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Entanglement of a nonlinear two two-level atoms interacting with deformed fields in Kerr medium

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Abstract. In this paper we investigate the entanglement dynamics between two two-level atoms interacting with two coherent fields in two spatially separated cavities which are filled with a Kerr-like medium. We examine the effect of nonlinear medium on the dynamical properties of entanglement and atomic occupation probabilities in the case of even and odd deformed coherent states. The results show that the deformed fields play important roles in the evolution of entanglement. Also, the results demonstrate that entanglement sudden death, sudden birth and long-distance can be controlled by the deformation and nonlinear parameters.

Keywords. Entanglement; Kerr-like medium; sudden birth; sudden death.

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Experimental realisation of a quantum computer has become one of the main focus areas of current quantum information research [1]. In this approach, entanglement plays a great role in information theory [1–5]. The nonlocal nature of entanglement has also been identified as an essential resource for many novel tasks in quantum information processing such as quantum teleportation [4,5], superdense coding [6], quantum cryptography [7,8] and quantum metrology [9]. These quantum information tasks cannot be carried out by classical resources and they rely on entangled states. This recognition led to an intensive search for mathematical tools that would enable a proper quantification of this resource. It is of primary importance to test whether a given quantum state is entangled or separable. Different entanglement measures have been used for pure and mixed states such as concurrence [6,7], entanglement of formation [8], negativity [9,10], Fisher information [11–13] and quantum Fisher information [14–16]. In this regard, the concurrence and negativity are used as good entanglement measures for mixed state, the von Neumann entropy has been proposed for pure state entanglement [17], and all these measures are used to test whether a given quantum state is separable or entangled. Also, some interesting physical phenomena are observed as a result of entanglement, such as entanglement sudden death (ESD) and entanglement sudden birth (ESB) [18,19].

Quantum groups (QG) have been introduced as a natural extension of the notation of coherent states and interpreted as nonlinear harmonic oscillator with a very specific type of linearity [20]. A q-deformed harmonic oscillator was defined in terms of q-boson annihilation and creation operators, the latter satisfying quantum Heisenberg–Weyl algebra [21] which plays an important role in QG. The deformed coherent states have been used to describe a large class of quantum systems prepared from several potentials (by a proper choice of the q-deformed parameter) such as infinite, modified Poschl–Teller, Morse potential [22] and from the finite-range potentials [23].

Over the last two decades, there have been several experimental demonstrations of nonclassical effects. Some important physical concepts of the corresponding coherent states exhibit many non-classical properties such as photon antibunching, sub-Poissonian photon statistics, squeezing, etc. (for a review, see [24]). It has been experimentally observed that the real laser, bunched and antibunched light, possesses a photon number statistics which can be super-Poissonian or sub-Poissonian. The physical importance of the deformed coherent states lies in the fact that they offer the best description for non-ideal physical devices such as lasers [25] (i.e. real lasers). The deformation parameters then play the role of a tuning parameter defining how far the realized device is from the ideal one.

More recently, the entangled coherent states for systems with $su(2)$ and $su(1,1)$ symmetries are studied [26], where a method for generating the entangled $su(2)$ and $su(1,1)$ coherent states is discussed and degrees of entanglement calculated. Also, entanglement between moving three-level atom with nonlinear deformed field has been investigated [27]. It is shown that the deformation effects play important roles in the evolution of entanglement. Moreover, the effect of a beam splitter on the nonlinear deformed coherent states has been examined [28]. It is found that these states are useful for generating quantum entanglement as the deformation parameter gets farther from unity and for strong input field regimes. Finally, the nonlocal correlation between two two-level atoms and optical field initially in deformed bosonic coherent states has been discussed [29]. The concurrence and negativity are measures of atom–atom entanglement. Important results related to this new kind of interaction such as ESD, ESB and entanglement stabilization have been explored. These results are obtained when the nonlinear Kerr medium and the two atoms are in the same cavity field. So the main aim is to study the interaction between two identical two-level atoms and the effects of nonlinear deformed field and Kerr-like medium on the entanglement. We study dynamical properties of the von Neumann entropy and concurrence. The paper is organized as follows: In §1, we introduce the derivation and main formula of the deformed even and odd coherent states. In §2, we present the model of the two nonlinear deformed fields and the two two-level atoms. The entanglement quantifiers in terms of the atomic density matrix will be presented in §3. In §4, we present the numerical results and discuss the different effects on the dynamics of the system entanglement. Finally, we summarize the main conclusions in §5.

1. Deformed even and odd coherent states

The deformed states, besides being of great importance both from a theoretical point of view and in practical applications, share peculiar features that make their structural properties amenable to accurate and detailed theoretical analysis. Deformation Lie algebras (or deformed quantum algebras) have turned into a fertile area of research in the last years, and many application ideas can be found in the literatures. These deformations exhibit a rich structure which present important results in the literature. Also, it can be used for the characterisation of the photon statistics of laser outputs reasonably close to threshold, single-atom resonance fluorescence, the micromaser field and absorption by two-level atoms.

It is known that the ordinary operators $\{1; \hat{a}; \hat{a}^\dagger; N\}$ form the Lie algebra of the Heisenberg–Weyl group and the linear harmonic oscillator can be connected with the generators of the Heisenberg–Weyl Lie group. The q-deformed quantum oscillators (nonlinear harmonic oscillators) are defined by the algebra generated by the operators $\{1; \hat{a}_q; \hat{a}_q^\dagger; N\}$ which obey the following commutation relations:

$$[\hat{a}_q, \hat{a}_q^\dagger] = [N+1]_q - q\,[N]_q = q^{-N-1},$$
$$[N, \hat{a}_q] = -\hat{a}_q \quad \text{and} \quad [N, \hat{a}_q^\dagger] = \hat{a}_q^\dagger. \tag{1}$$

The 'box' function [ ] represents the standard deformation defined by [21]

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\sinh(\gamma n)}{\sinh(\gamma)}, \quad q = \mathrm{e}^\gamma \text{ and } \gamma \in R, \tag{2}$$

where q is the deformation parameter by taking the classical $q \to 1$ limit (or $\gamma \to 0$); the deformed algebra reduces to the classical $SU(2)$ algebra. Also, a_q, $a_q^\dagger$ are deformed annihilation and creation operators acting on the Fock states, such that

$$\hat{a}_q|n\rangle = \sqrt{[n]_q}|n-1\rangle, \quad a_q^\dagger|n\rangle = \sqrt{[n+1]_q}|n+1\rangle,$$
$$\hat{a}_q^\dagger\hat{a}_q = [N]_q, \qquad \hat{a}_q\hat{a}_q^\dagger = [N+1]_q\,. \tag{3}$$

The Fock space of the q-boson states is introduced according to the construction

$$|n\rangle = \frac{(\hat{a}_q^\dagger)^n}{\sqrt{[n]_q!}}|0\rangle \tag{4}$$

and the q-factorial is defined as

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q; \quad [0]_q! = 1. \tag{5}$$

The deformed bosonic coherent states are coherent states that are constructed using a formally analogous scheme as the one allowing the construction of the Glauber coherent states starting from the Heisenberg–Weyl algebra. The deformed nonlinear coherent states which are defined as the eigenstates of the annihilation operator of a q-deformed bosonic field $\hat{a}_q$ as

$$|\alpha\rangle_q = \frac{1}{\sqrt{\sum_{n=0}^{\infty} \frac{\alpha^{2\ell}}{[\ell]_q!}}} \sum_{n=0}^{\infty} \frac{(\alpha^n + r(-\alpha)^n)}{(1+r^2)\sqrt{[n]_q!}}|n\rangle, \tag{6}$$

where $\ell = n(r^2+1) + (r^2-r)/2$ and r is an arbitrary constant. In this case, $r = 0, -1$ and 1 correspond to deformed coherent, odd coherent and even coherent states, respectively.

2. Model and its solution

The system consists of two non-interacting two-level atoms with the upper state $|+\rangle$ and the lower state $|-\rangle$, each atom is trapped inside a single cavity which is filled with a nonlinear Kerr-like medium. The Hamiltonian of the whole system can be written as

$$H = \chi_1^2 \hat{a}_q^{\dagger 2} \hat{a}_q^2 + \chi_2^2 \hat{b}_q^{\dagger 2} \hat{b}_q^2 + \lambda \big[\hat{a}_q \sigma_1^+ + \hat{a}_q^{\dagger} \sigma_1^- + b_q \sigma_2^+ + b_q^{\dagger} \sigma_2^- \big], \tag{7}$$

where a_q and $a_q^{\dagger}$ are the annihilation and creation operators of the deformed field mode. The operators σ_j^+ and σ_j^- are the usual raising and lowering operators for the jth two-level atom, χ_j is the Kerr nonlinear medium (we consider $\chi_1 = \chi_2 = \chi$) [30] and λ is the coupling constant between the atom and the field.

Let us assume that the initial state of the two atoms, $|\psi_{\mathrm{AB}}(0)\rangle$, is the maximally entangled quantum state or Bell state and $|\psi_{\mathrm{F}}(0)\rangle$ is the initial state of the input field. So the initial state of the combined system can be written as

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (|++\rangle + |--\rangle) \otimes |\alpha\rangle_q, \tag{8}$$

where $|\alpha\rangle_q$ is the deformed nonlinear coherent state given by eq. (6).

$$|\Psi(t)\rangle = \exp\{-iHt\} |\Psi(0)\rangle, \quad \hbar = 1. \tag{9}$$

The reduced density matrix of the two atoms can be obtained by tracing out the field basis as $\rho^{\mathrm{AB}}(t) = \mathrm{Tr}_{\mathrm{F}}(|\Psi(t)\rangle\langle\Psi(t)|)$.

3. Entanglement

The time evolution of entanglement for a composite system has been widely studied in recent years [31]. A lot of discussion has been devoted to the problem of disentanglement of a composite system in a finite time in the presence of dissipation, despite the fact that all the matrix elements of the system decay only asymptotically, i.e. when time goes to infinity. Yu and Eberly [32,33] coined the name ESD to the process of finite-time disentanglement. ESD has been confirmed experimentally [34]. On the other hand, the entanglement can also be created during the evolution or one can observe revival of the entanglement as well as ESB [35–37]. Among various studies on entanglement, for a pure bipartite state, the Schmidt decomposition [38] can be used to judge whether the state is entangled and the degree of entanglement can be quantified by partial von Neumann entropy. We use the von Neumann entropy to measure the entanglement between two two-level atoms and two nonlinear optical fields. The expression of the von Neumann entropy takes the form [17]

$$S = -\mathrm{Tr}(\hat{\rho} \ln \hat{\rho}). \tag{10}$$

For pure states, the von Neumann entropy is zero, i.e., for states that satisfy the condition $\hat{\rho}^2 = \hat{\rho}$, where $\hat{\rho}$ is the density operator describing a given quantum state. For this reason, this entropy cannot distinguish between various pure states, and it is rather a measure of the purity of quantum states. For the system under consideration, i.e. $\hat{\rho} = \hat{\rho}_{\mathrm{AB}}$ and the von Neumann entropy can be written as [39,40]

$$S_{\mathrm{AB}}(t) = -\sum_{j=1}^{4} E_j(t) \ln E_j(t), \tag{11}$$

where $E_j(t)$ are the eigenvalues of $\hat{\rho}_{\mathrm{AB}}(t)$.

On the other hand, it is well known that, both concurrence and negativity are considered as optimal measure for quantifying the atom–atom entanglement. The concurrence of the two atoms based on the atomic density matrix $\hat{\rho}_{\mathrm{AB}}(t)$ is given by [19,41,42]

$$C_{\mathrm{AB}}(t) = \max\{0, \mu_1 - \mu_2 - \mu_3 - \mu_4\}, \tag{12}$$

where μ_i $(i = 1-4)$ are the eigenvalues of the square roots of the density matrix, matrix $R = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ and σ_y is the Pauli matrix. ρ^* is the complex conjugate of ρ. The concurrence has zero value, i.e. $C_{\mathrm{AB}}(t) = 0$ for unentangled atoms, whereas $C_{\mathrm{AB}}(t) = 1$ for the maximally entangled atoms.

In the following section, we shall discuss numerically the effect of deformation and Kerr-like medium parameters on the dynamical behaviour of the von Neumann entropy, concurrence and the atomic population probabilities $\rho_{ii}(t)$ $(i = 1-4)$, where

$$\rho_{ii} = \sum_{n,m=0}^{\infty} |\psi_i(n,m,t)|^2, \quad i = 1,2,3,4 \tag{13}$$

are the atomic occupation probabilities of the states $|++\rangle$, $|+-\rangle$, $|-+\rangle$ and $|--\rangle$, respectively.

4. Results

In this section, we present the numerical results of the dynamical behaviour of von Neumann entropy as a measurement between the two atoms and a deformation

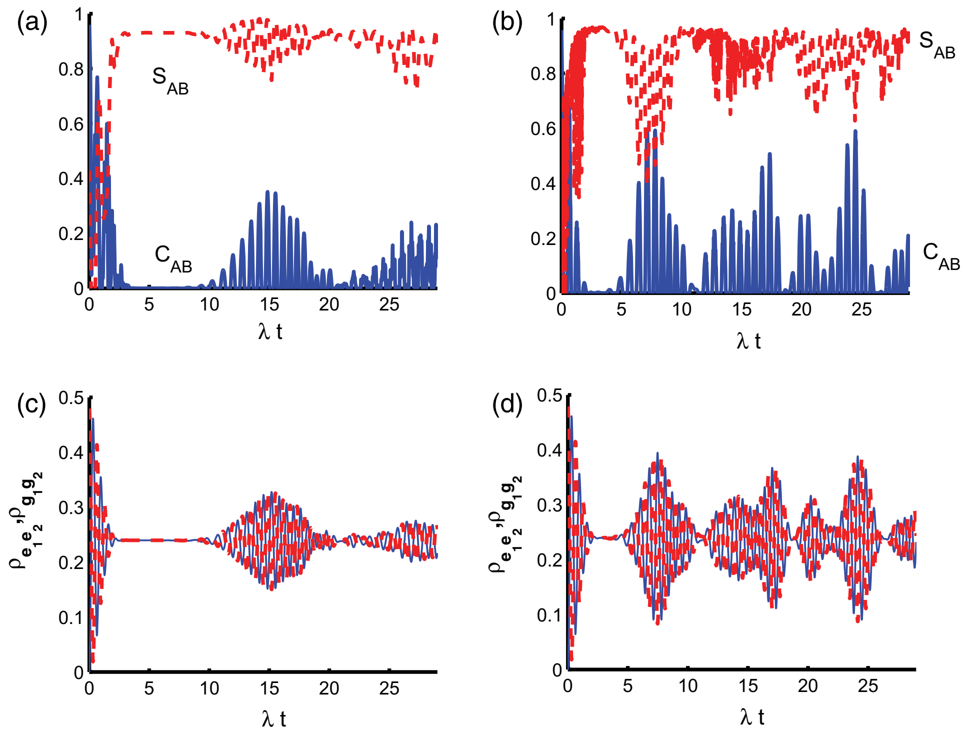

Figure 1. The time evolution of (**a**) von Neumann entropy S_{AB} (dashed line), concurrence C_{AB} (solid line) and (**c**) atomic population probabilities $\rho_{e_1e_2}$(solid line) and $\rho_{g_1g_2}$ (dashed line) for two two-level atoms interacting with a deformed nonlinear two-mode coherent field $r = 0$ for $\chi = 0$, $\bar{n}_1 = \bar{n}_2 = 5$ and $q = 1$. (**b**, **d**) The same as figures 1a and 1c, but the two fields are initially in the even deformed coherent states for $r = 1$.

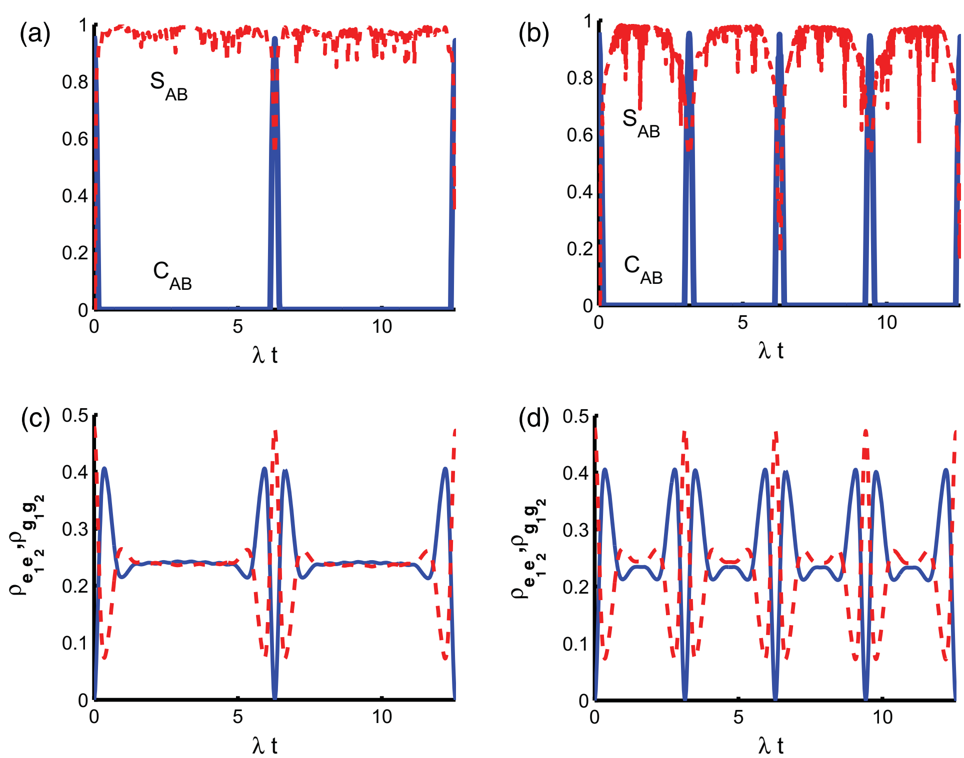

Figure 2. The same as figure 1 but for $\chi = 0.5$.

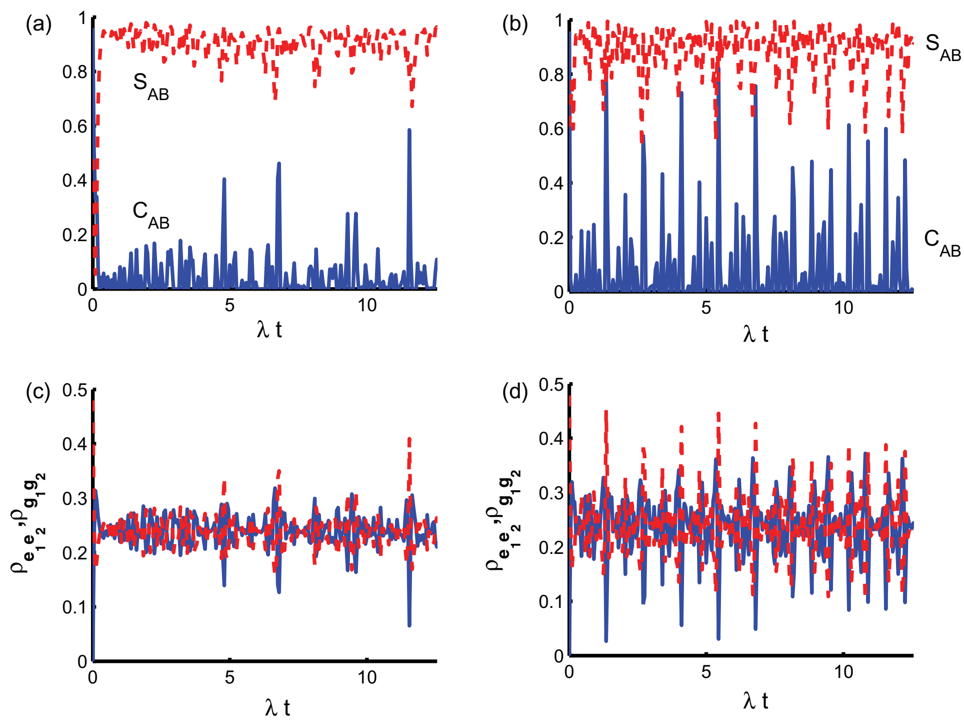

Figure 3. The same as figure 1 but for $q = 2$.

field. On the other hand, the atom–atom entanglement will be measured via concurrence which is given by eq. (12). In this case, we consider Bell states as the initial state of atoms and examine the effect of different parameters such as deformed parameter, q. This includes deformed coherent state when the parameter $r = 0$ and even deformed coherent state when $r = 1$. We also consider an important parameter that is the Kerr-like medium parameter χ on the entanglement. In all figures, we have considered the mean photon number, $\bar{n} = 5$ and the time is scaled as λt.

We start our discussion by considering the interaction system in the classical limit, i.e. $q \to 1$, and the deformed coherent state, $r = 0$, and in the absence of Kerr-like medium parameter, $\chi = 0$. In figure 1a, we plot the time evolution of $S_{\text{AB}}(t)$ and $C_{\text{AB}}(t)$ as functions of scaled time λt. We observed that $C_{\text{AB}}(t)$ and $S_{\text{AB}}(t)$ have approximately the same value in the beginning of the interaction, i.e. when $0 \le \lambda t \le \pi$, but after that, there is an opposite monotonic relation between the behaviour of $C_{\text{AB}}(t)$ and $S_{\text{AB}}(t)$. In other words, when the entanglement between the two atoms is zero, $C_{\text{AB}}(t) = 0$, then the entanglement with fields is maximum ($S_{\text{AB}}(t) = 1$) at $\pi \le \lambda t \le 3\pi$ and in this period of time the ESD appears clearly. When the time goes on, i.e. $\pi \le \lambda t \le 3\pi$, one can see the clearly opposite monotonic relation between $C_{\text{AB}}(t)$ and $S_{\text{AB}}(t)$, but ESD remains for a period of time less than the previous one.

In figure 1c, we plot the dynamics of the atomic population probabilities $\rho_{e_1e_2}$ (solid line) and $\rho_{g_1g_2}$ (dashed line), and we can see that the amplitudes of oscillations decrease during the time evolution and the population in the excited state are equal to the population in the ground state, i.e. $\rho_{e_1e_2} \simeq \rho_{g_1g_2}$ when $\pi \le \lambda t \le 3\pi$. In figure 1b, when $r = 1$, that means, we use even deformed coherent state as an initial state of the two fields, we observe that there is an opposite monotonic relation between $C_{\text{AB}}(t)$ and $S_{\text{AB}}(t)$, i.e. when the entanglement between two atoms reaches maximum value, we find that the entanglement between the atom and the field reaches a minimum value, and vice versa. This manner of $C_{\text{AB}}(t)$ and $S_{\text{AB}}(t)$ exhibits periodic oscillations. Also we can see that the greatest ESD occurs at $3\pi/4 \le \lambda t \le 3\pi/2$. As the time elapses, the ESD reduces. From figure 1d, we can see that the amplitude oscillations for populations $\rho_{e_1e_2}$, $\rho_{g_1g_2}$ are from 0 to $3\pi/4$, and $\rho_{e_1e_2} \simeq \rho_{g_1g_2}$ at $3\pi/4 \le \lambda t \le 3\pi/2$. We found that there is death entanglement between two atoms when $\rho_{e_1e_2} \simeq \rho_{g_1g_2}$.

Now, we discuss the influence of Kerr-like medium parameter ($\chi/\lambda = 0.5$) on the evolution of quantum entanglement for the system under considerations. We consider that the two fields initially in deformed coherent state for $r = 0$, and classical limit $q \to 1$ that all in figure 2a. It is clear that $C_{\text{AB}}(t)$ and $S_{\text{AB}}(t)$ are still in the same opposite monotonic relation, but the most

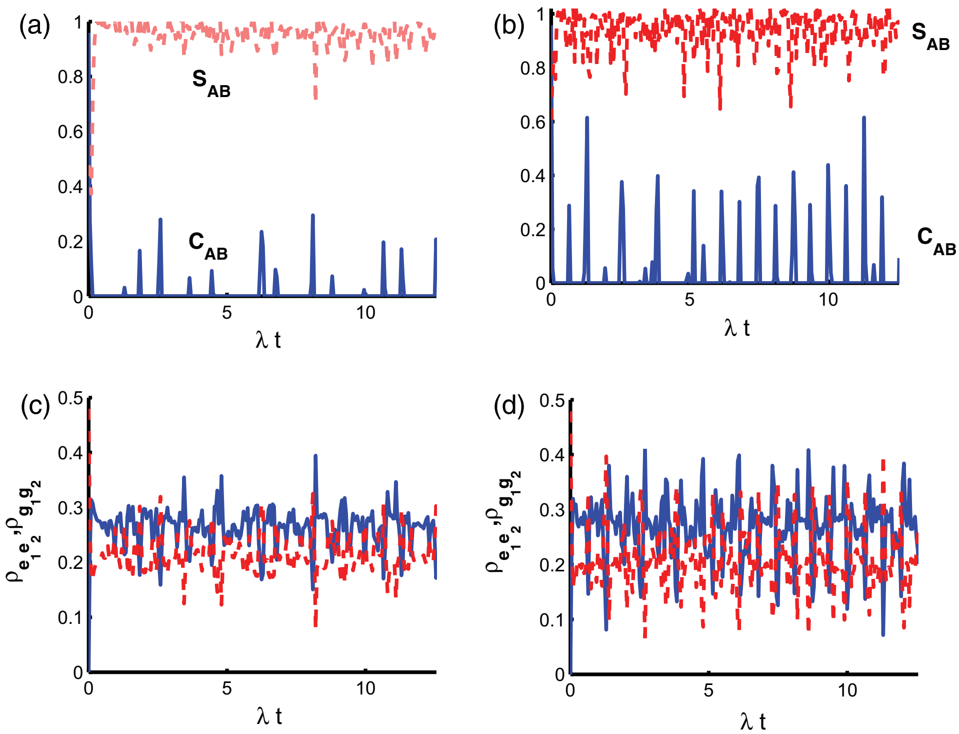

Figure 4. The same as figure 3 but for $\chi = 0.5$.

important observation here is that the ESD and ESB phenomena clearly appear in a regular and periodic manner. Also one can see that the length of period of time of death entanglement is 2π, and this is very useful in quantum systems. In figure 2c, we can see that the amplitudes of oscillations for populations $\rho_{e_1e_2}$, $\rho_{g_1g_2}$ decrease during the period from 0 to $\pi/2$ and $\rho_{e_1e_2} \simeq \rho_{g_1g_2}$ at $\pi/2 \leq \lambda t \leq 3\pi/2$, and these types of populations repeated regularly and periodically. Figures 2b and 2d are plotted under the same considerations as in figures 2a and 2c, but by considering the initial state of two fields is even deformed coherent state ($r = 1$). In figure 2b, $C_{\mathrm{AB}}(t)$ and $S_{\mathrm{AB}}(t)$ are the same as in figure 2a, but the length of the period of time of death entanglement is π. In figure 2d, $\rho_{e_1e_2}$, $\rho_{g_1g_2}$ are the same as in figure 2c, but $\rho_{e_1e_2}$ and $\rho_{g_1g_2}$ are not equal during the interaction.

In figure 3, we examine the effect of deformation parameter for $q = 2$ by taking the other parameters as in figure 1. Opposite monotonic relation is observed between $C_{\mathrm{AB}}(t)$ and $S_{\mathrm{AB}}(t)$, but taking the fields in a deformed coherent state ($r = 0$) and ($q = 2$) exhibits rapid and irregular oscillations, also there is no entanglement death. We can see that concurrence between two atoms does not exceed 0.6 at $\lambda t > 0$, and the entanglement between the fields and the atoms is not less than 0.4 at $\lambda t > 0$ (see figure 3a). To examine the effect of initial state of the two fields we set $r = 1$ in figure 3b, and one see an increase in some values of $C_{\mathrm{AB}}(t)$ and decrease of some values of $S_{\mathrm{AB}}(t)$ greater than the previous case in figure 3a, but with the same rapid and irregular oscillations; also there is no clear entanglement death.

Finally, in figure 4 we invoke again the effect of Kerr-like medium on the concurrence and von Neumann entropy under the same conditions as in figure 3. In figure 4a, when the initial state of the two fields is the deformed coherent state ($r = 0$), we have observed that the population probabilities and different entanglement measures show the same periodicity of the dynamics for each value of the q-deformation parameter with increase in the oscillation amplitudes as the q-deformation parameter gets far from the classical limit $q \to 1$.

5. Conclusion

We investigated the dynamics of the entanglement of two atoms interacting with two nonlinear deformed fields including the effect of a nonlinear medium. Using concurrence we have studied the entanglement evolution of the atom–atom and atom–field system. It is shown that in the absence of Kerr-like medium the populations are very sensitive to the q-deformation parameters of the field. For the entanglement phenomenon, the entanglement depends heavily on the q-deformation parameter. When $q \to 1$, the behaviour of entanglement changes by increasing and decreasing periodically with ordinary oscillatory behaviour, as the scaled time elapses, and there is ESD in this case. When $q \rightarrow 2$, the entanglement tends to irregular dense oscillatory behaviour as the scaled time elapses, and there is no clear entanglement death in this case. In the presence of Kerr-like medium, we have observed that the population probabilities and different entanglement measures show the same periodicity of the dynamics for each value of the q-deformation parameter with increase in the oscillation amplitudes as the q-deformation parameter gets far from 1. In this case, the entanglement dynamics is subjected to the same periodicity like the population one with sudden death and sudden birth phenomena. Finally, our results show that some new important and interesting features such as ESD and ESB can be obtained when the two atoms initially start from the entangled state. We have found that the system becomes separable, i.e. at the periodic time, our results also show that the interaction between two atoms and two fields in the presence of Kerr-like medium provides a much richer structure than in the absence of the Kerr-like medium parameter.

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