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Statistical distribution of quantum particles

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Abstract. In this work, the statistical distribution functions for boson, fermions and their mixtures have been derived and it is found that distribution functions follow the symmetry features of β distribution. If occupation index is greater than unity, then it is easy in the present approach to visualise condensations in terms of intermediate values of mixing parameters. There are some applications of intermediate values of mixing parameters.

Keywords. Bose statistics; Fermi statistics; Bose–Einstein condensation; degenerate gases; BCS-BEC cross-over.

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1. Introduction

Fermi–Dirac and Bose–Einstein distribution are the two well-known β distributions that describe respectively the number of indistinguishable fermions and bosons in different energy states [1]. All experimentally observed particles are either bosonic or fermionic; the general principles of quantum mechanics do not prevent the existence of some objects obeying intermediate statistics [2]. One of the motivation to study intermediate statistics is to construct fault tolerant quantum computer using an approach such as topological quantum computation [3] that relies on the existence of topological states of matter whose quasiparticle excitations are neither bosons nor fermions but are particles known as non-Abelian anyons obeying non-Abelian braiding statistics. In fact, there is a huge amount of work on intermediate statistics between Bose–Einstein and Fermi–Dirac statistics, mostly triggered by the quantum Hall effect and anyonic statistics [4–7]. The statistics of quasiparticles entering the quantum Hall effect, deduced by Danieal [8] from the adiabatic theorem, are also found to obey intermediate statistics.

The formulation presented in this work leads to intermediate statistics as a continuous interpolation between the Bose–Einstein and Fermi–Dirac statistics. The present approach is based on the assumption that every particle has a mixture property of both fermions and bosons. This is equivalent to the assumption that every particle is a boson with certain probability μ_{b} and a fermion with probability $\mu_{\mathrm{f}} = (1 - \mu_{\mathrm{b}})$. The present work attempts to generate the probability distribution plot (parametrised by μ_{b}), that lies between the graphical plots of Fermi–Dirac and Bose–Einstein distributions.

The rest of the paper is organised as follows. In §2, we introduce the basic definition of thermodynamic probability W. Section 3 gives the derivation and graphical plot for occupation index in terms of parameter μ_{b} using multivariate β distribution. The application of intermediate statistics is discussed in §4 and §5 gives the conclusion.

2. Definition and identities

We define the thermodynamic probability W in terms of β function $B(\alpha_1, \alpha_2)$ as

$$W(\alpha_1, \alpha_2) = \frac{1}{B(\alpha_1, \alpha_2)} = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2)}. \tag{1}$$

Based on the above definition, it is easy to prove the following identity:

$$W(\alpha_1, \alpha_2 + \alpha_3) \cdot W(\alpha_2, \alpha_3) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2) \cdot \Gamma(\alpha_3)} = W(\alpha_1, \alpha_2, \alpha_3). \tag{2}$$

For Bose–Einstein statistics, the thermodynamic probability W^i for the ith energy level, parametrised by the number of bosons N_{b}^i and the number of cells g^i is

$$W^i(N_b^i, g^i - 1) = \frac{\Gamma(N_b^i + g^i - 1)}{\Gamma(N_b^i)\cdot\Gamma(g^i - 1)}.$$

As N_b^i and g^i are considerably large numbers, we can neglect the -1 in the above expression. So, for Bose–Einstein statistics, the thermodynamic probability W^i becomes

$$W^i(N_b^i, g^i) = \frac{\Gamma(N_b^i + g^i)}{\Gamma(N_b^i)\cdot\Gamma(g^i)}. \tag{3}$$

Similarly, for Fermi–Dirac statistics, the thermodynamic probability W^i for the ith energy level, parametrised by the number of fermions N_f^i and number of free cells $g^i - N_f^i$ is

$$W^i(N_f^i, g^i - N_f^i) = \frac{\Gamma(g^i)}{\Gamma(N_f^i)\cdot\Gamma(g^i - N_f^i)}. \tag{4}$$

3. Formulation of thermodynamic probability W

In this section we consider a theoretical framework to generate Fermi–Dirac, Bose–Einstein and family of intermediate statistics from the thermodynamic probability W. The conventional method can be generalised further in the domain of γ function so as to include both statistics.

Consider a system of N identical and indistinguishable particles. These particles have definite energies and can occupy definite energy states. Hence, they can be represented as phase points in phase space. To determine the energy distribution of these particles in the most probable state, we divide the available volume in the phase space into a large number, say k, of compartments. Each compartment represents the small interval of energy. Further, we divide each compartment into elementary cells, each of size h^3. Suppose that the size of the compartment is very large compared to the size of the cell so that each compartment contains a very large number of cells. Let $(N^0, N^1, ..., N^k)$ be the number of particles having energy levels $(E^0, E^1, ..., E^k)$ in the compartments numbered as $0, 1, ..., k$ containing $g^0, g^1, ..., g^k$ cells, respectively in them. As total number of N particles in the system are distributed in k number of compartments, we have $N = \sum_{i=0}^{k} N^i$ and total energy $E = \sum_{i=0}^{k}(N^i \cdot E^i)$. For intermediate statistics, the thermodynamic probability W^i of the ith energy level, parametrised by the number of bosons N_b^i, number of fermions N_f^i and number of free cells $g^i - N_f^i$ is

$$\begin{aligned} W^i(N_b^i, N_f^i, g^i - N_f^i) &= W^i(N_b^i, g^i)\cdot W^i(N_f^i, g^i - N_f^i) \\ &= \frac{\Gamma(N_b^i + g^i)}{\Gamma(N_b^i)\cdot\Gamma(N_f^i)\cdot\Gamma(g^i - N_f^i)}. \end{aligned} \tag{5}$$

As probability is multiplicative, the probability W of the composite system is equal to the product of the probabilities of fermion–boson systems. The thermodynamic probability W^i for Bose–Einstein statistics in eq. (3) can be obtained by setting $N_f^i = 0$ in eq. (5). Similarly, the thermodynamic probability W^i for Fermi–Dirac statistics in eq. (4) can be obtained by setting $N_b^i = 0$ in eq. (5). The total number of different arrangements of all particles of the system gives the thermodynamic probability W.

$$\begin{aligned} W &= \prod_{i=1}^{k} W^i(N_b^i, N_f^i, g^i - N_f^i) \\ &= \prod_{i=1}^{k} \frac{\Gamma(N_b^i + g^i)}{\Gamma(N_b^i)\cdot\Gamma(N_f^i)\cdot\Gamma(g^i - N_f^i)}. \end{aligned} \tag{6}$$

Taking the natural log of both sides of eq. (6), we get an expression for dimensionless entropy S^*.

$$\begin{aligned} \ln W = \sum_{i=0}^{k} (&\ln\Gamma(N_b^i + g^i) - \ln\Gamma(N_b^i) \\ &- \ln\Gamma(N_f^i) - \ln\Gamma(g^i - N_f^i)). \end{aligned} \tag{7}$$

Using Stirling's approximation for $\Gamma(N)$, i.e. $\ln\Gamma(N) \cong N\cdot(\ln N - 1)$, we get

$$\begin{aligned} \ln W = \sum_{i=1}^{k} (&(N_b^i + g^i)\cdot(\ln(N_b^i + g^i) - 1) \\ &- N_b^i\cdot(\ln N_b^i - 1) - N_f^i\cdot(\ln N_f^i - 1) \\ &- (g^i - N_f^i)\cdot(\ln(g^i - N_f^i) - 1). \end{aligned} \tag{8}$$

The most likely distribution can be obtained by maximising the log-likelihood of thermodynamic probability W. The solution of this maximisation problem leads to the family of statistics. Fermi–Dirac and Bose–Einstein statistics are special cases of this general family of statistics. Consider the following optimisation problem:

$$\underset{(N^0, N^1, ..., N^k)}{\text{maximize}} \quad \ln W$$

$$\text{subject to} \quad \sum_{i=0}^{k} \delta N^i = 0, \; \sum_{i=0}^{k} E^i\cdot\delta N^i = 0.$$

To solve it, we start by defining the derivative of the generalised Lagrangian $\mathcal{L}_i$ for the case in which we have only one energy level i so that we can neglect the sum. We have

$$\delta\mathcal{L}_i = \left[\ln\left(\frac{g^i}{N_\mathrm{b}^i}+1\right)\cdot\delta N_\mathrm{b}^i + \ln\left(\frac{g^i}{N_\mathrm{f}^i}-1\right)\cdot\delta N_\mathrm{f}^i\right] - \alpha\cdot\delta N^i - \beta\cdot E^i\cdot\delta N^i. \tag{9}$$

Let $N_\mathrm{b}^i = \mu_\mathrm{b}\cdot N^i$ and $N_\mathrm{f}^i = \mu_\mathrm{f}\cdot N^i$ where μ_b, μ_f denote the probabilities for bosons and fermions respectively such that $\mu_\mathrm{b}+\mu_\mathrm{f}=1$. So eq. (9) becomes

$$\left(\mu_\mathrm{b}\cdot\ln\left(\frac{g^i}{\mu_\mathrm{b}\cdot N^i}+1\right) + \mu_\mathrm{f}\cdot\ln\left(\frac{g^i}{\mu_\mathrm{f}\cdot N^i}-1\right) - \alpha - \beta\cdot E^i\right)\cdot\delta N^i = 0. \tag{10}$$

The occupation index is defined as $N^i/g^i = y^i$. As $\delta N^i \neq 0$, we get

$$\mu_\mathrm{b}\cdot\ln\left(\frac{1}{\mu_\mathrm{b}\cdot y^i}+1\right)+\mu_\mathrm{f}\cdot\ln\left(\frac{1}{\mu_\mathrm{f}\cdot y^i}-1\right)-\ln C^i = 0, \tag{11}$$

where $C^i = \mathrm{e}^{\alpha+\beta\cdot E^i}$. The special cases for eq. (11) are as follows:

1. For $\mu_\mathrm{b}=0$, $y^i = 1/(\mathrm{e}^{\alpha+\beta\cdot E^i}+1)$, i.e., Fermi–Dirac statistics.
2. For $\mu_\mathrm{f}=0$, $y^i = 1/(\mathrm{e}^{\alpha+\beta\cdot E^i}-1)$, i.e., Bose–Einstein statistics.
3. For $\mu_\mathrm{b}=\mu_\mathrm{f}=1/2$, $y^i = 2/(\sqrt{(C^i)^2+1})$, i.e., intermediate statistics.

The plot of the solution for the occupation index y^i in eq. (11) for various values of μ_b is shown in figure 1. In the plot, the occupation index is along y-axis and (E^i/E_F) is along x-axis where $\alpha+\beta\cdot E^i = 8\cdot(X-1)$.

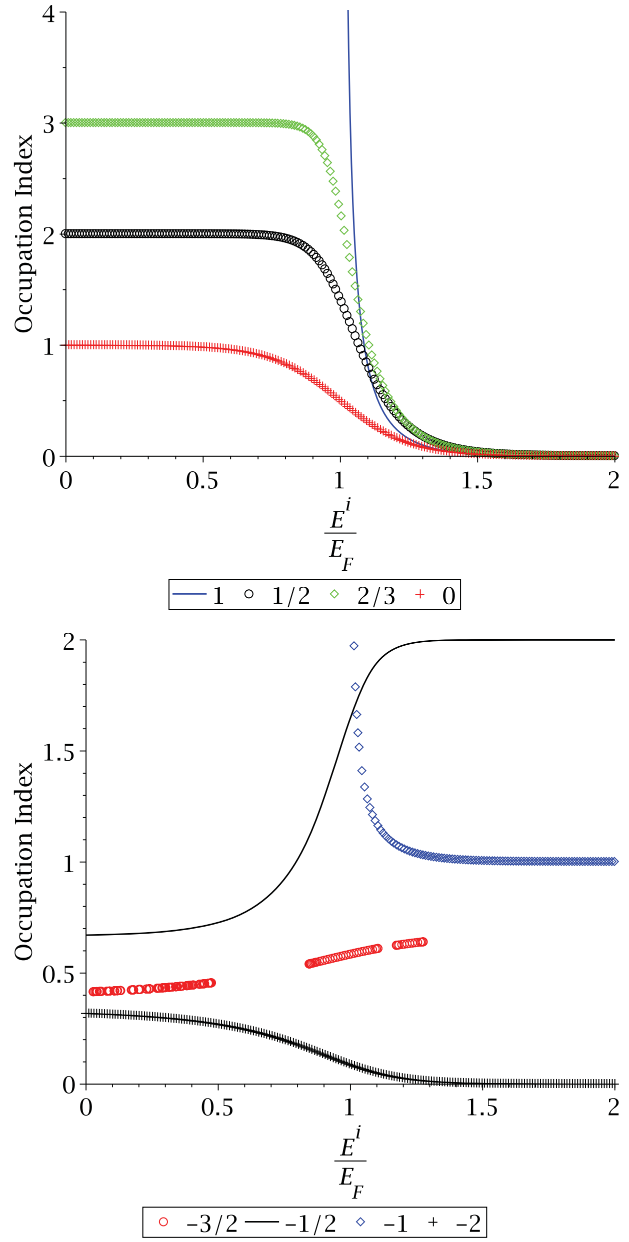

Figure 1. Distribution functions for various values of μ_b.

4. Discussion

For μ_b =0, 1/2, 2/3 and 1, the occupation index y^i in figure 1 along the y-axis is 1, 2, 3 and ∞ respectively, i.e., as $\mu_\mathrm{b}\to 1$, $y^i\to\infty$. As seen from both plots in figure 1, we can infer that the intercept term on the y-axis is $1/(1-\mu_\mathrm{b})$. For Bose–Einstein statistics, the occupation index is ∞ while the Fermi–Dirac statistics has a maximum occupancy index of unity. If occupation index is one, it means particle cannot go to condensation, whereas if the occupation index is infinity then particles can have the properties of the Bose–Einstein condensate.

All elementary particles are either bosons or fermions, and the spin-statistics identify the resulting quantum statistics that differentiates between them. Particles commonly associated with matter are fermions, and they have half-integer spin. On the other hand, the particles associated with fundamental forces are bosons, and they have integer spin. Wave–particle duality proposes that all particles exhibit both wave and particle properties. We may assume that every quantum particle has properties of both fermions and bosons, and may associate with matter or fundamental forces with a certain probability. Therefore, we found the form of statistical distribution which is the solution of eq. (11) using the computer algebra for the arbitrary values of μ_b. The generated distribution function can be used to study bosons, fermions and a mixture of fundamental particles.

The negative values of μ_b can be justified if it is defined as statistical weight. We know that statistical weights may be positive or negative [9]. Normally, as E^i/E_F increases, the occupation index decreases. But when we set μ_b to -0.5 or -1.5, the corresponding distribution function has a special property that as E^i/E_F increases, the occupation index shows increasing trend and may have special application in analysing certain properties.

5. Conclusion

We have taken a logical approach to derive the general distribution function for the mixture of bosons and fermions. We found that some distribution functions have surprising properties. We get a simple functional form of distribution function corresponding to $\mu_\mathrm{b} = 1/2$. However, for arbitrary values of μ_b, one has to take the help of computer algebra to obtain distribution functions. In future, we hope to extend this idea further to generate more distribution functions for the thermodynamic probability

$$W = \frac{\Gamma(g_\mathrm{b} + g_\mathrm{f} + N_\mathrm{b})}{\Gamma(g_\mathrm{b})\Gamma(N_\mathrm{b})\Gamma(N_\mathrm{f})\Gamma(g_\mathrm{f} - N_\mathrm{f})}.$$

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