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Volterra integral equation-factorisation method and nucleus–nucleus elastic scattering

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Abstract. An approximate solution for the nuclear Hulthén plus atomic Hulthén potentials is constructed by solving the associated Volterra integral equation by series substitution method. Within the framework of supersymmetry-inspired factorisation method, this solution is exploited to construct higher partial wave interactions. The merit of our approach is examined by computing elastic scattering phases of the $\alpha-\alpha$ system by the judicious use of phase function method. Reasonable agreements in phase shifts are obtained with standard data.

Keywords. Volterra integral equation; nuclear plus atomic Hulthén potentials; supersymmetry-inspired factorisation method; phase function method; $\alpha-\alpha$ elastic scattering phases.

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1. Introduction

Although all elementary and advanced parts of theoretical physics are formulated in terms of differential equations, the integral equation is compact and many turn out to be more convenient or powerful than the differential equation as the boundary conditions are built into the integral equation rather than imposed at the final stage of the solution. The initial value problem satisfies a Volterra integral equation [1] whereas for the boundary value problem the Fredholm integral equation is no doubt indispensable. There exist physical systems which involve two interactions: one long-range (Coulomb) and the other short-range. For example, the $\alpha-\alpha$ system [2–6] is described by a combination of Coulomb potential and some short-range interaction of nuclear origin. The $\alpha-\alpha$ system has been studied quite extensively and provides a large number of reliable phase-shift data [6–10]. There have been attempts to explain $\alpha-\alpha$ scattering phase shifts with various kinds of potentials like Woods–Saxon [11], Gaussian [12], energy-dependent [13] and so on. The nuclear Hulthén potential serves as a model for the interaction between nucleons in deuteron and for the α–nucleon systems [14–18]. Arnold and MacKellar [14] parametrised Hulthén potential to fit the deuteron binding energy and S-wave scattering lengths. We shall try to generate $\alpha-\alpha$ phase shifts up to 25 MeV by parametrising the two-body effective potential of Hulthén type [14–18].

The Coulomb field of the gold nuclei in the famous Rutherford experiment, for example, was completely shielded at a few Angstroms. This effect of screening should invariably affect the theory and the interpretation of data. In atomic and plasma physics screened and cut-off Coulomb potentials are important. Many standard results in non-relativistic scattering theory for the short-range potentials have to be modified for charged particle scattering as the particles interacting via the Coulomb potential never behave like free particles. Even the asymptotic condition for a well-behaved potential does not hold and as a consequence the concept of a phase shift is ill defined for Coulomb scattering. Therefore, the electromagnetic part of the interaction is described by a relatively short-range screened Coulomb potential, the atomic Hulthén one [19]. It has been frequently used in model calculation presumably because it is explicitly solvable. Also the traditional approach to the phase function method (PFM) [20] for the local potential does not hold good for the pure Coulomb interaction and it needs separate treatment. The Hulthén potential at small values of r behaves like a Coulomb potential, whereas for large values of r it decreases exponentially. Thus, to circumvent the difficulties for computing scattering phase shifts by using traditional PFM, the

electromagnetic interaction here is replaced by the atomic Hulthén one to validate the methodology of PFM for local potentials. This is the main motivation for considering screened Coulomb potential. One would expect the observable properties of the screened Coulomb interaction to approach those of the Coulomb potential at least under reasonable conditions for the screening function.

The α-particles are tightly bound and have no low lying excited state. In view of this, one can use the simple Schrödinger's equation with the Hulthén nuclear [15–18] plus Hulthén atomic interactions [19] to compute the phase shifts for the $\alpha-\alpha$ system. Here we shall follow two different approaches to the problem under consideration: (1) the direct computation of the scattering phase shifts by using the PFM [20] with the concerned interaction for $\ell = 0$ and 2 states, (2) constructing D-wave interaction from its S-wave counterpart by exploiting the methodology of supersymmetric quantum mechanics (SQM) [21–25] and PFM [20]. In nuclear physics both deep and shallow potentials have been studied to describe nucleus–nucleus interactions quite effectively. Within the framework of inverse scattering theory and the algebra of SQM, several reserachers [26–28] have studied the relation between shallow and deep nucleus–nucleus potentials which are phase equivalent. The construction of higher partial wave potential from its ground state requires the wave function of the ground state. The main objective of this paper is to develop a simple method for the Volterra integral equation [1] to obtain analytic wave function for the system involving electromagnetic plus short-range local potential. We shall see in the course of our study that the merit of integral equation approach is its simplicity as the required analytical solution will automatically generate the subject to a particular boundary condition. In §2 we describe our potential model for the system under consideration and generate the approximate analytic wave function by solving the concerned integral equation by power series method. Also we construct higher partial wave interactions through the SQM formalism. Section 3 is devoted to the results and discussions and finally in §4 some concluding remarks are given.

2. Integral equation method and SQM-generated potentials

At the centre of mass energy $E = k^2 + i\varepsilon$, the S-wave Schrödinger's equation for atomic plus nuclear Hulthén potential reads as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - V_1\frac{\mathrm{e}^{-\rho r}}{1-\mathrm{e}^{-\rho r}} - V_0\frac{\mathrm{e}^{-r/a}}{1-\mathrm{e}^{-r/a}}\right]\varphi(k,r) = 0 \tag{1}$$

with

$$V_1 = -(\beta^2 - \alpha^2) \tag{2}$$

and

$$\rho = \beta - \alpha. \tag{3}$$

The quantities V_1 and ρ represent the strength and inverse range parameters of the nuclear part [14,16] whereas V_0 and a stand for the strength and screening radius of the electromagnetic part of the interactions. In the limit $a \to \infty$, the Hulthén potential goes over to Coulomb potential if $V_0 a^2 = e^2 = 1$ (in atomic unit). In the unscreening limit, $a \to \infty$ and $V_0 \to 0$ such that their product remains a constant, $aV_0 = 2k\eta$, where η is the Sommerfeld parameter. Using the transformations for the dependent and independent variables

$$\varphi(k,r) = \frac{\mathrm{e}^{ikr}}{\rho}\left(1-\mathrm{e}^{-\rho r}\right)g(k,r) \tag{4}$$

and

$$z = 1 - \mathrm{e}^{-\rho r}. \tag{5}$$

Equation (1) becomes

$$\left[z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left\{2 - \left(3 - \frac{2ik}{\rho}\right)z\right\}\frac{\mathrm{d}}{\mathrm{d}z} + \frac{2ik}{\rho} - 1 - \frac{V_1}{\rho^2}\right]g(k,z) = \frac{V_0 z}{\rho^2}\left[\rho a - 1 + \frac{\rho a}{z}\right]g(k,z). \tag{6}$$

As z is less than one, we retain only the first-order terms in z at the right-hand side of the above equation. The corresponding integral equation with regular boundary condition reads as

$$\begin{aligned} g(k,z) = {} & {}_2F_1(b,c;d;z) - {}_2F_1(b,c;d;z) \\ & \times \int_0^z \frac{z'^{1-d}}{z'W}\frac{V_0 z'}{\rho^2}\left(\rho a - 1 + \frac{\rho a}{z'}\right)g(k,z')\,{}_2F_1 \\ & \times \left(b-d+1, c-d+1; 2-d; z'\right)\mathrm{d}z' \\ & + z^{1-d}\,{}_2F_1(b-d+1, c-d+1; 2-d; z) \\ & \times \int_0^z \frac{V_0 z'}{W\rho^2}\left(\rho a - 1 + \frac{\rho a}{z'}\right) \\ & \times g(k,z')\,{}_2F_1\left(b,c;d;z'\right)\mathrm{d}z' \end{aligned} \tag{7}$$

with the Wronskian [29]

$$W = (1-d)z^{-d}(1-z)^{d-b-c-1}. \tag{8}$$

The function ${}_2F_1(*)$ in the above equation represents the Gaussian hypergeometric function [29]. To solve eq. (7) we expand $g(k,z)$ in power series as

$$g(k,z) = \sum_{n=0}^{\infty} a_n z^n. \tag{9}$$

Substitution of eqs (8) and (9) in (7), some algebraic manipulation and use of standard integrals leads to

$$\sum_{n=0}^{\infty} a_n z^n = {}_2F_1\,(b, c; d; z) + \frac{V_0}{\rho^2}\,(\rho a - 1) \times \sum_{n=0}^{\infty} a_n \left[f_{n+2}\,(b, c; d; z) - f_{n+3}\,(b, c; d; z)\right] + \sum_{n=0}^{\infty} a_n \frac{V_0 a}{\rho} \left[f_{n+1}\,(b, c; d; z) - f_{n+3}\,(b, c; d; z)\right]. \tag{10}$$

In deriving the above equation we have used the following integral [29]:

$$f_\sigma(b, c; d; z) = \frac{1}{(d-1)} \Bigg[{}_2F_1(b, c; d; z) \times \int_0^z z'^{\sigma-1}(1-z')^{b+c-d} \times {}_2F_1(b-d+1, c-d+1; 2-d; z')\mathrm{d}z' - z^{1-d}\,{}_2F_1(b-d+1, c-d+1; 2-d; z) \times \int_0^z z'^{\sigma+d-2}(1-z')^{b+c-d}\,{}_2F_1(b, c; d; z')\mathrm{d}z'\Bigg]. \tag{11}$$

Here $f_\sigma(*)$, the particular solution of the non-homogeneous Gaussian hypergeometric equation [29] is given by

$$z(1-z)\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + [c_1 - (a_1 + b_1 + 1)z]\frac{\mathrm{d}y}{\mathrm{d}z} - a_1 b_1 y = z^{\sigma-1}, \tag{12}$$

where a_1, b_1, c_1 and σ are constants. The particular integral of eq. (11) is [29]

$$f_\sigma(b, c; d; z) = \frac{z^\sigma}{\sigma(\sigma + d - 1)}\,{}_3F_2(1, \sigma + b, \sigma + c; \sigma + 1, \sigma + d; z). \tag{13}$$

The series converges when $|z| < 1$; it converges when $|z| = 1$ provided that $\mathrm{Re}(c - a - b) > 0$.

Equating the coefficients of various power of z on both side of eq. (10) we have

$$a_0 = 1, \tag{14}$$

$$a_1 = \frac{X}{2}; \quad X = 1 + \frac{V_1}{\rho^2} - \frac{2ik}{\rho} + \frac{V_0 a}{\rho}, \tag{15}$$

$$a_2 = \frac{1}{12}\left\{\left(3 - \frac{2ik}{\rho} + X\right)X + 2Y\right\};$$

$$Y = \frac{V_0}{\rho^2}\,(\rho a - 1)\,, \tag{16}$$

$$a_3 = \frac{1}{48}\left\{\left(8 - \frac{4ik}{\rho} + X\right)4a_2 + X^2 Y\right\}, \tag{17}$$

$$a_4 = \frac{1}{20}\left\{\left(15 - \frac{6ik}{\rho} + X\right)a_3 + 20Y a_2\right\}, \tag{18}$$

and so on.

Substitution of eqs (5), (9), (14)–(18) in eq. (4) yields

$$\varphi(k, r) = \frac{\mathrm{e}^{ikr}}{\rho}(1 - \mathrm{e}^{-\rho r})\left[{}_2F_1\left(b, c; d; 1 - \mathrm{e}^{-\rho r}\right) + \frac{V_0 a}{2\rho}(1 - \mathrm{e}^{-\rho r})\left(1 + \frac{1 - \mathrm{e}^{-\rho r}}{3}\right) + \frac{V_0 a}{2\rho}\left[2bc + b + c + 1\right] f_2\left(b, c; d; 1 - \mathrm{e}^{-\rho r}\right) + \frac{V_0 a}{12\rho^2}\left\{a\rho\left[4b^2c^2 + bc\,(b + c + 3) + 4\,(b + c + 2)\right] - \left[2(b+2)(c+2) + 3(bc)^2\right]\right\} \times f_3\left(b, c; d; 1 - \mathrm{e}^{-\rho r}\right) + \cdots\right] \tag{19}$$

with

$$b = 1 - ik\rho + i\rho\left(k^2 + V_1\right)^{1/2}, \tag{20}$$

$$c = 1 - ik\rho - i\rho\left(k^2 + V_1\right)^{1/2} \tag{21}$$

and

$$d = 2. \tag{22}$$

The Hamiltonian hierarchy problems in SQM [21–25] lead to the addition of appropriate centrifugal barriers and consequently, higher partial wave potentials are generated fairly accurately. Following the approach of SQM [21–25] the supersymmetric partner potential $V_n(x)$ (higher partial wave interaction) generated from the ground-state interaction $V_0(x)$ is given by

$$V_n(x) = V_{n-1}(x) - \frac{\mathrm{d}^2}{dx^2}\left(\ln \varphi^{(0)}_{n-1}(x)\right); n = 1, 2, 3\ldots, \tag{23}$$

where $\varphi^{(0)}_{n-1}(x)$ represents near the origin behaviour of the eigenfunction of the appropriate angular momentum state. Therefore, the P-wave potential $V_p(r)$ for the system under consideration is written as

$$V_p(r) = V_0\frac{\mathrm{e}^{-r/a}}{1 - \mathrm{e}^{-r/a}} + V_1\frac{\mathrm{e}^{-\rho r}}{1 - \mathrm{e}^{-\rho r}} - \frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\ln \varphi^{(0)}(k, r)\right) \tag{24}$$

with

$$\varphi^{(0)}(k, r) = r\left[\mathrm{e}^{\omega\rho r/2} + \frac{V_0 a r}{2}\left(1 + \frac{\rho r}{3}\right)\right];$$

$$\omega = 1 + V_1\rho^2. \tag{25}$$

The quantity $\varphi^{(0)}(k, r)$ represents near the origin behaviour of the solution given in eq. (19). In view of eqs (24)

and (25) the desired expression for P-wave potential is obtained as

$$V_p(r) = V_0 \frac{e^{-r/a}}{1-e^{-r/a}} + V_1 \frac{e^{-\rho r}}{1-e^{-\rho r}} + \frac{1}{r^2} - \frac{\frac{\omega^2\rho^2}{4} e^{\omega\rho r/2} + \frac{V_0 a\rho}{3}}{e^{\omega\rho r/2} + \frac{V_0 ar}{2}\left(1+\frac{\rho r}{3}\right)} - \left(\frac{\frac{\omega\rho}{2} e^{\omega\rho r/2} + \frac{V_0 a}{2}\left(1+\frac{2\rho r}{3}\right)}{e^{\omega\rho r/2} + \frac{V_0 a}{2}\left(1+\frac{\rho r}{3}\right)}\right)^2. \quad (26)$$

Any second-order differential operator of the form

$$H_n = -\frac{d^2}{dr^2} + V_n(x) \quad (27)$$

can be factorised in terms of the following two first-order differential operators:

$$O_n^{(\pm)}(x) = \pm\frac{d}{dx} + \frac{d}{dx}\left(\ln \varphi_n^{(0)}(x)\right). \quad (28)$$

In the above equation $O_n^{(\pm)}(x)$ represents the lowering and raising operators. These relations are valid both for discrete and continuous eigenvalues of H_n. For the positive energy states the wave functions are related by

$$\varphi_n(x) = O_{n-1}^{(-)}\varphi_{n-1}(x). \quad (29)$$

To construct D-wave interaction one needs the knowledge of the P-state wave function. To that end the raising operator is

$$O^{(-)}(r) = -\frac{d}{dr} + \frac{1}{r} + ik + \frac{\frac{\omega\rho}{2} e^{\omega\rho r/2} + \frac{V_0 a}{2}\left(1+\frac{2\rho r}{3}\right)}{e^{\omega\rho r/2} + \frac{V_0 ar}{2}\left(1+\frac{\rho r}{3}\right)} \quad (30)$$

and the P-wave solution for the potential $V_p(r)$ is obtained as

$$\varphi^p(k,r) = \frac{e^{ikr}}{\rho}\left(1-e^{-\rho r}\right)\left[\frac{1}{r} - \frac{\rho e^{-\rho r}}{1-e^{-\rho r}} - \frac{d}{dr} + ik + \frac{\frac{\omega\rho}{2} e^{\omega\rho r/2} + \frac{V_0 a}{2}\left(1+\frac{2\rho r}{3}\right)}{e^{\omega\rho r/2} + \frac{V_0 ar}{2}\left(1+\frac{\rho r}{3}\right)}\right] \times \frac{\rho e^{-ikr}}{(1-e^{-\rho r})}\varphi(k,r). \quad (31)$$

The behavior of $\varphi^p(k,r)$ near the origin, denoted by $\varphi^{p(0)}(r)$, does not depend upon energy explicitly. Combination of eqs (23), (26) in conjunction with $\varphi^{p(0)}(r)$ yields

$$V_d(r) = V_p(r) - \frac{S''r)}{S(r)} + \left(\frac{S'(r)}{S(r)}\right)^2 \quad (32)$$

with

$$S(r) = re^{-\rho r}\left[\frac{\omega\rho}{2} e^{\chi\rho r/3} + \frac{V_0 a\rho r}{4} - \frac{V_0 r}{3}\right] + \frac{V_0 ar}{6}\left[\rho^2 r^2 + 5\rho r + 1\right] + re^{\omega\rho r/2}\left[1+\omega/2\right]; \quad \chi = 4 + \frac{V_1}{\rho^2} \quad (33)$$

and $S'(r)$ and $S''(r)$ represent the first and second derivatives of $S(r)$ with respect to r.

3. Results and discussions

The scattering phase shifts for local potentials [20] are generally computed by the application of the phase equation

$$\delta'_\ell(k,r) = -k^{-1}V(r)\left[\hat{j}_\ell(kr)\cos\delta_\ell(k,r) - \hat{\eta}_\ell(kr)\sin\delta_\ell(k,r)\right]^2. \quad (34)$$

Here the quantities $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$ are the Riccati–Bessel functions.

The D-wave phase shifts are computed by applying two different interactions: one obtained by adding proper centrifugal barrier to S-wave part, known as conventional method, while the other one generated through the algebra of SQM [21–25]. It is well known that the approximate higher partial wave generalisation of the Hulthén potential belongs to Eckart class of potential [30] in which the higher partial wave interactions are generated by the addition of centrifugal barrier like term $\ell(\ell+1)e^{-r/a}/(2a^2(1-e^{-r/a})^2)$. We have chosen to work with $V_0 a = 0.2758\,\text{fm}^{-1}$, $a = 20\,\text{fm}$ and $\hbar^2/2m = 10.3675\,\text{MeV fm}^2$. As the screening radius is considered to be $a = 20\,\text{fm}$, much more than nuclear range, the atomic Hulthén potential reproduces the effect of the Coulomb potential. Thus, in our subsequent discussions we designate the physical observables by the phrase 'Coulomb–nuclear' related to electromagnetic plus nuclear situation. The nuclear part of our potential is defined by two parameters, namely $\beta = 4.32\,\text{fm}^{-1}$ and $\alpha = 3.824\,\text{fm}^{-1}$. With these parameters and the interactions given in eqs (1) and (31), we portray the $\alpha-\alpha$ scattering phase shifts (both nuclear and Coulomb–nuclear) in figure 1 and the corresponding effective potentials $V(r)$ (both nuclear and Coulomb–nuclear) for the partial wave states $\ell = 0$ and 2 in figures 2 and 3 respectively along with the standard potentials of [12].

As expected, the S-wave scattering phase shifts δ_0 (Coulomb–nuclear) for the $\alpha-\alpha$ system are positive at low energies (up to 19.7 MeV) and beyond that it changes its sign. This is in excellent agreement with

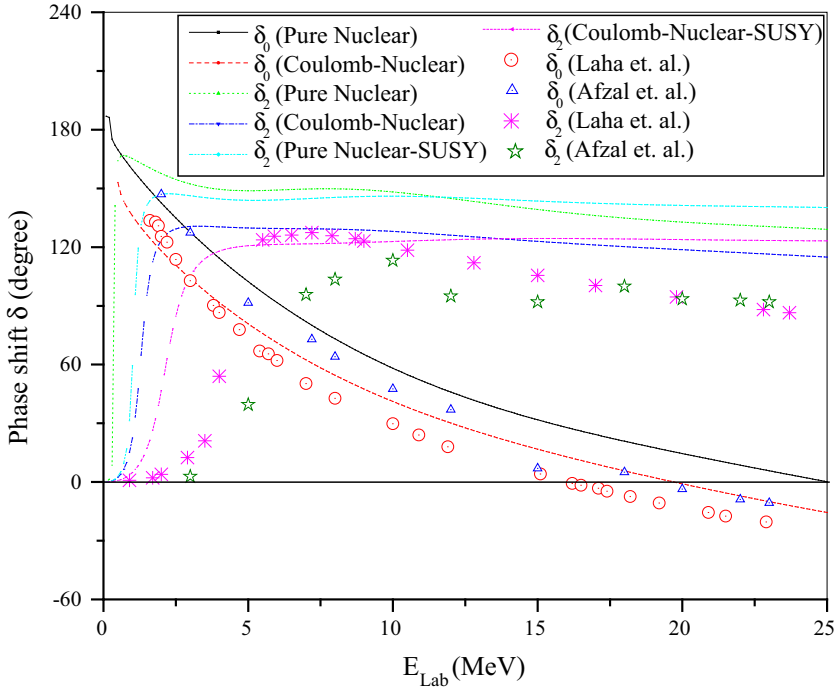

Figure 1. $\alpha-\alpha$ Scattering phase shifts for S- and D-waves.

standard result of Afzal *et al* [4]. Except in the very low-energy range (0–3 MeV) the numerical values of our phase shifts coincide with the experimental data of ref. [4] over the entire energy range under consideration. On the other hand, they match quite well with those of ref. [10] up to 5 MeV and beyond that they differ slightly. For the D-wave phase shifts δ_2 (Coulomb–nuclear) for both the interactions under consideration, our results differ from those of refs [4,10] at very low energies (up to 5 MeV), compare well in the energy range 5–15 MeV and again differ slightly beyond 15 MeV. However, they show the correct trend of the D-wave phase shifts. It is observed that the supersymmetry (SUSY)-generated potential is able to produce comparable phase shifts at low energies whereas the other one is significant at higher energies. On the other hand, the pure nuclear phase shifts produce higher phase shift values due to the absence of Coulomb repulsion. We have computed the pure nuclear phase shifts by turning off the electromagnetic interaction in our numerical routine for Coulomb plus nuclear phase shifts.

Our S-wave effective potential $V(r)$ (Coulomb plus nuclear) as shown in figure 2 is attractive in nature up to 10.3 fm and beyond that it becomes repulsive due to the Coulomb potential. Friedrich [12] proposed a two-parameter angular momentum and energy-independent local Gaussian potential and was able to reproduce the scattering phase shifts up to 40 MeV. Marquez [4] successfully described the $\alpha-\alpha$ system by considering a Woods–Saxon type potential for the nuclear part of the interaction. The potentials derived in refs [4] and [12] are virtually identical. Thus, to make a comparison we also plot the effective potentials of ref. [12] in figure 2. The D-wave potentials are computed (i) by simply adding the repulsive centrifugal barrier term [30] to its S-wave nuclear part along with the atomic Hulthén interaction and (ii) by using the potentials in eq. (32). In the range $r = 1-6$ fm the attractive strength of our effective S-wave potential varies between 62 and 1.8 MeV while the variation of the same for ref. [12] is between 98 and 0.006 MeV.

The D-wave potentials for both the cases have repulsive cores followed by attractive parts. It is observed that the SUSY-generated potential is much deeper than its conventional counterpart but is phase-equivalent. The effective D-wave potential of ref. [12], obtained by incorporating proper centrifugal barrier to its S-wave potential, is also shown in figure 3. It is seen that the size of the repulsive core as well as the depth of this potential is larger than our conventional potential but the depth is smaller than the SUSY case. It is noticed that our potentials resemble more or less the same nature as of refs [4,12]. The larger size of the repulsive core ensures less and less overlap of the particles. The $\alpha-\alpha$ potentials that have been constructed from the experimental phase shifts show some common features: A static $\alpha-\alpha$ potential common to all partial waves does not exist and, the ranges of the inner repulsive part and the outer attractive part are of the order of 1.5 and 5 fm respectively. It is found that both of our D-wave potentials, conventional

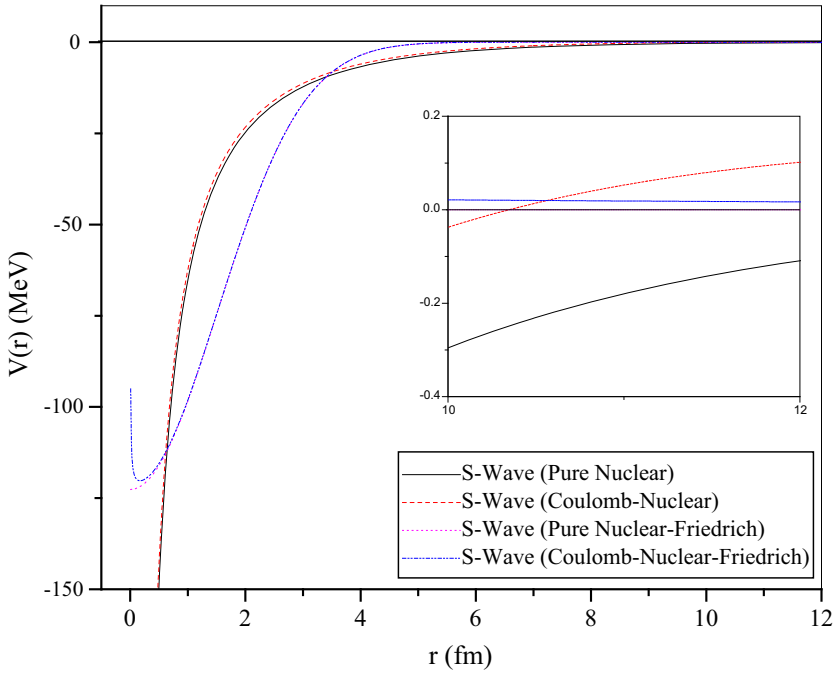

Figure 2. The effective $\alpha-\alpha$ potentials for S-wave.

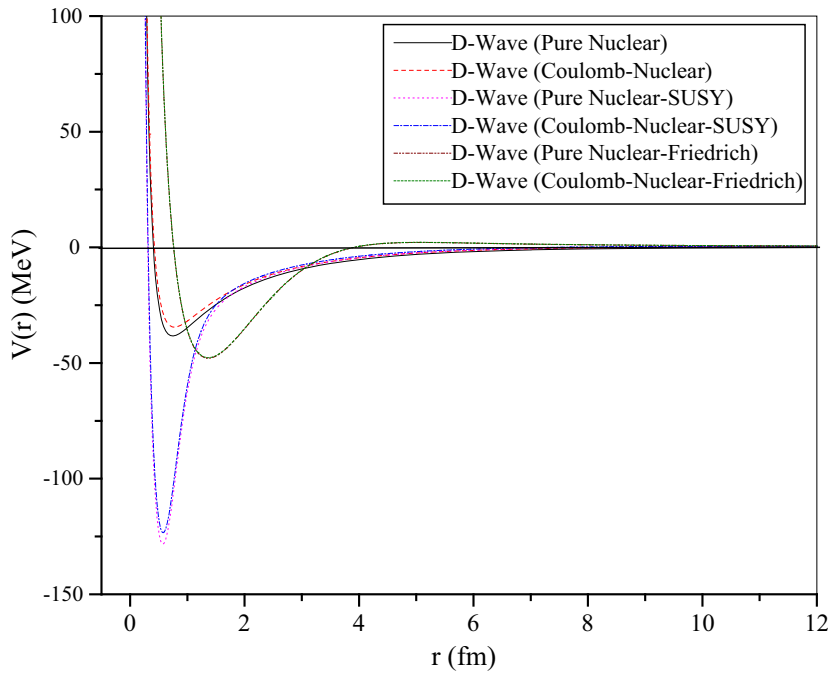

Figure 3. The effective $\alpha-\alpha$ potentials for D-wave.

and SUSY-generated, possess repulsive cores, followed by an attractive part. The repulsive part is a consequence of the ℓ-dependence of the interaction.

In nuclear physics, both shallow and deep potentials have been successfully applied to describe the nucleus–nucleus systems [26–28]. Interestingly, the deep and shallow nature of the nucleus–nucleus potential was a controversial question for a long time [31]. Baye [26] has applied the algebra of SQM for the construction of shallow nucleus–nucleus interactions which are phase equivalent to a deep potential. Our SQM-generated deep D-wave potential is approximately phase equivalent to its conventional strong ℓ-dependent shallow part (phase shift may differ by an integral multiple of π). The bound

states of the shallow potentials are related to the actual physical states of the fused nucleus. Michel and Reidemeister [32] nicely explained that it is possible to construct phase equivalent shallow potentials for $\alpha+^{16}$O deep potential through SQM.

4. Conclusion

In the phenomenological approach one generally attempts to construct an interaction which reproduces the standard values of the low-energy scattering parameters and the phase shifts for a particular system. Various $\alpha-\alpha$ potentials like square well, Gaussian shapes, Woods–Saxon have been proposed earlier [3,4,11,12] for the computation of phase shifts. In the recent past, we have parameterised nuclear Hulthén-type potentials and found reasonable agreement in the phase shift values with the earlier calculations for the $\alpha-\alpha$ and $\alpha-$He^3 systems [17,33]. In this paper, we have constructed ground-state wave function for the nuclear plus atomic Hulthén potentials by solving the Volterra integral equation by series substitution method. The higher partial wave interactions have been developed by exploiting the algebra of SUSY quantum mechanics and applied them to compute scattering phase shifts for the $\alpha-\alpha$ system. We have restricted ourselves up to the partial wave $\ell = 2$ as for the partial wave $\ell = 4$, inordinate mathematical complications are involved in SQM formalism. One of the reasons for undertaking this calculation is to examine how far the algebra of SQM becomes effective for deducing higher partial wave potentials from its ground-state information in subatomic realm. As we have noticed, even though the SUSY-generated D-wave potential is not capable of producing accurate phase shifts over the entire energy range under consideration, it can still consider to be useful as a source of rough trial potentials. Our method can easily be extended for other complex nucleus–nucleus and nucleon–nucleus elastic scattering. It is also possible to have other types of interactions that might produce better results, but answer to this point needs further investigation and goes beyond the scope this paper.

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