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Stochastic evolution of the Universe: A possible dynamical process leading to fractal structures

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Abstract. In this paper, we propose a stochastic evolution of the early Universe which can lead to a fractal correlation in galactic distribution in the Universe. The stochastic equation of state, due to fluctuating creation rates of various components in a many-component fluid, leads to a fluctuating expansion rate for the Universe in the early epochs. It provides persistent fluctuations in the number count vs. apparent magnitude relation, as expected from the observation of a fractal distribution of the galaxies. We also present a stochastic evolution of density perturbations in the early Universe.

Keywords. Cosmology; stochastic equations; Fokker–Planck equation.

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1. Introduction

The basic assumption of standard cosmological model is the Einstein's cosmological principle [1,2] which, in fact, is the hypothesis that the Universe is spatially homogeneous and isotropic on large scales. This implies linear deterministic Hubble's law $v = Hr$ (or a linear red-shift–apparent magnitude relation) which is valid at scales where matter distribution can be considered on average uniform and the law is well established within local scales. Many recent analyses (red-shift surveys such as cfA, Las Campanas, SSRS, LEDA, IRAS and ESP for galaxies, and Abell and ACO for galactic clusters) have revealed that the three-dimensional distribution of galaxies and clusters are characterised by large-scale structures (hierarchical) and huge voids. Such a distribution shows fractal correlations [3–6] up to the limits of available samples. This has confirmed the de-Vaucouleurs power law density–distance relation, $\rho(r) \propto (r/r_0)^{D_1-3}$ with fractal dimension $D_1 \approx 2$ at least in the range of scales 1 to 200 h^{-1} Mpc, that is sheet-like distribution of galaxies. In the above expression, $D_1 = 3$ corresponds to perfect homogeneous distribution of galaxies and a perfect linear Hubble expansion.

A fractal is a geometric shape that is not homogeneous, yet preserves the property that each part is a reduced-scale version of the whole [7]. That is, fractal structures are self-similar structures or possess scale-invariant properties. Most of the natural fractals are random fractals (for example, the galactic distribution). The reason is that, they consist of random shapes or patterns that are formed stochastically at any length scale. Because of randomness, the self-similarity of natural fractal is only statistical.

There are two views regarding the structure of the Universe. One is that the Universe shows inhomogeneities on all scales and the other is that the Universe is inhomogeneous and essentially fractal on the scales of galaxies and clusters of galaxies, but on larger scales this inhomogeneities are constrained by the observed smallness of CMB fluctuations and by X-ray background probes and hence it becomes isotropic and homogeneous [8–12], even though the cross-over scale to homogeneity is not well identified [6]. Wu and others [8–10,12] argue that D_1 is not constant for all scales and it varies from 2 to 3 as we go from local to large scales. The isotropy of CMB is the strongest evidence in support of this. Hence they claim that even if luminous matter exhibits this fractal character, the mass distribution of the Universe approaches homogeneity on large scales so that FRW model is valid. However, it is to be noted that the isotropy of CMB refers to background radiation. The relation between CMB radiation and matter distribution, corresponding to galactic clustering, required more detailed investigations and assumptions, because their origin and evolution are entirely different.

Labini *et al* [13] have analysed a number of red-shift surveys and found $D_1 = 2$ for all the data they looked at, and argued that there is no transition to homogeneity for scales upto 1000 Mpc, way beyond the expected turn-over. A controversy exists among cosmologists regarding this switch-over scale to homogeneity. This fractal behaviour of galaxy distribution within a scale of $200\, h^{-1}$ Mpc (this scale may be even deeper) is a challenge for the standard cosmology, where the linear Hubble law is a strict consequence of homogeneity of the expanding Universe. The presence of dark matter (distributed homogeneously) may save the cosmological principle even at small scales [14]. In this way, one may save the usual FRW metric (which needs a homogeneous density), while a substantial revision to the models of galaxy formation is required. On the contrary, if dark matter is found to have the same distribution as that of the luminous one, then a basic revision of the theory must be considered.

In fact, from a theoretical point of view, one would like to identify the dynamical processes which can lead to such a fractal distribution, because no dynamical theory till now explained how such a fractal Universe has arisen from a pretty smooth initial state. Here, we propose a stochastic model and argue that the Universe can exhibit fractal correlation due to a stochastic evolution in the early epochs.

The paper is organised as follows: Section 2 gives a brief review of stochastic evolution of the Universe. Section 3 describes the evolution of density perturbations under stochastic approach and determination of PDF for density contrast δ at any epoch. The results and conclusions derived from it are discussed in §4.

2. Stochastic evolution

Several recent measurements [15–23] indicate that the present Universe contains components other than ordinary matter and radiation, like vacuum energy, quintessence etc. Let the equation of state for each component, with density ρ_i, be written as $p_i = \omega_i \rho_i$ ($i = 1, 2, \ldots$), where p_i is the corresponding pressure, $\omega_i = 0$ for matter, 1/3 for radiation, -1 for vacuum energy, etc.; in general, $-1 \le \omega_i \le +1$. The total energy density $\rho = \sum_i \rho_i$ and the total pressure $p = \sum_i p_i$. From the law of conservation of total energy density, expressed in the form

$$\dot{\rho} = \sum_i \dot{\rho}_i = -3\frac{\dot{a}}{a}\rho(1+\omega) = -3\frac{\dot{a}}{a}\sum_i \rho_i(1+\omega) \tag{1}$$

which follows from the Einstein equation, one can write the total ω factor as

$$\omega = \frac{p}{\rho} = -\frac{\sum_i \dot{\rho}_i}{3(\dot{a}/a)\rho} - 1 = \sum_i \frac{\rho_i}{\rho}(1+\omega_i) - 1, \tag{2}$$

where a is the scale factor. The conservation of individual components, which may be expressed as $\dot{\rho}_i = -3(\dot{a}/a)\rho_i(1+\omega_i)$, is only an extra assumption because it does not follow from the Einstein's field equation. Equivalently, it can be stated that in a many-component cosmic fluid as in the above case, the Einstein equations, along with the equations of state of individual components, are insufficient to determine the individual $\dot{\rho}_i$'s. Thus, it is more general to assume that only the total energy density is conserved and this will lead to the creation of one component at the expense of other components. Since they are not uniquely determined by the field equations, such creations can be considered as sporadic events, like those occurring in galactic nuclei, which can result in fluctuations in the ratio ρ_i/ρ. This, in turn, will lead to a stochastic equation of state, as can be seen from eq. (2). Consequently, the expansion rate also will be fluctuating and the equation for the Hubble parameter will appear as a Langevin-type equation.

$$\dot{H} = -\frac{3}{2}H^2(1+\omega), \tag{3}$$

where we assumed the background is flat which is well-established. With fluctuating ω, eq. (3) is a Langevin-type equation, describing the evolution of the stochastic variable H; that is, the fluctuating character of ω leads to a random behaviour for the Hubble parameter. To simplify further, we make use of the transformation $x = 2/3H$. Hence eq. (3) becomes

$$\dot{x} = 1 + \omega. \tag{4}$$

This equation is a non-deterministic, stochastic, first-order differential equation. For the sake of simplicity, it was assumed that ω is a Gaussian δ-correlated stochastic force with zero mean. We used stochastic methods [24] for the analysis of this problem and the probability distribution function of the stochastic variable is calculated by solving the corresponding Fokker–Planck equation (FPE):

$$\frac{\partial W(x,t)}{\partial t} = \left[-D^1\frac{\partial}{\partial x} + D^2\frac{\partial^2}{\partial x^2}\right]W(x,t). \tag{5}$$

We find the drift coefficient $D^1 = 1$, and the diffusion coefficient $D^2 \equiv D$ is assumed to be a constant. We get a Gaussian distribution function $W(x,t)$ [25]. Thus, a fluctuating ω-factor, in turn, will lead to fluctuations in the time evolution of the Hubble parameter, that is, the expansion rate of the Universe becomes a stochastic quantity. We argue that such a fluctuating expansion rate might have led to a randomness in the recession velocities of the objects, in addition to peculiar velocities induced by density inhomogeneities.

The structures that we see today, are formed by a process known as gravitational instability, from primordial fluctuations in the cosmic fluid [26]. But because the strength of clustering is expected to increase with time (evolution of density contrast is proportional to some power of scale factor in the linear approximation according to the Standard Model), the galaxies must deviate from the smooth Hubble expansion. According to standard Friedmann model [1,27], $\delta v \propto \Omega_0^{0.6}\delta\rho$, where $\delta\rho$ is the density perturbation and Ω_0 is the present value of the ratio between critical density and density of the Universe. The peculiar velocity is one of the independent probes of inhomogeneities in the gravitational field induced by the density fluctuations. Another probe is the fluctuations in the background radiation. Observations show a very nearly linear Hubble expansion for local scales, and deviations from this deterministic Hubble's law increases with red-shift, which is obvious from the Hubble diagram or redshift (z)–apparent magnitude (m) diagram [28]. Despite rigorous attempts to control random errors in measurements, there is a clear scatter in it. Conventional explanation for the scatter is in terms of the peculiar velocities induced by the observed density fluctuations [27]. But density fluctuations are evolving phenomena, they cannot induce large randomness observed at high red-shifts. The scatter in the Hubble diagram or deviations from the linear Hubble expansion may also arise due to the inherent stochastic nature of the Hubble parameter [25] apart from the peculiar velocities which are significant only in the late Universe. The large scatter or randomness at high red-shifts can be explained on the basis of this fluctuating expansion rate of the Universe.

The evolution of the Universe becomes stochastic, where time evolution of cosmological parameters are described by Langevin-type equations [25,29] or stochastic differential equations (SDE). We argue that, such dynamical processes may lead to a fractal distribution (or a scale invariant inhomogeneous distribution) of galaxies, because dynamical equations containing stochastic quantity describe fractal growth [30,31].

In Standard Model, the density perturbations evolve deterministically, proportional to some power of scale factor in the linear approximation [27]. However, under stochastic equation of state, dynamical equations describing the evolution of density and its perturbation must be stochastic. In ref. [29], we have shown that the evolution of density is stochastic. In the following section we shall show that under these circumstances, the time evolution of density perturbations are described by Langevin-type equations.

3. Evolution of density perturbations

In the early Universe, when cosmic fluid is not unicomponent, a stochastic equation of state emerges, that is, the factor ω in the equation of state becomes a fluctuating quantity. Hence, the evolution of density perturbations is a stochastic process. In the early Universe, the energy density in any region can be written as a perturbation equation [32].

$$\rho(x,t) = \rho_{\rm b}(t) + \delta\rho(x,t), \tag{6}$$

where $\rho_{\rm b}$ is the background density, which at any time t is independent of the location. However, ρ at different regions are slightly different in the early Universe, and hence $\delta\rho$ also. The evolution of density contrast ($\delta = \delta\rho/\rho$) according to Standard Model is a deterministic one, proportional to some power of the scale factor in the linear approximation. In the stochastic approach, due to a fluctuating equation of state, its evolution is a stochastic process. We assume that the total energy density is conserved, and so

$$\dot{\rho} = -3H(\rho + p). \tag{7}$$

After some simplifications we get the evolution equation of $\delta\rho$ as [26]

$$\delta\dot{\rho} = -3(\rho_{\rm b} + p_{\rm b})\delta H - 3H_{\rm b}\delta\rho. \tag{8}$$

Here the suffix 'b' stands for the background. Using $\delta\rho = \rho_{\rm b}\delta$, eq. (8) becomes

$$\dot{\delta}(t) = -3H_{\rm b}\omega(t)\delta - 3[1+\omega]\delta H. \tag{9}$$

Now,

$$H_{\rm b}^2 = \frac{8\pi G}{3}\rho_{\rm b} \tag{10}$$

$$(H - \delta H)^2 = \frac{8\pi G}{3}(\rho - \delta\rho). \tag{11}$$

Equating both sides, we have

$$H^2 = \frac{8\pi G}{3}\rho \tag{12}$$

and

$$2H\delta H = \frac{8\pi G}{3}\delta\rho. \tag{13}$$

Using eqs (12), (13) and (6), we get

$$\delta H = \frac{1}{2}\sqrt{\frac{8\pi G}{3}}\delta\rho(\rho_{\rm b})^{-1/2}\left[1 + \frac{\delta\rho}{\rho_{\rm b}}\right]^{-1/2}. \tag{14}$$

Retaining only first-order terms, this becomes

$$\delta H = \frac{1}{2}\sqrt{\frac{8\pi G}{3}}\frac{\delta\rho}{\sqrt{\rho_{\rm b}}}. \tag{15}$$

Substituting for H_b and δH in eq. (9), it becomes

$$\dot{\delta} = \sqrt{6\pi G\rho_b}[\omega - 1]\delta. \tag{16}$$

Using the transformation

$$y = \ln\delta \tag{17}$$

eq. (16) becomes

$$\dot{y} = \sqrt{6\pi G\rho_b}[\omega - 1]. \tag{18}$$

This is a Langevin-type equation. As it has another stochastic variable (ρ_b), we have to write the stochastic equation for ρ_b also and the resulting system of equations has to be treated together, which leads to a two-variable distribution function. For this, we define a transformation

$$x = \frac{1}{\sqrt{6\pi G\rho_b}}. \tag{19}$$

Therefore, from eq. (18) and the conservation equation for density, we have

$$\dot{y} = -\frac{1}{x} + \frac{\omega(t)}{x} \tag{20}$$

and

$$\dot{x} = 1 + \omega(t). \tag{21}$$

To write the FPE, we need the drift and diffusion terms. From the general equation

$$\dot{x_i} = h_i + g_{ij}\Gamma_{ij}. \tag{22}$$

The drift coefficients are

$$D_i^{(1)} = h_i + g_{kj}\frac{\partial}{\partial x_k}g_{ij}. \tag{23}$$

The diffusion terms are

$$D_{ij}^{(2)} = g_{ij}g_{jk}. \tag{24}$$

From eqs (20) and (21) the drift terms are

$$D_y^{(1)} = -\frac{1}{x} - \frac{1}{x^2} \approx -\frac{1}{x^2}; \quad D_x^{(1)} = 1, \tag{25}$$

where we have used the fact that in the early Universe ρ_b is very high. The diffusion terms are

$$D_{yy}^{(2)} = \frac{1}{x^2}; \quad D_{yx}^{(2)} = D_{xy}^{(2)} = \frac{1}{x}; \quad D_{xx}^{(2)} = D. \tag{26}$$

Here the diffusion terms arise from fluctuations in the mean ω factor alone and D is a constant. Now we write the FPE for the distribution function $W(y, x, t)$ as

$$\frac{\partial W(y,x,t)}{\partial t} = \left[-\frac{\partial}{\partial y}D_y^{(1)} - \frac{\partial}{\partial x}D_x^{(1)} + \frac{\partial^2}{\partial y^2}D_{yy}^{(2)} + \frac{\partial^2}{\partial x^2}D_{xx}^{(2)} + 2\frac{\partial}{\partial y}\frac{\partial}{\partial x}D_{yx}\right]W(y,x,t). \tag{27}$$

Substituting for the drift and diffusion coefficients

$$\frac{\partial W(y,x,t)}{\partial t} = \left[-\frac{1}{x^2}\frac{\partial}{\partial y} - \frac{\partial}{\partial x} + \frac{1}{x^2}\frac{\partial^2}{\partial y^2} + D\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial^2}{\partial x \partial y}\right]W(y,x,t) \tag{28}$$

and neglecting the crossed term and applying a separation ansatz, $W = u(x)v(y)\mathrm{e}^{-\lambda t}$ we get (by variable separation)

$$\frac{1}{v}\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} - \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}y} = x^2\left[\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{D}{u}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \lambda\right]. \tag{29}$$

Here both sides can be equated to a constant (say c). We have a set of equations for $u(x)$ and $v(y)$. They are

$$\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} - \frac{\mathrm{d}v}{\mathrm{d}y} - cv = 0 \tag{30}$$

and

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \frac{1}{D}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{D}\left[\lambda + \frac{c}{x^2}\right]u = 0. \tag{31}$$

Due to the essential singularity of eq. (31) at $x = 0$, a series solution is possible. However, if we take $c = 0$, then we shall get a compact solution (to understand a general behaviour). After solving eqs (30) and (31) we can write the complete solution as

$$W(y,x,t) = a\exp\left[y + \frac{x}{2D} + ikx - \lambda t\right], \tag{32}$$

where

$$k^2 = \frac{\lambda}{D} - \frac{1}{4D^2}. \tag{33}$$

Physically reasonable solution exists for $\lambda \geq 1/4D$. A is the normalisation constant, chosen as $1/2\pi$. The general solution is obtained by integrating eq. (32) in the range $-\infty \leq k \leq +\infty$ and we get

$$W(y,x,t) = \frac{1}{\sqrt{4\pi Dt}}\mathrm{e}^y \exp\left[-\frac{(x-t)^2}{4Dt}\right]. \tag{34}$$

This is Gaussian in x. We can write the distribution function in terms of the original variables $y = \ln\delta$ and $x = 1/\sqrt{6\pi G\rho_b}$. The distribution function in terms of δ and ρ_b is

$$W(\delta,\rho_b,t) = \frac{1}{\sqrt{96\pi^2 GD\rho_b^3 t}} \times \exp\left[-\frac{(1 - t\sqrt{6\pi G\rho_b})^2}{24\pi G\rho_b Dt}\right]. \tag{35}$$

Thus, the density perturbation evolves in a non-deterministic way.

4. Discussion and conclusions

When the Universe is approximated by a many-component fluid, the fluctuations in creation rates are certainly physical processes which can lead to stochastic fluctuations in the mean ω factor of the equation of state (that is, a stochastic equation of state). The evolution of the Universe becomes stochastic (or the expansion rate of the Universe fluctuates), where the time evolution of the cosmological parameters are described by the Langevin equations or SDE. We argue that such dynamical processes may lead to fractal distribution (or a scale-invariant inhomogeneous distribution) of galaxies, because the fluctuations in the evolution process never lead to structures with perfect symmetry and most natural fractals are formed through stochastic process.

For a homogeneous distribution of galaxies, Hubble's count law is

$$N(<m) \propto 10^{0.2D_1 m}, \tag{36}$$

where m is the apparent magnitude of the object and N represents the number of galaxies brighter than the magnitude m. $D_1 = 3$ corresponds to homogeneous distribution. Equivalently, one can express (1) in terms of red-shift (z) also, that is, $N(<z)$. Apparent magnitude is related to z in the following way:

$$m - M = 5\log\left(\frac{D_{\mathrm{L}}}{1\,\mathrm{Mpc}}\right) + 25, \tag{37}$$

where M is the absolute magnitude of the galaxy. The luminosity distance D_{L} is given by

$$D_{\mathrm{L}} = r_j a(t_0)[1+z], \tag{38}$$

where $a(t_0)$ is the present scale factor and r_j is the radial coordinate of the object which emitted the light at some time t_j in the past. In flat FRW models, r_j is found as

$$r_j = \int_{t_j}^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \frac{c t_j{}^{2/3}}{a(t_j)} \int_{t_j}^{t_0} \frac{\mathrm{d}t}{a(t)}. \tag{39}$$

Integrating

$$r_j = \frac{2c}{a(t_j)} H_j[\sqrt{1+z} - 1] \tag{40}$$

and

$$D_{\mathrm{L}} = \frac{2c}{H_j}(1+z)^2[\sqrt{1+z} - 1], \tag{41}$$

where the Hubble parameter H_j corresponds to the epoch t_j. Thus, the luminosity distance is related to red-shift [1], which depends on the Hubble parameter. The fluctuations on this number counts around the average behaviour as a function of m or z can discriminate between a genuine fractal distribution and a homogeneous one [33]. Number count vs. apparent magnitude can be used to test whether the large-scale distribution of galaxies can be compatible with a fractal or with a homogeneous behaviour [13,34]. In a fractal distribution, one expects to find persistent scale-invariant fluctuations around the average behaviour, which do not decay with m or z. On the other hand, in a homogeneous distribution, on large enough scales, the relative variance of the counts should decrease exponentially with m [1,34]. Labini *et al* [33,34] claim that, the relative fluctuations in the counts as a function of m has a constant magnitude (for $z \geq 0.1$), which cannot be due to any smooth correction to the data as evolution effects, but they can be the outcome of an inhomogeneous distribution of galaxies.

In the previous section, we have shown that a fluctuating ω-factor, in turn, will lead to fluctuations in the time evolution of the Hubble parameter, that is, the expansion rate of the Universe becomes a stochastic quantity. We argue that such a fluctuating expansion rate might have led to a randomness in the recession velocities of the objects, in addition to peculiar velocities induced by density inhomogeneities. We also argue that, this randomness in the recession velocities of galaxies led to a fractal distribution of galaxies and clusters of galaxies. From eqs (36) and (41) we can see that a fluctuating or stochastic expansion rate may provide the constant fluctuations observed in the number count vs. m or z relation.

As fluctuations are always present in physical processes, they never lead to structures with perfect symmetry. For example, random walk of particles, diffusion-limited aggregation etc. lead to fractal structures. In the preceding sections we have shown that the evolution of the Universe becomes stochastic under a stochastic equation of state. A stochastic equation of state emerges due to fluctuations in the creation rates. Under these conditions, the expansion rate of the Universe becomes stochastic and induce randomness in the recession velocities of galaxies, especially in the early epochs.

This stochastic evolution of the Universe may provide a dynamical process leading to a self-similar structure in the Universe and also produce the scatter observed in the Hubble diagram at high red-shifts, where peculiar velocities are inadequate. We predict that the persistent fluctuations found in the number counts vs. apparent magnitude relation, especially for high red-shifts (early epochs), which do not decay with m, may be due to the stochastic nature of the Hubble parameter. As both density and Hubble parameter are stochastic in the early epochs, time evolution of density perturbations must be non-deterministic and hence described by SDE as we have done in §3. In this paper, we have not made any attempt to characterise the statistically scale-invariant structures observed in the range of scales 1–200 h^{-1} Mpc by measuring either the correlation function or power spectrum; we have attempted only to provide a possible stochastic process which can produce an inhomogeneous distribution of galaxies. Here we have only argued that a stochastic evolution of the Universe may lead to a scale-invariant large-scale structure.

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