum mechanics. The classical (integer order) Klein–Gordon equation was developed on the basis of homogeneous space and time. In mesoscopic and microscopic scales, space and time are not homogeneous enough to be treated with classical calculus. Here, coarse-grained approach is called for, that gives rise to fractional differentials in space and time. Classically, the Klein–Gordon equations use fine-grained approach. The basic problem occurring in the rough and non-homogeneous plane is that the infinitesimal quantities cannot be made arbitrarily zero, instead they have a finite spread or length [1–3], and this is the coarse-grained approach. In classical homogeneous space–time, an infinitesimal quantity of space length $(\mathrm{d}x)$ and time-scale $(\mathrm{d}t)$ is arbitrarily small such that they are taken to be zero. To sustain the finite spread, we can introduce the infinitesimal space and time quantities as $(\mathrm{d}x)^\alpha$ and $(\mathrm{d}t)^\alpha$ in the region. As $(\mathrm{d}x)$ is very small, we have $(\mathrm{d}x)^\alpha > (\mathrm{d}x)$ and $(\mathrm{d}t)^\alpha > (\mathrm{d}t)$ if $0 < \alpha < 1$. In this way, the finite spreads can be taken into account and differentiation and integration with respect to $(\mathrm{d}x)^\alpha$ and $(\mathrm{d}t)^\alpha$ make us to use fractional derivatives and fractional integration. In this paper, we used fractional calculus to study Klein–Gordon equation in the non-homogeneous plane. The non-homogeneous plane can be treated as porous surface or rough surface. Thus, we apply coarse-grained phenomenon, in space and time by using $(\mathrm{d}x)^\alpha$ and $(\mathrm{d}t)^\alpha$. Though we are using a new mathematical tool to study Klein–Gordon equation, we consider that all basic conservation laws are satisfied in this fractional space–time. We modified the quantities as we are dealing with fractional space and time. Fractional calculus was first proposed by Riemann–Liouville [4]. According to the theory proposed by Riemann–Liouville [5], the fractional derivative of a constant is non-zero. This was a contradictory idea from classical calculus. To overcome this difficulty, Caputo [6] developed a new definition of fractional derivative for differentiable function only and for continuous but not necessarily differentiable functions, Jumarie [7–9] gave another definition. This modification supports the derivative of a constant to be zero, as in conjugation to classical calculus. To solve the linear fractional-order differential equations, authors in [4,10–12] used 'one-parameter' Mittag–Leffler function. In this paper, to develop fractional Klein–Gordon equation (FKGE), we shall modify relativistic energy equation [13–19], quantum operators [19] and will use Jumarie fractional derivative and finally we shall obtain two types of solutions. Zero-mass solution satisfies photon's criteria and non-zero mass satisfies general theory of relativity. Further, we develop rest mass condition which leads us to a concept of hidden wave. The hidden wave may help us to understand the physics of particle decay.

Organisation of the paper is as follows: In §2 we describe some basic definitions and assumptions for fractional Klein–Gordon equation, §3 and 4 are used to derive the fractional Klein–Gordon equation and fractional equation of continuity, respectively. Sections 5 and 6 are devoted to describe the negative fractional probability density situation and the physically acceptable conditions. Sections 7–10 are devoted for the generalisation and interpretation of the parameters $\mu(\alpha)$, $B(\alpha)$ and the hidden wave. Finally, some conclusions are given in §11.

2. Some basic definitions and assumptions for fractional Klein–Gordon equation

Mathematicians defined fractional derivatives in various ways. The definitions most used are Riemann–Liouville (R–L) definition [5,14], Caputo definition [6] and modified R–L definition [2,9]. All the above definitions are of non-local type.

2.1 *Riemann–Liouville (R–L) definition of fractional derivative*

According to Riemann–Liouville, the fractional derivative of an integrable function can be defined as

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha+m+1)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m+1} \times \int_a^b (x-\xi)^{m-\alpha} f(\xi)\mathrm{d}\xi, $$

where $m \le \alpha \le m+1$, m is a positive integer. According to this definition, a fractional derivative of a constant function is non-zero [2,5,14]. This is a contradiction of the classical concept of derivative.

2.2 *Jumarie-modified definition of fractional derivative*

To eliminate the problem of R–L fractional derivative mentioned above, Jumarie modified [2,7–9] R–L definition for continuous but not necessarily differentiable functions. Consider such a function $f(x)$ in the range $0 \le x \le a$. Jumarie suggested the fractional αth-order derivative in the form [7–9]

$$ f^{\alpha}(x) = {}_0^JD_x^{\alpha} = \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)\displaystyle\int_a^b (x-\xi)^{m-\alpha} f(\tau)\mathrm{d}x, \ \alpha<0, \\ \dfrac{1}{\Gamma(1-\alpha)}\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{m+1}\displaystyle\int_a^b (x-\tau)^{-\alpha} f(\xi)\mathrm{d}\xi, 0<\alpha\le 1, \\ \left(f^{(\alpha-n)}(x)\right)^{(n)}, \ n \le \alpha < n+1, n>1. \end{cases} $$

From this point of view, Jumarie defined fractional derivative [4,18] as

$$ {}_0^JD_x^{\alpha} f(x) = \frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x_{\alpha}} = f_+^{(\alpha)}(x) = \lim_{h\to 0} \frac{\Delta_+^{(\alpha)[f(x)-f(0)]}}{h^{\alpha}}. $$

In terms of this definition of fractional derivative, the problems of R–L derivative overcome and the following formulae are valid in terms of this modified definition [1,7–9]

$$ D_x^{\alpha} x^{\gamma} = (x^{\gamma})^{(\alpha)} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}; \quad \gamma > -1, $$

$$ \begin{aligned} D_x^{\alpha}[f(x)g(x)] &= (f(x)g(x))^{(\alpha)} \\ &= g(x)D_x^{\alpha}f(x) + f(x)D_x^{\alpha}g(x) \\ &= g(x)f^{(\alpha)}(x) + f(x)g^{(\alpha)}(x)\tau, \\ D_x^{\alpha}[f(g(x))] &= (f(g(x)))^{(\alpha)} = f_g'(x)D_x^{\alpha}[g(x)] \\ &= D_g^{\alpha} f[g(x)](g_x')^{\alpha}. \end{aligned} $$

2.3 *Mittag–Leffler function*

The one-parameter Mittag–Leffler function is a special function which is defined as an infinite series [5] as

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \quad z \in C $$

which coincides with normal exponential function when $\alpha = 1$. To solve the linear fractional differential equations, the one-parameter Mittag–Leffler function plays an important role [4].

2.4 *Basic assumptions for fractional Klein–Gordon equation*

Before developing fractional Klein–Gordon equation, we are making three assumptions.

(1) Total fractional energy (refer §A.6), i.e. ϵ_{α} of a particle of fractional mass (refer §A.1), i.e. m_{α} and fractional momentum (refer §A.1), i.e. p_{α} is related by the

following relations:

$$\epsilon_\alpha^2 = A(\alpha) p_\alpha^2 c^2 + B(\alpha) m_\alpha^2 c^2, \tag{1}$$

where $A(\alpha)$ and $B(\alpha)$ are two parameters which depend on the values of α such that, when $\alpha = 1$, $A(\alpha) = 1$, $B(\alpha) = 1$. Equation (1) transforms to classical energy equation [18], i.e., $\epsilon_1^2 = p_1^2 c^2 + m_1^2 c^4$ or $\epsilon^2 = p^2 c^2 + m^2 c^4$, if we use $A(1) = 1$, $B(1) = 1$.
(2) The momentum operator in fractional sense is defined as [18]

$$\hat{p}_\alpha \to -i\hbar_\alpha \frac{\partial^\alpha}{\partial x^\alpha}. \tag{2}$$

If we consider a fractional wave function as $\Psi_\alpha = \phi(x^\alpha, t^\alpha)$ then we have

$$\hat{p}_\alpha \phi(x^\alpha, t^\alpha) = -i\hbar_\alpha \frac{\partial^\alpha \phi(x^\alpha, t^\alpha)}{\partial x^\alpha} = p_\alpha \phi(x^\alpha, t^\alpha). \tag{3}$$

Here, p_α is the fractional momentum which is the eigenvalue of the momentum operator $\hat{p_\alpha}$. $\hbar_\alpha$ is the reduced Planck constant (refer §A.5) of order α. For $\alpha = 1$, eq. (3) is transformed to one-dimensional momentum operator, i.e.,

$$\hat{p}_1 \phi(x, t) = -i\hbar_1 \left(\frac{\partial}{\partial x} \phi(x, t) \right)$$

or simply

$$\hat{p}\phi = -i\hbar \frac{\partial \phi}{\partial x}.$$

In one-dimensional notation, we have $p_1 = p_x$, $\hbar_1 = \hbar$ and p_x is the momentum in x direction.
(3) Energy operator in fractional sense is defined as [18]

$$\hat{\epsilon}_\alpha \to i\hbar_\alpha \frac{\partial^\alpha}{\partial t^\alpha}. \tag{4}$$

For fractional wave function $\Psi_\alpha = \phi(x^\alpha, t^\alpha)$, we write

$$\hat{\epsilon}_\alpha \phi(x^\alpha, t^\alpha) = i\hbar_\alpha \frac{\partial^\alpha \phi(x^\alpha, t^\alpha)}{\partial t^\alpha} = \epsilon_\alpha \phi(x^\alpha, t^\alpha). \tag{5}$$

Here, ϵ_α is the eigenvalue of the energy operator $\hat{\epsilon_\alpha}$. For $\alpha = 1$, eq. (5) coincides with one-dimensional energy equation, i.e.,

$$\hat{\epsilon}_1 \phi = i\hbar \frac{\partial \phi}{\partial t}.$$

In the next section, we shall develop the fractional Klein–Godron equation using the above defined operators and assumptions.

3. Derivation of fractional Klein–Gordon equation

From eq. (5) we have

$$\epsilon_\alpha \phi(x^\alpha, t^\alpha) = i\hbar_\alpha \frac{\partial^\alpha \phi(x^\alpha, t^\alpha)}{\partial t^\alpha}.$$

Operating $i\hbar_\alpha(\partial^\alpha/\partial t^\alpha)$ on both sides of (5), we get

$$\epsilon_\alpha \left(i\hbar_\alpha \frac{\partial^\alpha \phi(x^\alpha, t^\alpha)}{\partial t^\alpha} \right) = i\hbar_\alpha \left(i\hbar_\alpha \frac{\partial^{2\alpha} \phi(x^\alpha, t^\alpha)}{\partial t^{2\alpha}} \right).$$

Using eq. (4) we write following expression:

$$\epsilon_\alpha^2 \phi(x^\alpha, t^\alpha) = -\hbar_\alpha^2 \frac{\partial^{2\alpha} \phi(x^\alpha, t^\alpha)}{\partial t^{2\alpha}}. \tag{6}$$

From eq. (3) we have

$$p_\alpha \phi(x^\alpha, t^\alpha) = -i\hbar_\alpha \frac{\partial^\alpha \phi(x^\alpha, t^\alpha)}{\partial t^\alpha}.$$

Operating with $-i\hbar_\alpha(\partial^\alpha/\partial x^\alpha)$ on both sides of the above we get

$$p_\alpha \left(-i\hbar_\alpha \frac{\partial^\alpha \phi(x^\alpha, t^\alpha)}{\partial x^\alpha} \right) = -i\hbar_\alpha \left(-i\hbar_\alpha \frac{\partial^{2\alpha} \phi(x^\alpha, t^\alpha)}{\partial x^{2\alpha}} \right)$$

or

$$p_\alpha^2 \phi(x^\alpha, t^\alpha) = -\hbar_\alpha^2 \frac{\partial^{2\alpha} \phi(x^\alpha, t^\alpha)}{\partial x^{2\alpha}}, \tag{7}$$

where $\phi(x^\alpha, t^\alpha)$ is an eigenfunction of the fractional energy operator and the fractional momentum operator. From eq. (1) we get

$$\epsilon_\alpha^2 \phi\left(x^\alpha, t^\alpha\right) = A\left(\alpha\right) p_\alpha^2 c^2 \phi\left(x^\alpha, t^\alpha\right) + B\left(\alpha\right) m_\alpha^2 c^4 \phi\left(x^\alpha, t^\alpha\right).$$

Using (6) and (7) in the above equation, we get the following equation:

$$-\hbar_\alpha^2 \frac{\partial^{2\alpha} \phi(x^\alpha, t^\alpha)}{\partial t^{2\alpha}} = -A(\alpha)\hbar_\alpha^2 \frac{\partial^{2\alpha} \phi(x^\alpha, t^\alpha)}{\partial x^{2\alpha}} c^2 + B(\alpha) m_\alpha^2 c^4 \phi(x^\alpha, t^\alpha)$$

or

$$-\frac{1}{c^2} \frac{\partial^{2\alpha} \phi(x^\alpha, t^\alpha)}{\partial t^{2\alpha}} + A(\alpha) \frac{\partial^{2\alpha} \phi(x^\alpha, t^\alpha)}{\partial x^{2\alpha}} = \frac{B(\alpha)}{\hbar_\alpha^2} m_\alpha^2 c^2 \phi(x^\alpha, t^\alpha). \tag{8}$$

Equation (8) is the fractional Klein–Gordon equation.

4. Fractional equation of continuity

Taking complex conjugate of eq. (8) we have the following equation:

$$-\frac{1}{c^2}\frac{\partial^{2\alpha}\phi^*(x^\alpha,t^\alpha)}{\partial t^{2\alpha}}+A(\alpha)\frac{\partial^{2\alpha}\phi^*(x^\alpha,t^\alpha)}{\partial x^{2\alpha}} =\frac{B(\alpha)}{\hbar_\alpha^2}m_\alpha^2c^2\phi^*(x^\alpha,t^\alpha), \tag{9}$$

where $\phi^*(x^\alpha,t^\alpha)$ is the complex conjugate of $\phi(x^\alpha,t^\alpha)$. Multiplying eq. (8) with $\phi^*(x^\alpha,t^\alpha)$ from left and eq. (9) with $\phi(x^\alpha,t^\alpha)$ from left, and subtracting, we shall get the following equation:

$$\begin{aligned}A(\alpha)(\phi^*(x^\alpha,t^\alpha)&D_x^{2\alpha}[\phi(x^\alpha,t^\alpha)]\\&-\phi(x^\alpha,t^\alpha)D_x^{2\alpha}[\phi^*(x^\alpha,t^\alpha)])\\=-\frac{1}{c^2}\Big(&\phi^*(x^\alpha,t^\alpha)D_t^{2\alpha}[\phi(x^\alpha,t^\alpha)]\\&-\phi(x^\alpha,t^\alpha)D_t^{2\alpha}[\phi^*(x^\alpha,t^\alpha)]\Big).\end{aligned}$$

Here, we defined the operators ${}^JD_x^\alpha\equiv\partial^\alpha/\partial x^\alpha$,${}^JD_x^{2\alpha}={}^JD_x^\alpha.{}^JD_x^\alpha$ and ${}^JD_t^\alpha\equiv\partial^\alpha/\partial t^\alpha$ as Jumarie fractional derivative and for the sake of simplicity, we shall use the notation D_x^α in place of ${}^JD_x^\alpha$. Using Jumarie chain rule [2,7], we have the following steps:

$$\begin{aligned}D_x^\alpha[\phi^*(x^\alpha,t^\alpha)&D_x^\alpha\phi(x^\alpha,t^\alpha)-\phi(x^\alpha,t^\alpha)D_x^\alpha\phi^*(x^\alpha,t^\alpha)]\\=\ &\phi^*(x^\alpha,t^\alpha)D_x^\alpha D_x^\alpha\phi(x^\alpha,t^\alpha)\\&-D_x^\alpha\phi(x^\alpha,t^\alpha)D_x^\alpha\phi^*(x^\alpha,t^\alpha)\\&+D_x^\alpha\phi^*(x^\alpha,t^\alpha)D_x^\alpha\phi(x^\alpha,t^\alpha)\\&-\phi(x^\alpha,t^\alpha)D_x^\alpha D_x^\alpha\phi^*(x^\alpha,t^\alpha).\end{aligned}$$

Using the above equation we can write the following result:

$$\begin{aligned}A(\alpha)D_x^\alpha[\phi^*(x^\alpha,t^\alpha)&D_x^\alpha\phi(x^\alpha,t^\alpha)\\&-\phi(x^\alpha,t^\alpha)D_x^\alpha\phi^*(x^\alpha,t^\alpha)]\\=-\frac{1}{c^2}&D_t^\alpha[\phi^*(x^\alpha,t^\alpha)D_t^\alpha\phi(x^\alpha,t^\alpha)\\-&\phi(x^\alpha,t^\alpha)D_x^\alpha\phi^*(x^\alpha,t^\alpha)].\end{aligned}\tag{10}$$

We define the probability current density (j_α) and the fractional probability density (ρ_α) in terms of fractional derivatives in the following form:

$$j_\alpha=\frac{i\hbar_\alpha}{2^\alpha m_\alpha}[\phi^*(x^\alpha,t^\alpha)D_x^\alpha\phi(x^\alpha,t^\alpha)-\phi(x^\alpha,t^\alpha)D_x^\alpha\phi^*(x^\alpha,t^\alpha)]$$

and

$$\rho_\alpha=\frac{i\hbar_\alpha}{2^\alpha m_\alpha c^2A(\alpha)}[\phi^*(x^\alpha,t^\alpha)D_t^\alpha\phi(x^\alpha,t^\alpha)-\phi(x^\alpha,t^\alpha)D_t^\alpha\phi^*(x^\alpha,t^\alpha)].$$

Then eq. (10) reduces to fractional equation of continuity in the form

$$D_x^\alpha[j_\alpha]=D_t^\alpha[\rho_\alpha]. \tag{11}$$

The expression of ρ_α can be rearranged in the following form:

$$\rho_\alpha=\frac{1}{2^\alpha m_\alpha c^2A(\alpha)}[\phi^*(x^\alpha,t^\alpha)i\hbar_\alpha D_t^\alpha\phi(x^\alpha,t^\alpha)-\phi(x^\alpha,t^\alpha)i\hbar_\alpha D_t^\alpha\phi^*(x^\alpha,t^\alpha)]. \tag{12}$$

According to fractional Schrödinger equation [18], we have the following equation:

$$D_t^\alpha\phi(x^\alpha,t^\alpha)=-\frac{1}{\hbar_\alpha}i\epsilon_\alpha\phi(x^\alpha,t^\alpha)$$

or

$$i\hbar_\alpha D_t^\alpha\phi(x^\alpha,t^\alpha)=\epsilon_\alpha\phi(x^\alpha,t^\alpha)$$

and the corresponding conjugate equation is

$$-i\hbar_\alpha D_t^\alpha\phi^*(x^\alpha,t^\alpha)=\epsilon_\alpha\phi^*(x^\alpha,t^\alpha).$$

Using these conditions in eq. (12) we get the following equation:

$$\begin{aligned}\rho_\alpha&=\frac{1}{2^\alpha m_\alpha c^2A(\alpha)}\left[\phi^*(x^\alpha,t^\alpha)\epsilon_\alpha\phi(x^\alpha,t^\alpha)+\phi(x^\alpha,t^\alpha)\epsilon_\alpha\phi^*(x^\alpha,t^\alpha)\right]\\&=\frac{2\epsilon_\alpha}{2^\alpha m_\alpha c^2A(\alpha)}\left[\phi^*(x^\alpha,t^\alpha)\phi(x^\alpha,t^\alpha)\right],\end{aligned}$$

$$\rho_\alpha=\frac{2\epsilon_\alpha}{2^\alpha m_\alpha c^2A(\alpha)}|\phi(x^\alpha,t^\alpha)|^2. \tag{13}$$

Thus, we can find the fractional probability density in terms of fractional energy. We can see that, if the fractional energy is positive (i.e. $\epsilon_\alpha>0$), the fractional probability density is positive. If fractional energy is negative (i.e. $\epsilon_\alpha<0$), the system collapses because fractional probability density must be positive. Thus, the expression for ρ_α in (13) cannot be fractional positional probability density. Thus, fractional Klein–Gordon equation fails like the classical integer-order Klein–Gordon equation for negative energy situation.

5. Negative fractional probability density problem

Due to a negative fractional probability density, the physical solution is not possible. For the physical solution, we need some extra condition which are described below.

(a) *Pauli–Weisskopf interpretation*

In this interpretation [15], we can multiply both sides of (13) with an electric charge (e), and then we shall get the following modified expression (for electric charge density):

$$e\rho_\alpha = \frac{2e\epsilon_\alpha}{2^\alpha m_\alpha c^2 A(\alpha)} |\phi(x^\alpha, t^\alpha)|^2.$$

As the electrical charge is sign dependent, if fractional energy ϵ_α is negative, the term $e\epsilon_\alpha$ will be positive for the negative charge e. Clearly, as $e\rho_\alpha > 0$, that will be always positive. Thus, we write the following rules:

$$e\rho_\alpha > 0,\ \epsilon_\alpha > 0,\ e > 0,$$
$$e\rho_\alpha > 0,\ \epsilon_\alpha < 0,\ e < 0.$$

Here, $e\rho_\alpha$ is defined as the fractional electric charge density. Thus, we need a system of particles which contains both signs of charges. This situation can also be treated as a 'particle–antiparticle' system.

(b) *Single-particle interpretation*

While treating the system with a single particle, we cannot consider both signs of charges at the same time. To resolve this problem, we can use another interpretation. For a single particle with positive energy state, we can consider the parameter $A(\alpha)$ as a positive quantity and negative energy state $A(\alpha)$ is negative. If we make these assumptions, the fractional position probability (ρ_α) given in (13) is always positive. It is valid for both chargeless or charged particle. This interpretation needs more fundamental investigation.

6. Physically acceptable conditions

Equation (8) will be a wave equation in space variable under the following two conditions:

(a) *Zero mass condition*

If we consider a particle of fractional rest mass, $m_\alpha = 0$, we shall get from eq. (8), the following expression:

$$-\frac{1}{A(\alpha)c^2}\frac{\partial^{2\alpha}\phi(x^\alpha, t^\alpha)}{\partial t^{2\alpha}} + \frac{\partial^{2\alpha}\phi(x^\alpha, t^\alpha)}{\partial x^{2\alpha}} = 0. \quad (14)$$

Equation (14) is the form of general fractional wave equation and the corresponding fractional velocity of this wave is $V_\alpha = c\sqrt{A(\alpha)}$. Special theory of relativity suggests [16] that $V_\alpha \le c$. Thus, the condition we obtain is $\sqrt{A(\alpha)} \le 1$. Now taking the above expression, we define smoothness parameter $\mu(\alpha)$ as

$$\frac{V_\alpha}{c} = \sqrt{A(\alpha)} = \mu(\alpha), \quad (15)$$

where $\mu(\alpha)$ is a special parameter which defines the characteristic of fractional space. When $\mu(\alpha) = 1$, we have $V_\alpha = c$. This means that the space is homogeneous without any porosity or roughness. This happens because the fractional mass of the particle is zero, which means that the particle we treat is a photon in free space (vacuum) according to the special theory of relativity [8]. If $\mu(\alpha) < 1$, the velocity of the photon is less than the speed of light in free space. That is, the space we consider is not free, but there are some other particles which make the space heterogeneous. From eq. (15) we can see that, if $A(\alpha)$ is negative, the velocity of the massless particle (photon) is a complex quantity. Thus, we can predict another particle with velocity c. Here, we can predict a complex photon with a velocity of the same magnitude.

(b) *Non-zero mass in motion*

In this case, $m_\alpha \neq 0$ and $B(\alpha) = 0$, eq. (8) reduces to

$$-\frac{1}{A(\alpha)c^2}\frac{\partial^{2\alpha}\phi(x^\alpha, t^\alpha)}{\partial t^{2\alpha}} + \frac{\partial^{2\alpha}\phi(x^\alpha, t^\alpha))}{\partial x^{2\alpha}} = 0 \quad (16)$$

which gives the same equation as (14). Thus, we get the same situation if we put either fractional mass zero or $B(\alpha) = 0$. This implies that the conditions (i) $B(\alpha) = 0$, but mass of the particle is non-zero and (ii) $B(\alpha) \neq 0$, but mass of the particle is zero, lead to the same equation despite the different physical situations.

From the discussion of the previous subsection, we shall also find here $(V_\alpha/c) = \sqrt{A(\alpha)} = \mu(\alpha)$. There is a different boundary situation, i.e. $\mu(\alpha) < 1$. This occurs due to the presence of non-zero mass. As space becomes non-homogeneous and rough, we can say it as the self-distortion of space. Now, the probability current density in this condition suggested by eq. (13) is

$$\rho_\alpha = \frac{2\epsilon_\alpha}{2^\alpha m_\alpha c^2 A(\alpha)} |\phi(x^\alpha, t^\alpha)|^2. \quad (17)$$

To satisfy the positive fractional probability density, we need $A(\alpha)$ to be positive for positive fractional energy and negative for negative fractional energy. Thus, we can predict the existence of another particle with real velocity for positive fractional energy and a particle with complex velocity for negative fractional energy. Hence, we can write

$$\begin{cases} \epsilon_\alpha > 0,\ A(\alpha) > 0, & \text{positive velocity } = c|\sqrt{A(\alpha)}| \\ \epsilon_\alpha < 0,\ A(\alpha) < 0, & \text{complex positive velocity} \\ = ic|\sqrt{A(\alpha)}|. \end{cases}$$

For negative energy, $\mu(\alpha)$ is a complex quantity and its value is $\mu(\alpha) = i|\sqrt{A(\alpha)}|$. So, it can be treated as $\mu(\alpha) = |\sqrt{A(\alpha)}| \exp(i\pi/2)$. That means, there is a phase shift of angle $\pi/2$. Thus, smoothness parameter $\mu(\alpha)$ has phase shifted through an angle $\pi/2$ for a complex particle keeping the magnitude the same. Thus, we have $|\mu(\alpha)_{\text{complex particle}}| = |\mu(\alpha)_{\text{particle}}|$. Consequently, velocity of the negative energy particle also suffers a phase shift through an angle $\pi/2$.

7. Generalisation

Thus, the parameter $\mu(\alpha)$ can be defined as follows:

$$\begin{cases} |\mu(\alpha)| \leq 1 \text{ for } m_\alpha = 0 \\ |\mu(\alpha)| < 1 \text{ for } m_\alpha \neq 0. \end{cases}$$

For a photon, the roughness parameter is always one, only if there exists no other cause of roughness in space. If any other bodies are present on the path of the photon, we can have a lower velocity of photons with respect to the velocity of photons in free and smooth space. For the massive body, which is the cause of maximum roughness of space (may be like a black hole); the speed of light is zero. Clearly, in this case, photon seized its motion.

8. A hidden wave

Now, we consider the situation when $\mu(\alpha) = \sqrt{A(\alpha)} = 0$, $B(\alpha) \neq 0$ and $m_\alpha \neq 0$, then from (8) we get

$$-\frac{1}{c^2}\frac{\partial^{2\alpha}\phi(x^\alpha, t^\alpha)}{\partial t^{2\alpha}} = \frac{B(\alpha)}{\hbar_\alpha^2} m_\alpha^2 c^2 \phi(x^\alpha, t^\alpha). \quad (18)$$

Rearranging eq. (18) we have

$$\frac{\partial^{2\alpha}\phi(x^\alpha, t^\alpha)}{\partial t^{2\alpha}} = \frac{B(\alpha)}{\hbar_\alpha^2} m_\alpha^2 c^4 \phi(x^\alpha, t^\alpha). \quad (19)$$

As $\mu(\alpha) = \sqrt{A(\alpha)} = 0$, the particle is at rest, and consequently, there is no space variation. Then the fractional wave function of eq. (19) varies with time only. Thus, eq. (19) reduces to the following form:

$$\frac{\mathrm{d}^{2\alpha}\phi(t^\alpha)}{\mathrm{d}t^{2\alpha}} = \frac{B(\alpha)}{\hbar_\alpha^2} m_\alpha^2 c^4 \phi(t^\alpha). \quad (20)$$

The solution of eq. (20) is suggested by Ghosh *et al* [4] in the following form in terms of Mittag–Leffler function of complex arguments.

$$\phi(t^\alpha) = \Lambda E_\alpha \left(\pm i \frac{1}{\hbar_\alpha} \sqrt{B(\alpha) m_\alpha^2 c^4 t^\alpha} \right). \quad (21)$$

This implies that the particle is oscillating in time, but localised in space which means that the particle is localised in space, but getting older in time. We call this concept a hidden wave. If a particle is localised in space, it does not mean that it is localised in time also. The particle is moving on a timeline. For example, if we took a particle muon at rest in space i.e. localised with respect to its position, it still decays. This decay says that there is always a dynamic system in time though the particle is at rest (with respect to space). This type of phenomenon can occur due to the porosity or roughness of the structure of time. The roughness or bumpy character is due to the non-zero mass of the particle.

9. Discussion on $\mu(\alpha)$

The parameter $\mu(\alpha)$ has a deeper significance. For a photon, the rest mass is zero. From the theory of relativity [8,14], the speed of a photon in free space is maximum and its value is c. Thus, we have $\mu(\alpha) = \frac{c}{c} = \sqrt{A(\alpha)} = 1$ which suggests that corresponding space is homogeneous and without any roughness. For non-zero mass, $V_\alpha < c$ implies $\mu(\alpha) < 1$ space is non-homogeneous and rough with heterogeneity. According to general theory of relativity [13,16], mass deforms the space–time. This curved space–time has an effect on the motion of the particle and leads to the theory of gravity. A photon has no rest mass and so it does not bend or distort or deform the space. That is why the light moves in a straight line in free space. If we consider the non-zero mass condition, we may have $\mu(\alpha) = 0$. This means that particle is at rest. This situation may explain as the infinitely massive particle which largely distorts the space and creates an infinite well; like probable Black Holes! The particle hence is trapped in it and $\frac{v_\alpha}{c} = \sqrt{A(\alpha)} = \mu(\alpha)$ defines the roughness of the space. For more rough space, the value of $\mu(\alpha)$ decreases. A schematic diagram is drawn as shown in figure 1.

In figure 1 we can see that a massive particle is trapped. Here, velocity of the particle is zero due to the roughness created by the mass. Secondly, we can see a smooth straight line due to the zero-mass particle. For a particle with non-zero mass (not infinitely massive) and non-zero velocity, the space has roughness, but not like the case of infinitely massive mass. Here, $\mu(\alpha) = 0$ means there is a black hole. Space is so much curved that the body is trapped inside its own created roughness. This analysis is also supported by general theory of relativity [13,16]. For no mass condition, we have $m_\alpha = 0$, $\mu(\alpha) = 1$ then we can say that $B(\alpha) = 0$.

10. Discussion on $B(\alpha)$

From eq. (20), we write

$$\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}t^{2\alpha}}\phi(t^\alpha) + \left(\frac{B(\alpha)}{\hbar_\alpha^2}\right) m_\alpha^2 c^4 \phi(t^\alpha) = 0.$$

Here, we assumed that $A(\alpha) = 0$. Thus, the total energy eq. (1) gives $\epsilon_\alpha^2 = B(\alpha) m_\alpha^2 c^4$ or equivalently $\epsilon_\alpha = \sqrt{B(\alpha)} m_\alpha c^2$. This implies that the total energy in rest condition is not exactly equal to the fractional mass multiplied by the square of the velocity of light. There is an extra factor involving the parameter $B(\alpha)$ such that $\sqrt{B(\alpha)} = (\epsilon_\alpha / m_\alpha c^2)$. This is the fraction of energy ratio. We may also define this as the smoothness parameter of time. More study is needed to know

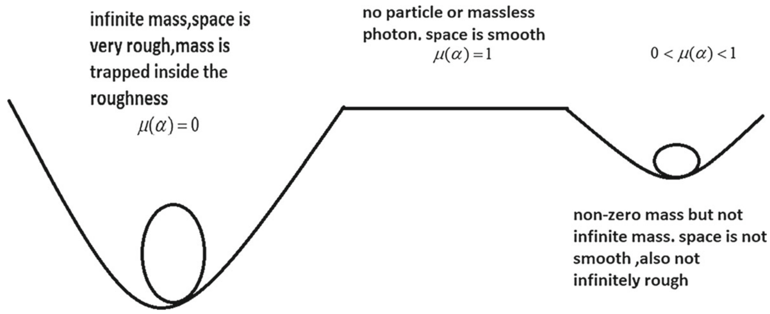

Figure 1. Effect of smoothness parameter on space.

the properties of the parameter. When $\alpha \to 1$, we have $\lim_{\alpha\to 1}\sqrt{B(\alpha)} = \lim_{\alpha\to 1}(\epsilon_\alpha/m_\alpha c^2) = 1$. When $\alpha = 1$, then Einstein's theory of relativity is satisfied. Now eq. (20) can be written as the following fractional differential equation:

$$\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}t^{2\alpha}}\phi(t^\alpha) + \omega\alpha^2\phi(t^\alpha) = 0.$$

Here ω_α is the fractional angular frequency and further we have the following expressions:

$$\omega_\alpha^2 = \frac{B(\alpha)}{\hbar_\alpha^2} m_\alpha^2 c^4, \tag{22}$$

$$\omega_\alpha = \pm\frac{\sqrt{B(\alpha)}}{\hbar_\alpha} m_\alpha c^2. \tag{23}$$

From the definition of fractional angular frequency [18], that is,

$$\omega_\alpha = \frac{2\pi}{N_\alpha} = \frac{\sqrt{B(\alpha)}}{\hbar_\alpha} m_\alpha c^2,$$

where N_α is the fractional time period. From (23) and using the value of ω_α in terms of N_α, we get

$$N_\alpha = \frac{2\pi\hbar_\alpha}{\sqrt{B(\alpha)}m_\alpha c^2} \tag{24}$$

which is the time period of a particle in the rest condition. The rest particle 'oscillate in time' with an angular frequency which is

$$\omega_\alpha = \pm\frac{\sqrt{(B(\alpha))}}{\hbar_\alpha} m_\alpha c^2.$$

For $\alpha = 1$, the time period is $N_1 = 2\pi\hbar_1/\sqrt{B(1)}m_1c^2$ and here we define $\hbar_1 = \hbar$, $B(1) = 1$, $m_1 = m$. If we use the energy hypothesis of Planck [16], $\epsilon_1 = \epsilon = \hbar\omega$ and equate it with the rest mass energy mc^2, we shall get

$$N_1 = \frac{2\pi\hbar_\alpha}{\hbar\omega} = \frac{2\pi}{\omega}$$

which is a classical time period, as expected. From eq. (23), we can see that the fractional angular frequency may have two values – negative or positive. As a result, the time period may be negative or positive. Thus, we define two time periods using (23) and (24) as

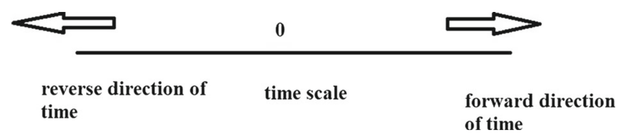

Figure 2. A schematic diagram of the fractional time period.

$$N_{\alpha\pm} = \pm\frac{2\pi\hbar_\alpha}{\sqrt{B(\alpha)}m_\alpha c^2}. \tag{25}$$

A schematic diagram of the fractional time period is shown in figure 2. From eq. (25), we can get $|N_{\alpha+}| = |N_{\alpha-}|$. In eq. (25) if we put $B(\alpha) = 0$, we shall encounter an unavoidable singularity, but this is not unavoidable. As $B(\alpha) = 0$, the solution of eq. (20) becomes

$$\phi(t^\alpha) = C_1 + C_2 t^\alpha, \tag{26}$$

where C_1, C_2 are arbitrary constants. The solution (26) is not a wave function. It is a power-law function and the solution must satisfy the boundary condition, i.e., at infinity, it must be vanished. Hence, if we put $B(\alpha) = 0$ in eq. (20) then its solution is not a wave function, but actually a power-law function. Hence, if we put both the parameters zero, the particle will be seized in space and time. But its internal energy will not allow the seize. Stopping a particle in time and space scale means it is independent of time and space. This sounds absurd. So we cannot have a physical system with $B(\alpha) = 0$ and $A(\alpha) = 0$ at the same time. Hence, to get a wave-like solution, we should not use $B(\alpha) = 0$ when $A(\alpha) = 0$. If we do, there will be an inescapable situation and the situation is not physically valid. Moreover, the total energy will be zero, and so we cannot consider both the parameters as zero simultaneously.

Figure 2 suggests that from the beginning of time the oscillation has two time scales with the same magnitude.

One is in the forward direction and the other is in the reverse direction. Further study is needed for physical interpretability.

11. Conclusion

In this paper, we have established relativistic fractional-order Klein–Gordon equation with Jumarie-type fractional derivative. We found two parameters, $A(\alpha)$ and $B(\alpha)$. The parameter $A(\alpha)$ is the square of the smoothness parameter of the fractional space and $B(\alpha)$ can be defined as roughness parameter of time. Fractional space is something like a porous or a bumpy space. To understand the physics of this space, we need to know the properties of space. The smoothness parameter plays this role. A similar role is played by the parameter $B(\alpha)$. The square root of the parameter $B(\alpha)$ is the smoothness parameter for time. Higher mass of particle lowers the smoothness parameters. As a result, space is bumpier due to the particles' own mass. If we increase the mass of the particle sufficiently high, we may have a black hole like situation. We studied fractional Klein–Gordon equation and found various interesting facts like hidden waves, smoothness parameter, negative time-line and a basic connection between our theory and general theory of relativity. The fractional K–G equation suggests all the possibilities which can be connected with general theory of relativity. Also, we found the smoothness parameter which decides the motion of a particle. Further study is needed to get more information.

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Appendix: Description of fractional quantities used in the paper

A.1 *Fractional mass and momentum*

Fractional mass m_α is defined as $m_\alpha = \int \rho \mathrm{d}x^\alpha$, where ρ is the fractional linear mass density in one dimension; for constant ρ, it is $m_\alpha = \rho \int \mathrm{d}x^\alpha$. This integration is with respect to coarse-grained space, i.e. $\mathrm{d}x^\alpha$. We have considered that the density is the same as in the case $\alpha = 1$ [18]. The fractional change of displacement, i.e. $\mathrm{d}^\alpha x (= \Gamma(1+\alpha)\mathrm{d}x)$ per unit change in fractional time differential $(\mathrm{d}t)^\alpha$ is the fractional velocity, i.e. for $0 < \alpha < 1$: $v_\alpha = \frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \alpha! \frac{\mathrm{d}x}{(\mathrm{d}t)^\alpha}$.

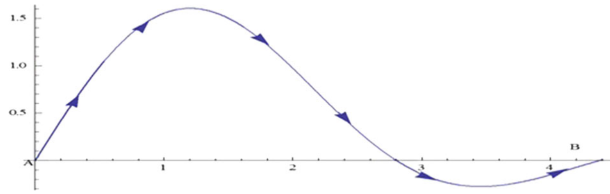

Figure 3. Plot showing fractional wavelength.

Then we can write [18]

$$\frac{\mathrm{d}^{1-\alpha}}{\mathrm{d}t^{1-\alpha}}(v_\alpha) = \frac{\mathrm{d}x}{\mathrm{d}t} = v; \quad 0 < \alpha < 1.$$

Now we shall define fractional momentum. When a body of fractional mass m_α is moving with fractional velocity v_α, we can define fractional momentum in the form $p_\alpha = m_\alpha v_\alpha$. It coincides with classical momentum at $\alpha = 1$. Now, in terms of fractional momentum, we write the following expressions for fractional mass:

$$m_\alpha = \frac{p_\alpha}{v_\alpha} = \frac{p_\alpha / c}{\mu(\alpha)}, \quad \text{where } \mu(\alpha) = \frac{v_\alpha}{c}.$$

Thus, smoothness parameter is directly related to the fractional mass of the system.

A.2 *Fractional wavelength*

Fractional wavelength is demonstrated by a plot (figure 3) of a fractional wave of the order $\alpha = 0.8$.

The wavelength is the distance, which is covered by a fractional wave in a full fractional cycle. Fractional wavelength is not a fixed quantity. It changes with the evolution of fractional time, like a damped oscillating wave.

A.3 *Fractional time period*

Consider a fractional wave propagating in a medium. It must repeat its initial phase after a certain time. This time of repetition for a fractional wave is defined as the fractional time period [18]. The time taken N_α for a wave to cover the distance AB (figure 3) is the fractional time period. We considered this as the first-order time period. As wavelength changes, the time period also changes with the wave propagation. We assume that $\lambda_\alpha = v_\alpha N_\alpha$.

A.4 *Fractional angular frequency*

Figure 3 is the polar plot of the fractional wave of the order $\alpha = 0.8$. In this polar plot, we can easily see that the wave returns to the same point after completing a fractional cycle, i.e. to its origin. By polar plot, we can say that the angle traversed in a full fractional cycle

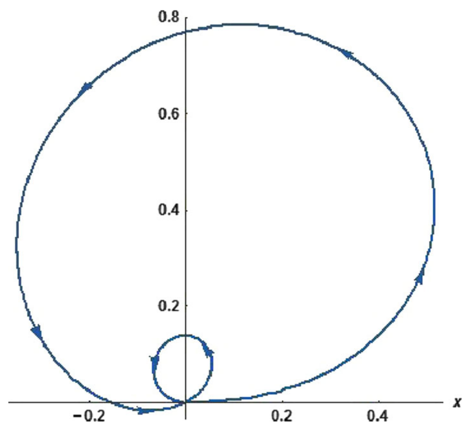

Figure 4. Plot showing the concept of fractional angular frequency.

is 2π. Thus, the fractional angular frequency can be assigned as $\omega_\alpha = 2\pi/N\alpha$.

In figure 4, the initial line is along the horizontal line. From this, we can see that the product of fractional angular momentum and fractional time period T_α is always 2π though both are varying. In limiting condition, we have

$$\lim_{\alpha\to 1} \omega_\alpha = \lim_{\alpha\to 1} \frac{\omega_\alpha}{\omega} = 1.$$

A.5 *Fractional reduced Planck constant*

In this section, we shall show that the fractional Planck constant and Planck constant are the same. Similarly, fractional reduced Planck constant and reduced Planck constant are also the same. Fractional reduced Planck constant $\hbar_\alpha$ is a basic constant for α ordered fractional system. At $\alpha = 1$, this constant is the same as the reduced Planck constant $\hbar$. This fractional reduced Planck constant $\hbar_\alpha$ is used for quantisation of fractional space phenomena at microscopic level. In this paper, we have introduced fractional Planck constant $\hbar_\alpha$ as a basic constant. For the limiting condition of α, this constant is of the form of reduced Planck constant $\hbar$. Consider integer (i.e. classical) and fractional energies $E = \hbar\omega = h\upsilon$ and $\epsilon_\alpha = \hbar_\alpha\omega\alpha = h_\alpha\upsilon_\alpha$. Then, we can write for the limiting condition, i.e. with $\alpha \to 1$, $E = h\upsilon = h_\alpha\upsilon_\alpha$. Here, υ is the integer-order frequency and υ_α is the fractional-order frequency. We write the following expression with above description $h = \lim_{\alpha\to 1}(\upsilon_\alpha/\upsilon)h_\alpha$ and

$$\hbar = \lim_{\alpha\to 1} \frac{\omega_\alpha}{\omega}\hbar_\alpha.$$

Using Appendix A.5, we have

$$\hbar = \lim_{\alpha\to 1} \hbar_\alpha.$$

We have energy proportional to angular frequency and this assumption does not depend upon the value of α. Therefore, we have $E \propto \omega$ and $\epsilon_\alpha \propto \omega_\alpha$. If we remove the proportionality, we can have a constant such that the following condition is satisfied:

$$\frac{\epsilon_\alpha}{\omega_\alpha} = \frac{\epsilon_\beta}{\omega_\beta} = \cdots = \hbar \quad (0 < \alpha < \beta < \cdots < 1).$$

Therefore, we have $\hbar_\alpha = \hbar$ and hence $h_\alpha = h$. We can conclude that the fractional Planck constant is nothing but Planck constant. The reduced fractional Planck constant is also the reduced Planck constant.

A.6 *Fractional energy*

Fractional energy is defined as the sum of fractional kinetic energy and fractional potential energy. We previously defined [18] fractional kinetic energy as

$$T_\alpha = \frac{1}{2^\alpha m_\alpha} p_\alpha^2$$

and fractional potential energy as in the general form of $V(x^\alpha)$. Thus, the total fractional energy is

$$\epsilon_\alpha = \frac{1}{2^\alpha m_\alpha} p_\alpha^2 + V(x^\alpha).$$

At $\alpha = 1$, total fractional energy is the same as the classical total energy.

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