Pramana – J. Phys. (2018) 90:31
https://doi.org/10.1007/s12043-018-1524-2

A new transiently chaotic flow with ellipsoid equilibria

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MS received 10 July 2017; revised 28 September 2017; accepted 10 October 2017;
published online 8 February 2018

Abstract. In this article, a simple autonomous transiently chaotic flow with cubic nonlinearities is proposed. This system represents some unusual features such as having a surface of equilibria. We shall describe some dynamical properties and behaviours of this system in terms of eigenvalue structures, bifurcation diagrams, time series, and phase portraits. Various behaviours of this system such as periodic and transiently chaotic dynamics can be shown by setting special parameters in proper values. Our system belongs to a newly introduced category of transiently chaotic systems: systems with hidden attractors. Transiently chaotic behaviour of our proposed system has been implemented and tested by the OrCAD-PSpise software. We have found a proper qualitative similarity between circuit and simulation results.

Keywords. Transiently chaotic flows; ellipsoid equilibria; hidden attractors.

PACS Nos 05.45.−a; 05.45.Ac

1. Introduction

Most of the known and analysed simple chaotic flows that have chaotic behaviour are associated with saddle points [1–6]. These saddle points are the main guides in finding the system's strange attractor and its basin of attraction. In these systems, by putting the initial condition of the flow in the neighbourhood of the saddle points, the trajectory is usually attracted to the attractor. Because of this reason, this kind of attractors is called self-excited.

Recently, many studies have been conducted around the topic of systems discovered with attractors that do not have a saddle point in their basin of attraction [7–13]. These attractors are called hidden attractors because finding them is more complicated than finding self-excited attractors [14–17]. There are no clues about their whereabouts or their basins of attraction [18–21]. Our interest is in hidden attractors that have significant roles in engineering and real-life applications. If they are ignored, there might be some unpredictable or possibly tragic responses to small perturbations in structures like bridges or airplane wings [19,20,22–24].

On the other hand, finding new chaotic systems with special properties is a very new and hot topic in the field of nonlinear dynamics. From such studies, we can point out systems with multistability [25–27], with extreme multistability [28–31], with multiple attractors, without linear terms [32], with stable equilibria [33], with no equilibria [34,35], with line of equilibria [36,37], with curves of equilibria [38–40], and with surfaces of equilibria, and systems which belong to many families of hidden attractors [41–43].

In this paper, we introduce a novel system with hidden transient attractor and an ellipsoid surface of equilibria. Some transiently chaotic systems are reported in [44–46]. Although three-dimensional systems with surfaces of equilibria [47] and four-dimensional systems with plane of equilibria [31,48] have been reported before, there are no four-dimensional dynamical system with a surface of equilibria in literature. The main goal of this paper is to introduce a new class of dynamical systems with surface of equilibria and hidden transient attractor and expand the list of known mathematically simple hidden chaotic attractors.

Every new chaotic system that is designed, should support at least one of the three principles that have been noted in [49]. These rules are listed below:

(1) The system should credibly model some important unsolved problem in nature and shed insight on that problem.
(2) The system should exhibit some behaviour previously unobserved.
(3) The system should be simpler than all other known examples exhibiting the observed behaviour.

The system that we are going to introduce in this paper satisfies the second and third conditions. The rest of the paper is as follows. Section 2 contains description of the new system with ellipsoid equilibrium and its dynamical analyses. Section 3 contains a circuit design of the new discussed system. Section 4 gives the conclusion of the paper.

2. New transiently chaotic system: Dynamical analysis

Consider system (1) which is a four-dimensional flow:

$$\begin{aligned}
\dot{x} &= w,\\
\dot{y} &= w \times \left(4y - z + xz - y^2 + 2yz + zw + 2w^2 - 3\right),\\
\dot{z} &= w \times (4y + 2z - w + xz + xw + yw),\\
\dot{w} &= bx^2 + \frac{y^2}{2^2} + \frac{z^2}{2^2}\\
&\quad -3^2 + w \times \left(3y^2 - 0.6zw - 19w^2\right),
\end{aligned} \tag{1}$$

where x, y, z, and w are state variables.

If we set the parameter $b = 1$, this system has an equilibrium in

$$w = 0 \quad \text{and} \quad \frac{x^2}{1^2} + \frac{y^2}{2^2} + \frac{z^2}{2^2} - 3^2 = 0$$

which is an ellipsoid in xyz-space.

The Jacobian matrix of these equations is

$$J = \begin{pmatrix} 0, 0, 0, 1 \\ wz,\, w \times (4 - 2y + 2z),\, w \times (x + 2y - 1),\, 4y - z + xz - y^2 + 2yz + 2zw + 6w^2 - 3) \\ w \times (z + w),\, w \times (4 + w),\, w \times (2 + x),\, 4y + 2z - 2w + xz + 2xw + 2yw \\ 2x,\, \frac{y}{2} + w \times (6y),\, \frac{z}{2} - 0.6w^2,\, 3y^2 - 1.2zw - 57w^2 \end{pmatrix}. \tag{2}$$

The characteristic equation of system (1) which is obtained from $\lambda I - J = 0$ evaluated on the equilibrium is

$$\lambda^2 \times \left(-\lambda^2 + 3\lambda y + y^2 + z^2 + \frac{3y}{2} + \frac{3yz}{2} + \frac{xz^2}{2} + y^2 z + \frac{xyz}{2}\right) = 0. \tag{3}$$

Thus, the eigenvalues are equal to

$$\lambda_0 = 0, \lambda_2 = 0, \lambda_{3,4} = \frac{3y^2}{2} \pm \frac{\sqrt{\left(8x - 6y + 6yz + 2xz^2 + 4y^2 z + 8y^2 - 2y^3 + 9y^4 + 4z^2 + 2xyz\right)}}{2}. \tag{4}$$

Various projections of the phase portrait diagram of this system are plotted in figure 1 with the initial condition of $(-2.15, -0.83, -4.29, -0.11)$ and $b = 1$.

Bifurcation diagram for parameter b is plotted in figure 2. The relative initial conditions are mentioned in the figure caption. This system will be unbounded suddenly, if the run time exceeds 1000 s. All the trajectories in this system escape to infinity in long runs. It means that the strange attractor is not stable and exists in transient. We are aware of no such phenomena in chaotic flows reported till now.

3. Circuit design

The long simulation time as well as the errors in numerical simulations, when investigating dynamics of chaotic systems, leads the researchers to the physical implementation of such systems as an effective approach of study [50]. In addition, the hardware implementation of mathematical chaotic models is an important topic from the point of practical applications [51–55]. Nonlinear dynamical systems can be realised by using commercially available amplifiers, integrated circuits, or FPGAs [56–58].

It is noted that when using commercially available amplifiers one should be aware of their limitations [59]. In this direction, an electronic circuit which emulates the mathematical model (1) is presented in this section to show its feasibility. Also, the designed circuit is effective for practically discovering the dynamics of the theoretical model. Considering the saturation property of the analog multipliers and the op-amps, usually the state variables with a linear transformation are reduced.

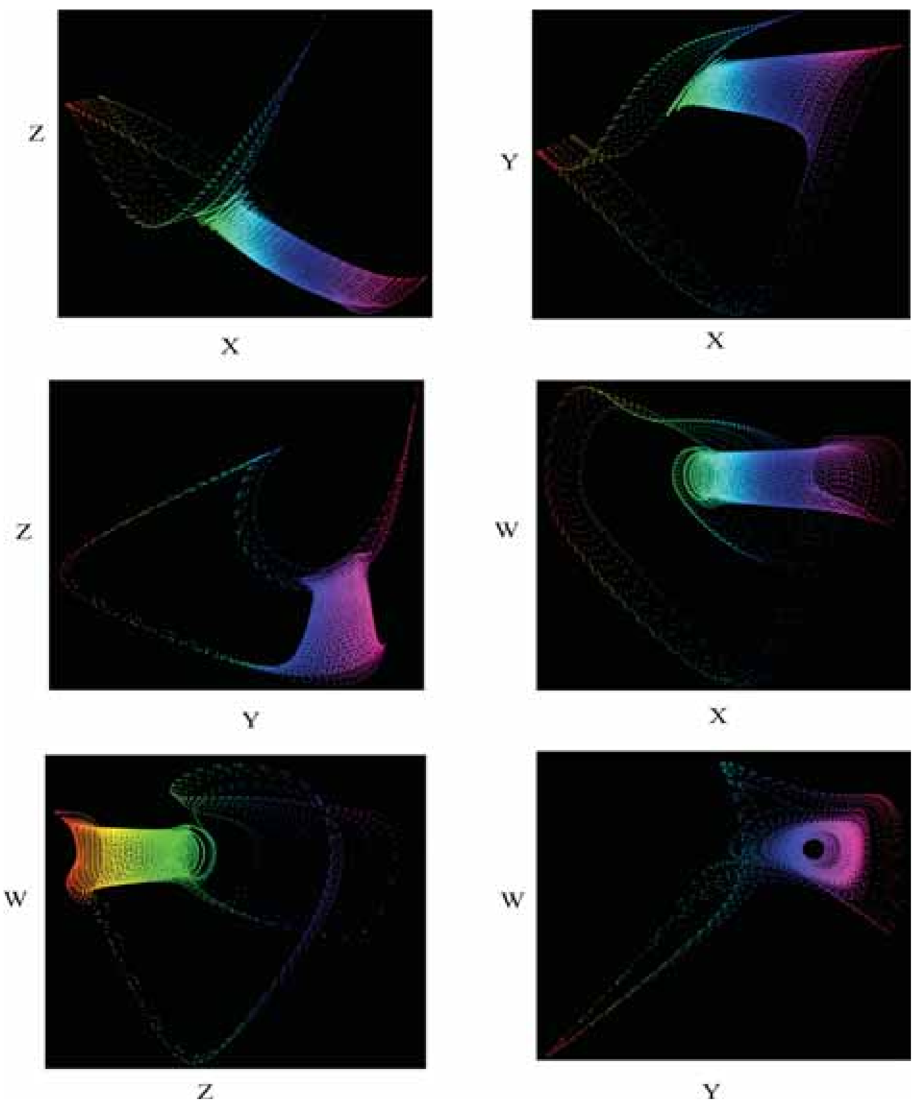

Figure 1. Various projections of the phase portrait of system (1) with the initial conditions of (−2.15, −0.83, −4.29, −0.11) and $b = 1$.

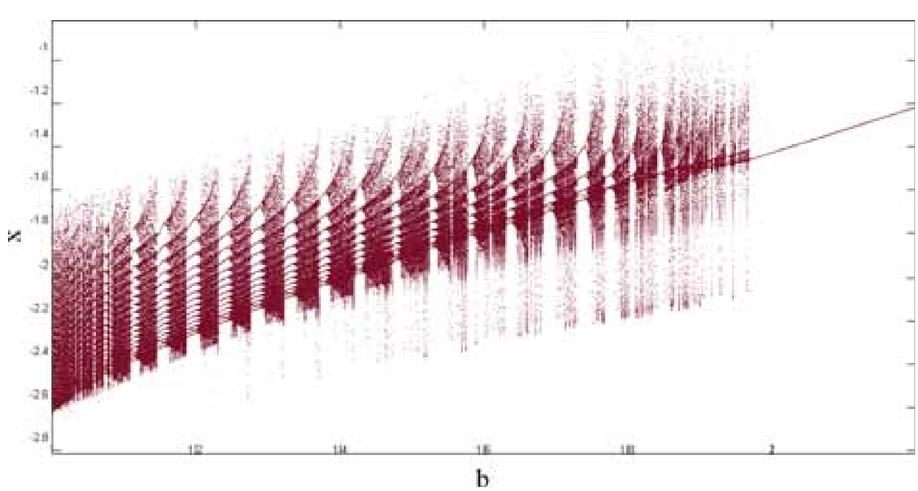

Figure 2. Bifurcation diagram for system (1) for different values of b with the initial conditions of (−2.15, −0.83, −4.29, −0.11).

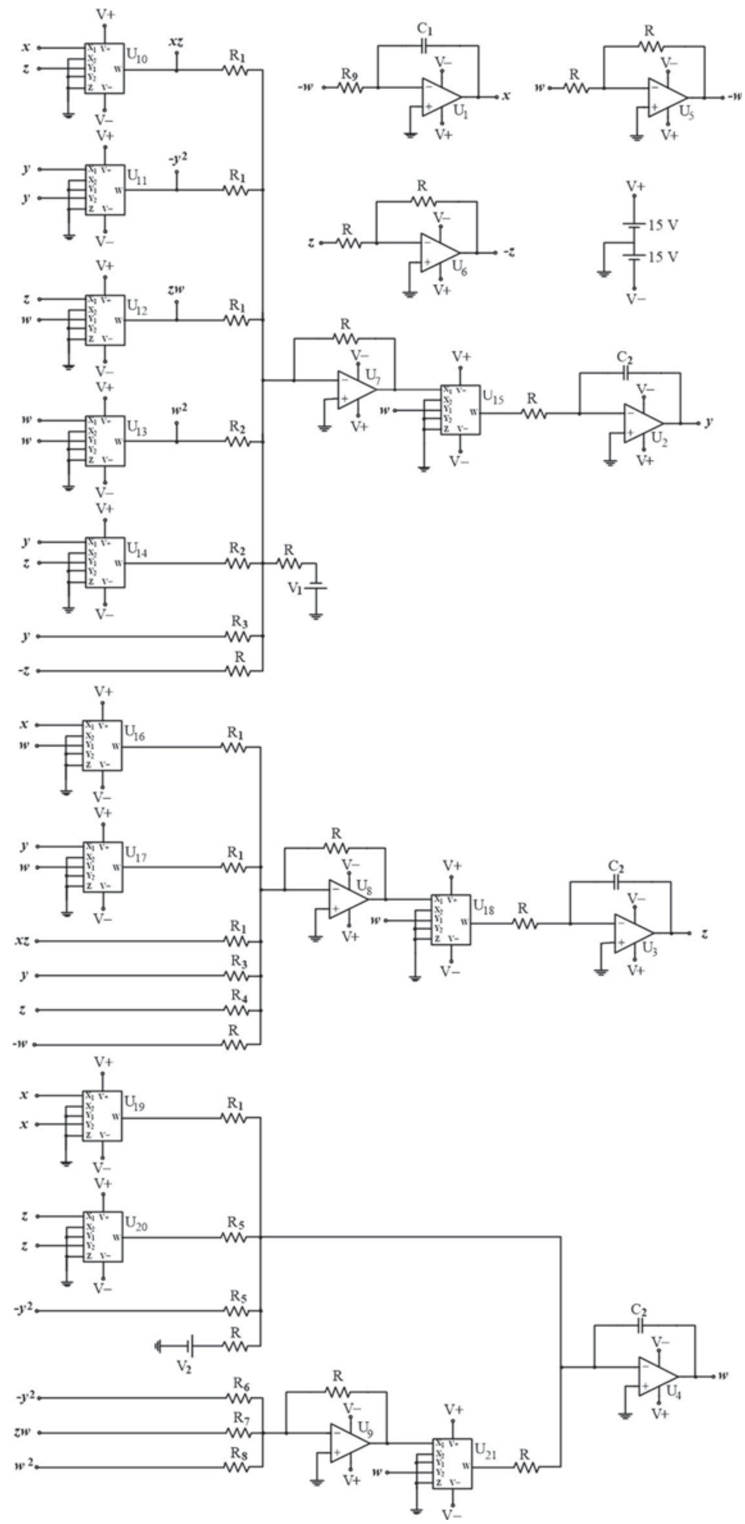

Figure 3. Schematic of the designed circuit that emulates the proposed system (1).

Figure 3 depicts the schematic of the circuit that emulates system (1). This circuit has four integrators (U_1−U_4) and five inverting amplifiers (U_5−U_9), which are implemented with the operational amplifier TL084, as well as twelve signal multipliers (U_{10}−U_{21}) by using the analog multiplier AD633.

By applying Kirchhoff's circuit laws, the corresponding circuital equations of the designed circuit can be written as

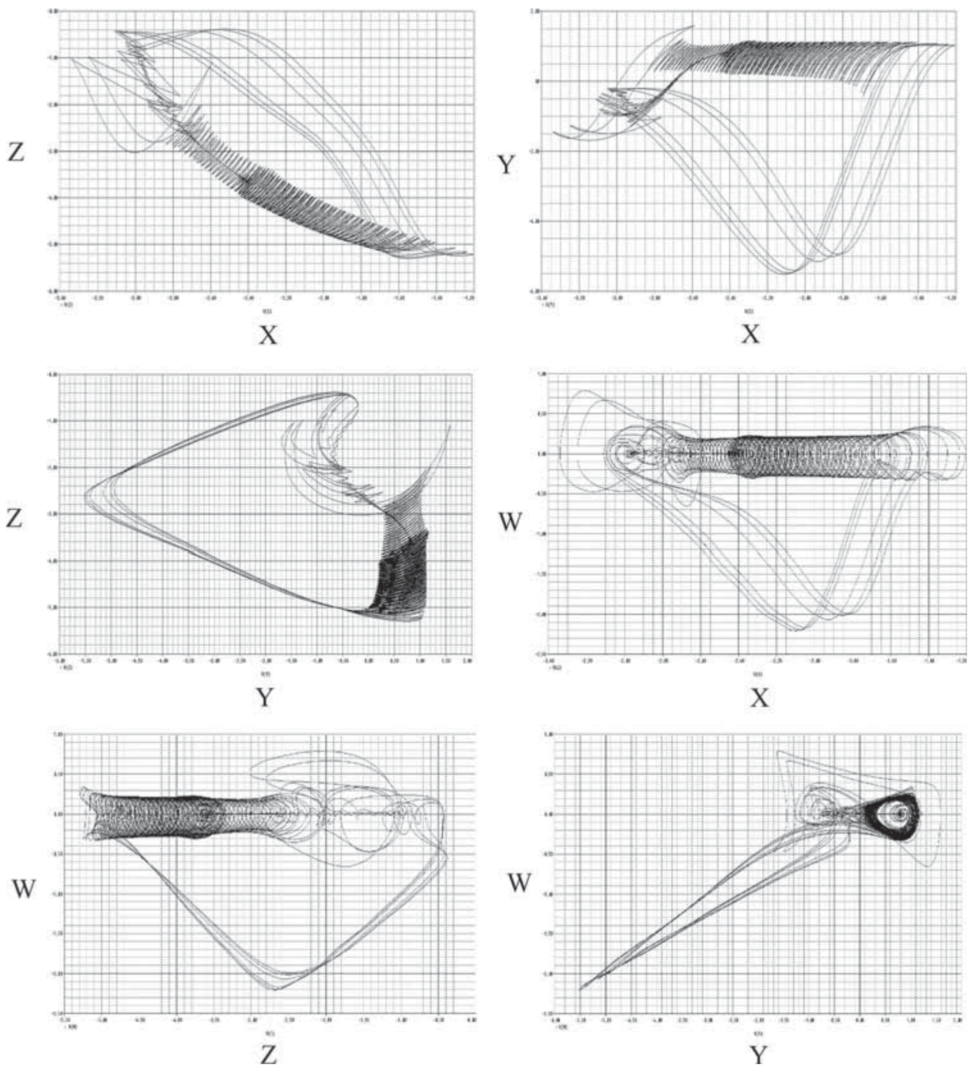

Figure 4. The obtained PSpice phase portraits in $x-y$ plane, $x-z$ plane, $x-w$ plane, $y-z$ plane, $y-w$ plane and $z-w$ plane.

$$\begin{cases} \dot{x} = \frac{1}{R_9C_1}w, \\ \dot{y} = \frac{1}{R_9C_1 10}\left[\frac{w}{1V}\left(\frac{R}{R_3}y - z + \frac{R}{R_1 10V}xz \right.\right. \\ \quad -\frac{R}{R_1 10V}y^2 + \frac{R}{R_2 10V}yz + \frac{R}{R_1 10V}zw \\ \quad \left.\left. +\frac{R}{R_2 10V}w^2 - V_1\right)\right], \\ \dot{z} = \frac{1}{RC_2 10}\left[\frac{w}{1V}\left(\frac{R}{R_3}y + \frac{R}{R_4}z - w + \frac{R}{R_1 10V}xz \right.\right. \\ \quad \left.\left. +\frac{R}{R_1 10V}xw + \frac{R}{R_1 10V}yw\right)\right], \\ \dot{w} = \frac{1}{RC_2 10}\left[\frac{x^2}{1V} + \frac{R}{R_5 1V}y^2 + \frac{R}{R_5 1V}z^2 \right. \\ \quad -10V_2 + \frac{w}{1V}\left(\frac{R}{R_6 10V}y^2 \right. \\ \quad \left.\left. -\frac{R}{R_7 10V}zw - \frac{R}{R_8 10V}w^2\right)\right], \end{cases} \tag{5}$$

where the variables x, y, z, and w correspond to the voltages in the outputs of the integrators U_1–U_4. Normalising the differential equations of system (2), by using

$$\tau = \frac{t}{R_9}, \quad C_1 = \frac{t}{10RC_2},$$

we can see that this system is equivalent to system (1), with the circuit components, which have been selected as: R= 10 kΩ, R_1=1 kΩ, R_2=0.5 kΩ, R_3 = 2.5 kΩ, R_4 = 5 kΩ, R_5 = 40 kΩ, R_6 = 0.333 kΩ, R_7 = 1.666 kΩ, R_8 = 0.0526 kΩ, R_9 = 100 kΩ, C_1 = 10 nF, C_2 = 1 nF, V_1 = 3 V DC, V_2 = 0.9 V DC, while the power supplies of all active devices are ±15 V DC.

In figure 4, various projections of phase planes, which are captured using the electronic simulation package

Cadence OrCAD, are presented. The comparison of these attractors with the respective ones from system's simulation process (figure 1) proved that the circuit emulates the proposed system well.

4. Conclusion

A four-dimensional autonomous transiently chaotic system with ellipsoid equilibria has been proposed in this paper. The system's characteristics such as equilibrium points and their stability are analysed. The proposed system exhibits transiently periodic and transiently chaotic behaviour by varying one parameter. It belongs to a new category of dynamical systems with hidden attractors. An analog circuit has been designed to realise the differential equations of the proposed transiently chaotic system. The designed circuit has been implemented and tested using the OrCad-PSpice software to verify the simulation results. Comparison of the experimental observations and numerical simulations showed good qualitative agreement between the transiently chaotic system and its circuitry realisation.

Acknowledgements

The authors would like to thank Professor J C Sprott for help and comments which enhanced the quality of this paper. Sajad Jafari, Shirin Panahi, and Zainab Aram were supported by Iran National Science Foundation (No. 96000815).

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