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Complex dynamics of a new 3D Lorenz-type autonomous chaotic system

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Abstract. This paper investigates a new three-dimensional continuous quadratic autonomous chaotic system which is not topologically equivalent to the Lorenz system. The dynamical behaviours of this system are further investigated in detail, including the ultimate boundedness, the invariant sets and the global attraction domain according to Lyapunov stability theory of dynamical systems. The innovation of the paper lies in the fact that this paper not only proves this chaotic system is globally bounded for the parameters of this system but also gives a family of mathematical expressions of global exponential attractive sets with respect to the parameters of this system. To validate the ultimate bound estimation, numerical simulations are also investigated. Numerical simulations verify the effectiveness and feasibility of the theoretical scheme.

Keywords. New autonomous chaotic system; chaotic attractors; Lyapunov stability theory; ultimate boundedness; numerical localization.

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1. Introduction

In 1963, Lorenz found the Lorenz chaotic attractor when he studied the atmospheric convection [1]. As a paradigm of chaotic systems, the Lorenz system has received extensive attention. This classical Lorenz system has been extensively studied in the fields of chaos theory, dynamical systems, chaos control and chaos synchronization. Chaotic systems are considered to be useful in many disciplines such as population dynamics, electrical engineering, information processing, cryptography, secure communication, information storage, etc. [2–22]. So, it is important to analyse the dynamical behaviours of the new chaotic systems.

In recent years, dynamical behaviours of chaotic systems, such as the stability of the equilibrium point, periodic solutions, circuit implementation, chaos control, chaos synchronization, chaos attractors and homoclinic orbits have been extensively studied [2,16,23–32]. However, little seems to be known about the ultimate boundedness and global attraction domains of chaotic systems. In fact, ultimate boundedness is an important concept in dynamical systems and it is also very important for the researchers to study the qualitative behaviours of a chaotic system. Ultimate boundedness of a chaotic system can be applied to study the stability of the equilibrium, the existence of periodic solutions, chaos control, the Lyapunov dimension of chaotic attractors, the Hausdorff dimension of the chaotic attractor and chaos synchronization. Furthermore, if one can show that a chaotic or a hyperchaotic system has a global attractive set, then we know that the system cannot possess equilibrium points, periodic solutions or quasiperiodic solutions, or other chaotic attractors outside the global attractive set. This is very important for engineering applications [10–12], because it is very difficult to predict the existence of hidden attractors and they can lead to crashes. It is very difficult to obtain the ultimate boundedness of a chaotic system as there is no regular way to find the proper Lyapunov functions for the given chaotic system. Leonov *et al* studied the boundedness of the famous Lorenz system with a large number of literature and obtained many fruitful results [6,7]. Motivated by Leonov's work, Yu and Zhang *et al* gave new results of the ultimate bound on the trajectories of the family of the Lorenz systems according to the stability theory of dynamical systems [33,34]. The ultimate bounds of other Lorenz-type chaotic systems were also studied [35–41].

Motivated by this discussion, we shall study the dynamics of a new Lorenz-like chaotic system which is not topologically equivalent to the Lorenz system. The rest of this paper is organized as follows. The chaotic model is given in §2. In §3, we use Lyapunov stability theory to study the ultimate boundedness and global attractive sets of this chaotic system. Numerical localization of the attractor is also presented in §3. Section 4 gives conclusions.

2. Dynamical system model

A new chaotic system derived from the Lorenz system is given by the following autonomous differential equations [42]:

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = a(y-x), \\ \mathrm{d}y/\mathrm{d}t = xz - y, \\ \mathrm{d}z/\mathrm{d}t = b - xy - cz, \end{cases} \tag{1}$$

where $X = (x, y, z)^T \in R^3$ is the state variable of system (1), $a > 0, b > 0, c > 0$ are the parameters of system (1). The authors have proved that system (1) is not topologically equivalent to the Lorenz system, the Chen system, the Lü system, and the unified system in [42]. If $a = 5, b = 16, c = 1$, there exists a chaotic attractor for the chaotic system (1) [42]. Figure 1 shows the chaotic attractor of system (1) plotted in the xOyz space. Figure 2 shows the projection of the chaotic attractor of system (1) onto the $x-y$ plane. Figure 3 shows the projection of the chaotic attractor of system (1) onto the $x-z$ plane. Figure 4 shows the projection of the chaotic attractor of system (1) onto the $y-z$ plane.

Remark 1. Currently, the question of the equivalence of various Lorenz-like systems and the possibility of universal consideration of their behaviour, are being actively discussed in view of the possibility of reduction of such systems to the same form with the help of various reversible transformations. In general, it is interesting and difficult to discuss the equivalence of various Lorenz-like systems (see the excellent paper [16]

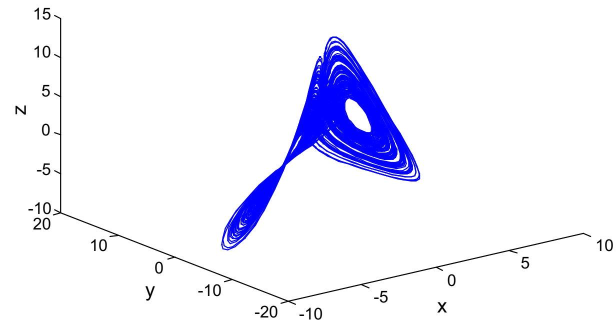

Figure 1. Chaotic attractor of system (1) in the xOyz space.

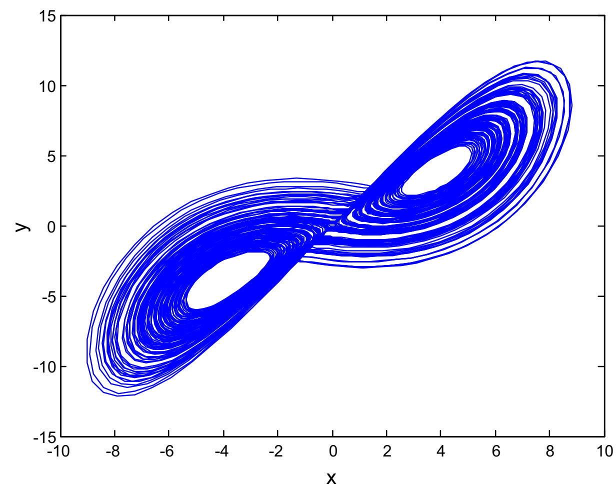

Figure 2. Projection of the chaotic attractor onto the $x-y$ plane.

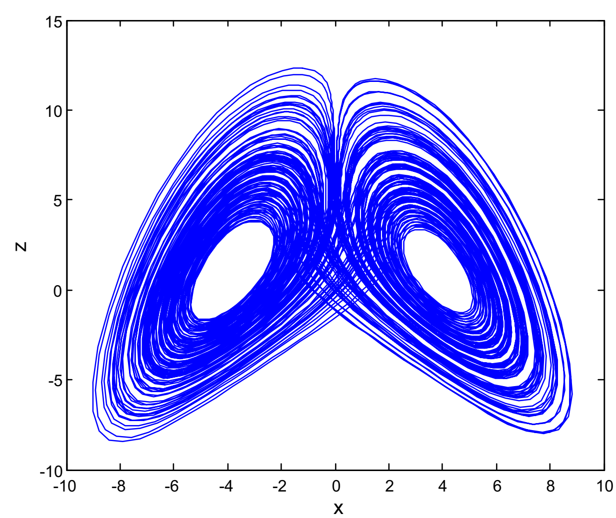

Figure 3. Projection of the chaotic attractor onto the $x-z$ plane.

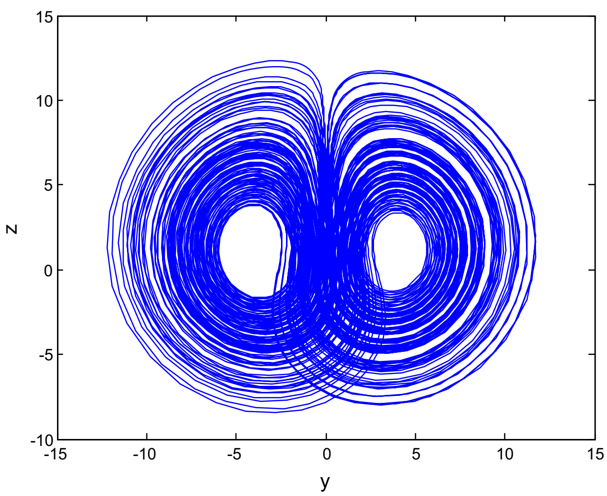

Figure 4. Projection of the chaotic attractor onto the y–z plane.

for a detailed discussion of the equivalence of various Lorenz-like chaotic systems).

2.1 *Dissipation*

The state space of system (1) is three-dimensional. The vector field on the right-hand sides of system (1) is defined by

$$f(X)=\begin{bmatrix} f_1(X)\\ f_2(X)\\ f_3(X)\end{bmatrix}=\begin{bmatrix} ay-ax\\ xz-y\\ b-xy-cz\end{bmatrix}.$$

Since the divergence of the vector field $f(X)$ is easily calculated as

$$\nabla V=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}=-a-c-1,$$

as long as $a+c+1>0$, system (1) is dissipative, and it converges to a set of measure zero with exponential rate $\mathrm{d}V/\mathrm{d}t=\mathrm{e}^{-(a+c+1)}$. It means $V(t)=V_0\mathrm{e}^{-(a+c+1)t}$ for the initial cubage V_0.

2.2 *Invariance*

The positive z-axis is invariant under the flow, that is to say, z-axis is positively invariant under the flow generated by system (1). However, this is not the case on the positive x-axis and y-axis.

The equilibria and stability, various chaotic attractors, Poincaré maps, bifurcations, and Lyapunov-exponent spectrum of this system (1) are studied in [42]. In the following, we shall discuss the ultimate boundedness and the domain of attraction of system (1) according to Lyapunov stability theory.

3. Dynamical behaviours of the new chaotic system

3.1 *Ultimate boundedness*

In this section, we shall discuss the ultimate boundedness of system (1) for $\forall a>0, b>0, c>0$. Before giving the main results of this paper, let us first introduce the following lemma.

Lemma 1. *Let us define a three-dimensional ellipsoid*

$$\Gamma_1=\left\{(x_1,y_1,z_1)\,|\,\frac{x_1^2}{n^2}+\frac{y_1^2}{k^2}+\frac{(z_1-l)^2}{l^2}=1, nkl\neq 0\right\} \tag{2}$$

and the function of three variables $H(x_1,y_1,z_1)=x_1^2+y_1^2+z_1^2$, $(x_1,y_1,z_1)\in\Gamma_1$. *Then we obtain*

$$\max_{(x_1,y_1,z_1)\in\Gamma_1} H=\begin{cases}\dfrac{n^4}{n^2-l^2}, & n\geq k, n\geq\sqrt{2}l,\\[2mm] \dfrac{k^4}{k^2-l^2}, & k>n, k\geq\sqrt{2}l,\\[2mm] 4l^2, & k<\sqrt{2}l, n<\sqrt{2}l.\end{cases}$$

Proof. It can be easily proved by the Lagrange multiplier method. □

Theorem 1. *For any* $a>0, b>0, c>0$, *the following set*

$$\Omega_{\lambda,m}=\left\{(x,y,z)\,|\,\lambda x^2+my^2+m\left(z+\frac{a\lambda}{m}\right)^2 \leq R^2, \forall\lambda>0, \forall m>0\right\} \tag{3}$$

is the ultimate bound set and positively invariant set of system (1), *where*

$$R^2=\begin{cases}\dfrac{(ac\lambda+bm)^2}{4am\,(c-a)}, & a\leq 1, c\geq 2a,\\[2mm] \dfrac{(ac\lambda+bm)^2}{4m\,(c-1)}, & a>1, c\geq 2,\\[2mm] \dfrac{(ac\lambda+bm)^2}{c^2m}, & c<2a, c<2.\end{cases}$$

Proof. Define the following Lyapunov-like function

$$V_{m,\lambda}(X)=V_{m,\lambda}(x,y,z)=\lambda x^2+my^2+m\left(z+\frac{a\lambda}{m}\right)^2, \quad \forall\lambda>0, \forall m>0. \tag{4}$$

Computing the derivative of $V_{m,\lambda}(x,y,z)$ along the trajectory of system (1), we have

$$\left.\frac{\mathrm{d}V_{m,\lambda}(x,y,z)}{\mathrm{d}t}\right|_{(1)}$$
$$= 2\lambda x\frac{\mathrm{d}x}{\mathrm{d}t} + 2my\frac{\mathrm{d}y}{\mathrm{d}t} + 2m\left(z+\frac{a\lambda}{m}\right)\frac{\mathrm{d}z}{\mathrm{d}t}$$
$$= 2\lambda x(ay-ax) + 2my\,(xz-y) + 2m\left(z+\frac{a\lambda}{m}\right)(b-xy-cz)$$
$$= -2a\lambda x^2 - 2my^2 - 2cmz^2 + 2(bm-ac\lambda)z + 2ab\lambda.$$

Obviously, the surface Γ_0 that is defined by

$$\Gamma_0 = \left\{(x,y,z)|\, a\lambda x^2+my^2+cm\left(z-\frac{bm-ac\lambda}{2cm}\right)^2 = \frac{(bm+ac\lambda)^2}{4cm}\right\} \quad (5)$$

is an ellipsoid in R^3 for $\forall a>0, b>0, c>0$, Outside Γ_0, $\frac{\mathrm{d}V_{m,\lambda}(X)}{\mathrm{d}t}<0$, while inside Γ_0, $\frac{\mathrm{d}V_{m,\lambda}(X)}{\mathrm{d}t}>0$. Thus, the maximum value of $V_{m,\lambda}(x,y,z)$ can only be reached at Γ_0. Since $V_{m,\lambda}(X)$ is a continuous function and Γ_0 is a bounded closed set, the function (4) can reach its maximum value $\max V_{m,\lambda}(X) = R^2, (X\in\Gamma_0)$ on the surface Γ_0 defined in (5).

Obviously, $\{(x,y,z)\,|V_{m,\lambda}(X)\le \max V_{m,\lambda}(X), X\in\Gamma_0\}$ contains the solutions of the chaotic system (1). By solving the following conditional extremum problem, one can get the maximum value of function (4) as follows:

$$\begin{cases} \max V_{m,\lambda}(X) = \max\{\lambda x^2+my^2+m(z+a\lambda/m)^2\}, \\ s.t. a\lambda x^2+my^2+cm\left(z-\frac{bm-ac\lambda}{2cm}\right)^2 \\ \quad = \frac{(bm+ac\lambda)^2}{4cm}, \end{cases} \quad (6)$$

that is,

$$\begin{cases} \max V_{m,\lambda}(X) = \max\{\lambda x^2+my^2+m(z+a\lambda/m)^2\}, \\ s.t.\ \frac{\lambda x^2}{(bm+ac\lambda)^2/4acm} + \frac{my^2}{(bm+ac\lambda)^2/4cm} \\ \quad + \frac{m\,(z-(bm-ac\lambda)/2cm)^2}{(bm+ac\lambda)^2/4c^2m} = 1. \end{cases} \quad (7)$$

Let $\sqrt{\lambda}x = x_1$, $\sqrt{m}y = y_1$, $\sqrt{m}z = z_1$,then the above conditional extremum problem (7) takes the form:

$$\begin{cases} \max V_{m,\lambda}(X) = \max\left\{x_1^2+y_1^2+\left(z_1+a\lambda/\sqrt{m}\right)^2\right\}, \\ s.t.\ \frac{x_1^2}{(bm+ac\lambda)^2/4acm} + \frac{y_1^2}{(bm+ac\lambda)^2/4cm} \\ \quad + \frac{\left(z_1-(bm-ac\lambda)/2c\sqrt{m}\right)^2}{(bm+ac\lambda)^2/4c^2m} = 1. \end{cases} \quad (8)$$

Let

$$x_1 = x_1,\ y_1 = y_1,\ z_1 + (a\lambda/\sqrt{m}) = w_1,$$

then the conditional extremum problem (8) takes the form:

$$\begin{cases} \max V_{m,\lambda}(X) = \max\left\{x_1^2+y_1^2+w_1^2\right\}, \\ s.t.\ \frac{x_1^2}{(bm+ac\lambda)^2/4acm} + \frac{y_1^2}{(bm+ac\lambda)^2/4cm} \\ \quad + \frac{\left(w_1-(bm+ac\lambda)/2c\sqrt{m}\right)^2}{(bm+ac\lambda)^2/4c^2m} = 1. \end{cases} \quad (9)$$

According to Lemma 1, we can easily get the conditional extremum problem (9) as

$$R^2 = \begin{cases} \frac{(ac\lambda+bm)^2}{4am\,(c-a)}, & a\le 1, c\ge 2a, \\ \frac{(ac\lambda+bm)^2}{4m\,(c-1)}, & a>1, c\ge 2, \\ \frac{(ac\lambda+bm)^2}{c^2m}, & c<2a, c<2. \end{cases}$$

Finally, it is easy to show that (3) is the ultimate bound and positively invariant set of chaotic system (1). This completes the proof. □

Remark 2.

(i) Let us take $\lambda = 1$, then we obtain

$$\Omega_{1,m} = \left\{(x,y,z)|\,x^2+my^2 + m\left(z+\frac{a}{m}\right)^2 \le l^2, \forall m>0\right\}$$

is the ultimate bound and positively invariant set of chaotic system (1), where

$$l^2 = \begin{cases} \frac{(ac+bm)^2}{4am\,(c-a)}, & a\le 1, c\ge 2a, \\ \frac{(ac+bm)^2}{4m\,(c-1)}, & a>1, c\ge 2, \\ \frac{(ac+bm)^2}{c^2m}, & c<2a, c<2. \end{cases}$$

(ii) Let us take $m = 1$, then we obtain

$$\Omega_{\lambda,1} = \left\{(x,y,z)\,|\lambda x^2+y^2+(z+a\lambda)^2 \le r^2, \forall\lambda>0\right\}$$

is the ultimate bound set and positively invariant set of chaotic system (1), where

$$r^2 = \begin{cases} \dfrac{(ac\lambda+b)^2}{4a\,(c-a)}, & a \le 1, c \ge 2a, \\ \dfrac{(ac\lambda+b)^2}{4\,(c-1)}, & a > 1, c \ge 2, \\ \dfrac{(ac\lambda+b)^2}{c^2}, & c < 2a, c < 2. \end{cases}$$

(iii) Let us take $m = 1$, $\lambda = 1$, then we obtain

$$\Omega_{1,1} = \left\{ (x, y, z) | \, x^2 + y^2 + (z+a)^2 \le L^2 \right\}$$

is the ultimate bound and positively invariant set of chaotic system (1), where

$$L^2 = \begin{cases} \dfrac{(ac+b)^2}{4a\,(c-a)}, & a \le 1, c \ge 2a, \\ \dfrac{(ac+b)^2}{4\,(c-1)}, & a > 1, c \ge 2, \\ \dfrac{(ac+b)^2}{c^2}, & c < 2a, c < 2. \end{cases}$$

When $a = 5$, $b = 16$, $c = 1$ [42], we obtain that

$$\Omega_{1,1} = \left\{ (x, y, z) | \, x^2 + y^2 + (z+5)^2 \le 21^2 \right\}$$

is the ultimate bound and positively invariant set of chaotic system (1). In figure 5, we give the localization of chaotic attractor of system (1) in the xOyz space defined by $\Omega_{1,1}$. Figure 6 shows the projection of $\Omega_{1,1}$ onto the $x-y$ plane. Figure 7 shows the projection of $\Omega_{1,1}$ onto the $x-z$ plane. Figure 8 shows the projection of $\Omega_{1,1}$ onto the $y-z$ plane.

(iv) An oscillation in a dynamical system can be easily localized numerically if initial conditions from its open neighbourhood lead to long-time behaviour that approaches the oscillation. Such oscillation (or a set of oscillations) is called an attractor, and

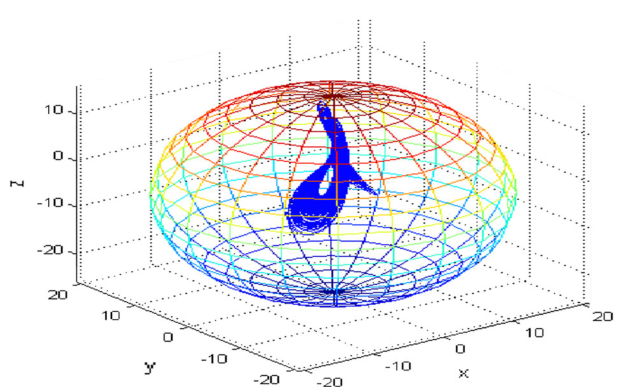

Figure 5. Localization of the chaotic attractor of system (1) defined by $\Omega_{1,1}$.

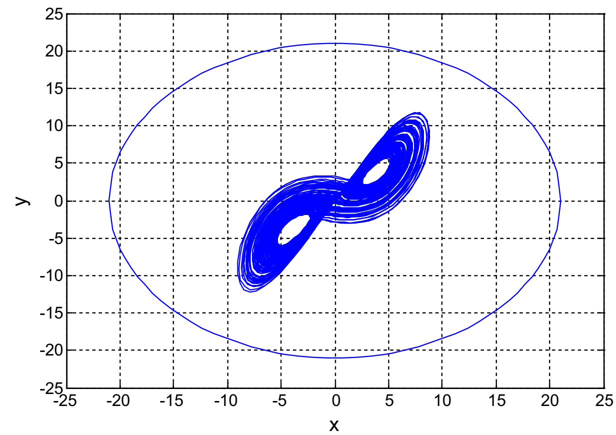

Figure 6. Projection of $\Omega_{1,1}$ onto the $x-y$ plane.

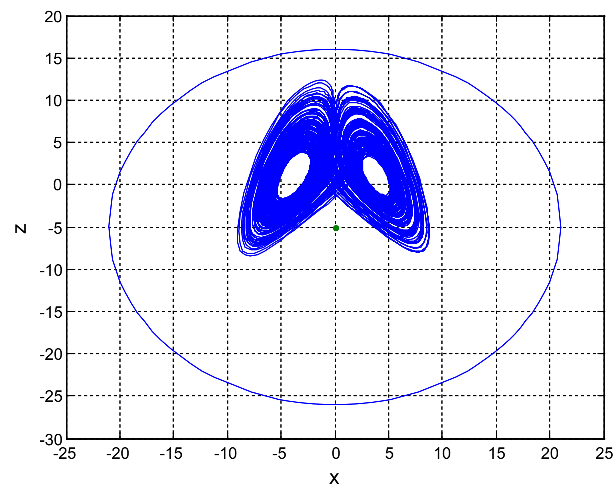

Figure 7. Projection of $\Omega_{1,1}$ onto the $x-z$ plane.

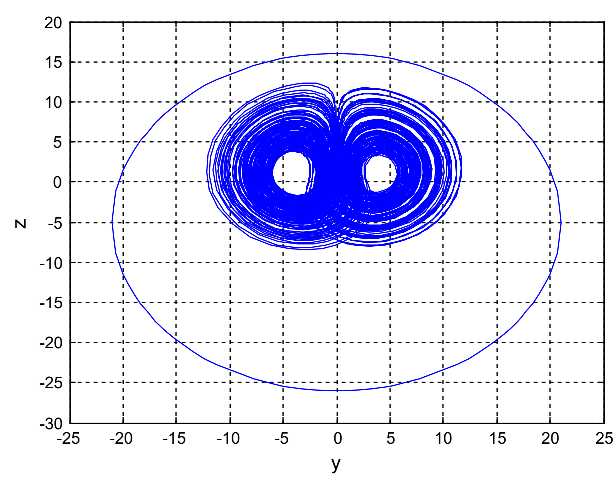

Figure 8. Projection of $\Omega_{1,1}$ onto the $y-z$ plane.

its attracting set is called the basin of attraction. Thus, from a computational point of view in applied problems of nonlinear analysis of dynamical models, it is essential to regard attractors depending on the simplicity of finding its basin of attraction in the phase space (see the excellent papers [43,44] for a detailed discussion of the chaotic attractor).

3.1.1 *Global attraction domain.* Though Theorem 1 claims that chaotic system (1) is ultimate boundedness, it does not give the trajectory rate of system (1) going from the exterior of the global attractive sets into the interior of the global attractive sets. The trajectory rate of system (1) going from the exterior of the global attractive sets into the interior of the global attractive sets is described by Theorem 2. In this section, we shall study the global attractive sets of chaotic system (1) for $\forall a > 0, b > 0, c > 0$.

Theorem 2. *For any* $\lambda > 0, m > 0, a > 0, b > 0, c > 0$, *with*

$$V_{m,\lambda}(X) = V_{m,\lambda}(x, y, z) = \lambda x^2 + my^2 + m\left(z + \frac{a\lambda}{m}\right)^2, \quad \forall\lambda > 0, \forall m > 0, \theta = \min\{a, c, 1\},$$

$$L_{m,\lambda} = \frac{(bm + ac\lambda)^2}{\theta cm},$$

$$X(t) = (x(t), y(t), z(t)),$$

$$X(t_0) = (x(t_0), y(t_0), z(t_0)).$$

When $V_{m,\lambda}(X(t)) > L_{m,\lambda}, V_{m,\lambda}(X(t_0)) > L_{m,\lambda}$, *we get the inequality of this system* (1), *given by*

$$V_{m,\lambda}(X(t)) - L_{m,\lambda} \le \left[V_{m,\lambda}(X(t_0)) - L_{m,\lambda}\right] \mathrm{e}^{-\theta(t-t_0)}.$$

Especially, the set

$$\Psi_{\lambda,m} = \left\{X | V_{m,\lambda}(X(t)) \le L_{m,\lambda}\right\} = \left\{(x, y, z) |\, \lambda x^2 + my^2 + m\left(z + \frac{a\lambda}{m}\right)^2 \le L_{m,\lambda}, \forall\lambda > 0, \forall m > 0\right\},$$

is the global exponential attractive set of system (1).

Proof. Let us define

$$f(z) = -cmz^2 + 2bmz + 2ab\lambda + \frac{ca^2\lambda^2}{m}, \quad \forall c > 0, \forall m > 0,$$

then we get

$$\max_{z \in R} f(z) = \frac{(ac\lambda + bm)^2}{cm}.$$

Define the following Lyapunov-like function:

$$V_{m,\lambda}(X) = V_{m,\lambda}(x, y, z) = \lambda x^2 + my^2 + m\left(z + \frac{a\lambda}{m}\right)^2, \quad \forall\lambda > 0, \forall m > 0.$$

Computing the derivative of $V_{m,\lambda}(X(t))$ along the trajectory of system (1), when

$$V_{m,\lambda}(X(t)) > L_{m,\lambda}, V_{m,\lambda}(X(t_0)) > L_{m,\lambda},$$

we have

$$\begin{aligned}
\left.\frac{\mathrm{d}V_{m,\lambda}(X)}{\mathrm{d}t}\right|_{(1)}
&= 2\lambda x\frac{\mathrm{d}x}{\mathrm{d}t} + 2my\frac{\mathrm{d}y}{\mathrm{d}t} + 2m\left(z + \frac{a\lambda}{m}\right)\frac{\mathrm{d}z}{\mathrm{d}t} \\
&= 2\lambda x(ay - ax) + 2my(xz - y) + 2m\left(z + \frac{a\lambda}{m}\right)(b - xy - cz) \\
&= -2a\lambda x^2 - 2my^2 - 2cmz^2 + 2(bm - ac\lambda)z + 2ab\lambda \\
&\le -a\lambda x^2 - my^2 - 2cmz^2 + 2(bm - ac\lambda)z + 2ab\lambda \\
&= -a\lambda x^2 - my^2 - cmz^2 - 2ac\lambda z - cmz^2 + 2bmz + 2ab\lambda \\
&= -a\lambda x^2 - my^2 - cm\left(z + \frac{a\lambda}{m}\right)^2 - cmz^2 + 2bmz + 2ab\lambda + \frac{ca^2\lambda^2}{m} \\
&\le -\theta V_{m,\lambda}(X) + f(z) \\
&\le -\theta V_{m,\lambda}(X) + \max_{z \in R} f(z) \\
&= -\theta V_{m,\lambda}(X) + \frac{(ac\lambda + bm)^2}{cm} \\
&\le -\theta\left(V_{m,\lambda}(X) - L_{m,\lambda}\right) < 0.
\end{aligned}$$

That is equivalent to say,

$$\left.\frac{\mathrm{d}V_{m,\lambda}(X(t))}{\mathrm{d}t}\right|_{(1)} \le -\theta\left(V_{m,\lambda}(X(t)) - L_{m,\lambda}\right) < 0. \tag{10}$$

Integrating both sides of (10) yields

$$\begin{aligned}
V_{m,\lambda}(X(t)) &\le V_{m,\lambda}(X(t_0))\,\mathrm{e}^{-\theta(t-t_0)} + \int_{t_0}^{t} \theta L_{m,\lambda}\mathrm{e}^{-\theta(t-\tau)}\mathrm{d}\tau \\
&= V_{m,\lambda}(X(t_0))\,\mathrm{e}^{-\theta(t-t_0)} + L_{m,\lambda}(1 - \mathrm{e}^{-\theta(t-t_0)}).
\end{aligned}$$

If $V_{m,\lambda}(X(t)) > L_{m,\lambda}, V_{m,\lambda}(X(t_0)) > L_{m,\lambda}$, we have the exponential inequality for system (1),

$$V_{m,\lambda}(X(t)) - L_{m,\lambda} \le \left[V_{m,\lambda}(X(t_0)) - L_{m,\lambda}\right] \mathrm{e}^{-\theta(t-t_0)}.$$

By the definition, taking upper limit on both sides of the above inequality as $t \to +\infty$ results in

$$\overline{\lim_{t\to+\infty}} V_{m,\lambda}(X(t)) \le L_{m,\lambda}.$$

That is to say, the following set

$$\Psi_{\lambda,m} = \left\{X \,|\, V_{m,\lambda}(X(t)) \le L_{m,\lambda}\right\} = \left\{(x, y, z) \,|\, \lambda x^2 + my^2 + m\left(z + \frac{a\lambda}{m}\right)^2 \le L_{m,\lambda}, \forall\lambda > 0, \forall m > 0\right\},$$

is the global exponential attractive set of system (1). This completes the proof. □

Remark 3. Lyapunov stability theory can also be extended to study the global exponential attractive set and the stability of the neural network and the time-varying dynamic networks [45–48].

4. Conclusions

In this paper, some dynamical behaviours of a new Lorenz-like chaotic system are analysed both theoretically and numerically, including the ultimate boundedness, the invariant sets and the global attraction domain. Theoretical results show that this system has complex and interesting dynamics. The results in this paper may give the readers a new insight into chaos and the formation mechanism of the chaos. In addition, the results obtained in this paper are also very useful in engineering applications such as chaos control, chaos synchronization and estimating the Lyapunov dimension of the chaotic attractors. The homoclinic orbits and heteroclinic orbits of this system will be taken into consideration in the future work. Therefore, further research on this system is still important and insightful.

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