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Zakharov equations for viscous flow and their use in the blood clot formation

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Abstract. For theoretical study, blood can be regarded as a viscous electrically conducting fluid of negative ions and protons. Zakharov equations including viscosity are relevant for describing the behaviour of blood plasma. The dispersion formula is derived from the perturbation method and is solved numerically. It turns out that the imaginary part of one root of the perturbation frequency is greater than zero, and modulation instability occurs. This would lead to the formation of blood clot. The viscous force can suppress the occurrence of instability and prevent thrombosis. One can find that the chaotic state of blood signals human health.

Keywords. Blood plasma; Zakharov equations; viscosity; modulation instability.

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1. Introduction

Blood, which mainly consists of blood plasma and blood corpuscle, is a typical biological fluid. Ninety percent of the blood plasma is water, and the other ten percent is plasma protein. Using salting-out method, the plasma protein can be divided into three categories; albumin, globulin and fibrinogen. Among them, the albumin has the lowest isoelectric point, and the pH $= 4.7$. In the body fluid of pH $= 7.4$, each albumin can carry more than 200 negative charges. Therefore, the blood plasma can be considered as a viscous, electrically conducting fluid containing protons and negative ions, and the rheological properties of blood have been widely studied in the past few years [1–3].

So far, many studied blood coagulation using theoretical models. For the abnormal blood vessel model and the metal-probe-in-blood model, Abraham-Shrauner attributed the possibility of blood clot formation to the force on the point charge [4]. From 2003 to 2008, Anand *et al* developed a model comprising reaction–diffusion equations to simulate the formation, growth, and lysis of clots in both flowing blood and quiescent plasma [5–7]. Recently, Govindarajan *et al* used a computational model to investigate thrombus formation as a process developing in a flow chamber [8]. It is believed that the blood flow is not steady everywhere in cardiovascular system [9]. Therefore, we attempt to explore the clot formation associated with modulation instability in this paper.

In 1972, Zakharov derived a set of nonlinear equations to describe the electromagnetic properties of plasmas, and the equations were called Zakharov equations [10]. Usually, Zakharov equations in three dimensions are modulationally unstable and the collapse of wave fields will lead to the local rarefaction of particle density [11]. Similarly, the modulation interaction can greatly change the rheological properties of the viscous blood when modulation instability occurs, resulting in the non-uniform patterns from initially uniform flow. Perhaps it is responsible for the pathological blood clot.

With the above discussion in mind, we first derive the Zakharov equations including viscosity based on two-component fluid mechanics in §2. In §3, both the longitudinal and transverse perturbations are taken into account, and the dispersion equation is obtained. Section 4 is devoted to the numerical solution of the unstable growth rate. The implications of these results will be discussed in the last section.

2. Basic equations

For a conductive fluid, the blood plasma consists of protons and anions (albumin), and we take the following two-fluid equations [12,13]:

$$\frac{\partial n_p}{\partial t} + \nabla \cdot (n_p \mathbf{v}_p) = 0, \tag{1}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = 0, \tag{2}$$

$$\frac{\partial \mathbf{v}_p}{\partial t} + (\mathbf{v}_p \cdot \nabla)\mathbf{v}_p = \frac{e}{m_p}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_p \times \mathbf{B}\right) - \frac{\nabla \cdot \vec{\mathbf{P}}_p}{m_p n_p}, \tag{3}$$

$$\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla)\,\mathbf{v}_i = \frac{-q}{m_i}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_i \times \mathbf{B}\right) - \frac{\nabla \cdot \vec{\mathbf{P}}_i}{m_i n_i}, \tag{4}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{5}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}(e n_p \mathbf{v}_p - q n_i \mathbf{v}_i), \tag{6}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{7}$$

where n_ς is the number density, $\mathbf{v}_\varsigma$ is the velocity, m_ς is the mass, $\mathbf{E}$ is the electric field, and $\mathbf{B}$ is the magnetic field. For the proton component, $\varsigma = p$ and for albumin $\varsigma = i$. The symbol $-q$ is used to denote the charge of albumin, and $-q = -\chi e < 0$, $e > 0$. The pressure tensor $\vec{\mathbf{P}}_\varsigma$ can be written as

$$\vec{\mathbf{P}}_\varsigma = p_\varsigma \vec{\mathbf{I}} - \vec{\mathbf{t}}_\varsigma.$$

With the help of Newton viscosity μ_ς, the viscous force tensor is

$$\vec{\mathbf{t}}_\varsigma = \mu_\varsigma\left(\nabla^2 \mathbf{v}_\varsigma + \frac{1}{3}\nabla\left(\nabla \cdot \mathbf{v}_\varsigma\right)\right). \tag{8}$$

Because of the large difference for proton and ion oscillation frequencies, we can take two-time-scale approximation: fast-time-scale $t_f \sim \omega_{pp}^{-1}$ and slow-time-scale $t_s \sim \omega_{pi}^{-1}$. Hence, we assume that all quantities can be divided into fast-time-scale and slow-time-scale components:

$$A = (n_p, n_i; \mathbf{v}_p, \mathbf{v}_i; p_p, p_i; \mathbf{E}, \mathbf{B}) = A_f + A_s$$

and

$$\langle A_f \rangle = 0,$$

where $\langle A_f \rangle$ denotes the ensemble average value of the fast-time-scale components over the slow-time-scale. On a slow-time-scale, a quasineutral condition is valid: $\langle n_s^p e - q n_s^i \rangle = 0$, and one gets for blood plasma,

$$n_s^p = \frac{q}{e} n_s^i = \chi n_s^i.$$

Substituting the corresponding fast and slow components into eq. (1), one obtains the average equation

$$\frac{\partial}{\partial t} n_s^p + \nabla \cdot \left(n_s^p \mathbf{v}_s^p + \langle n_f^p \mathbf{v}_f^p \rangle\right) = 0, \tag{9}$$

and the fast-component equation

$$\frac{\partial}{\partial t} n_f^p + \nabla \cdot \left(n_s^p \mathbf{v}_f^p + n_f^p \mathbf{v}_f^p + n_f^p \mathbf{v}_s^p - \langle n_f^p \mathbf{v}_f^p \rangle\right) = 0. \tag{10}$$

On the other hand, we obtain the lowest order equation of protons for the fast-time-scale from eq. (3),

$$\frac{\partial \mathbf{v}_f^p}{\partial t} \approx \frac{e}{m_p} \mathbf{E}_f. \tag{11}$$

On the basis of eq. (11), we can estimate the terms in eq. (10) as follows:

$$\left|\frac{\nabla \cdot \left(n_f^p \mathbf{v}_f^p\right)}{\partial n_f^p / \partial t}\right| \sim \frac{k n_f^p v_f^p}{\omega_f n_f^p} \sim \frac{k}{\omega}\frac{e E_f}{m_p \omega} \sim \frac{k}{k_d}\left(\frac{\omega_{pp}}{\omega_f}\right)^2 \bar{W}_f^{1/2}, \tag{12}$$

$$\left|\frac{\nabla \cdot \left(n_f^p \mathbf{v}_s^p\right)}{\partial n_f^p / \partial t}\right| \sim \frac{k}{k_d}\frac{\omega_{pp}}{\omega_f}\frac{v_s^p}{v_{Tp}}, \tag{13}$$

where $\omega_{pp} = \left(4\pi n_s^p e^2/m_p\right)^{1/2}$ is the proton oscillation frequency, k_d $(= \omega_{pp}/v_{Tp})$ is the Debye wave number, and $k_d > k$ is due to Landau damping [11]. For slow-time-scale fluid motion, the following relation is satisfied:

$$\left|\mathbf{v}_s^p\right| < v_{Tp}. \tag{14}$$

As long as the intensity of the wave is not large enough,

$$\bar{W}_f = \langle E_f^2 \rangle / 4\pi n_s^p T_p < 1 \tag{15}$$

eq. (10) can be simplified to

$$\frac{\partial}{\partial t} n_f^p + \nabla \cdot \left(n_s^p \mathbf{v}_f^p\right) = 0. \tag{16}$$

According to eqs (11) and (16), the following estimations are valid:

$$v_f^p \sim \frac{|e| E_f}{m_p \omega_f} \sim \frac{\omega_{pp}}{\omega_f} \bar{W}_f^{1/2} v_{Tp}, \tag{17}$$

$$\frac{n_f^p}{n_s^p} \sim \frac{k}{\omega_f} v_f^p \sim \frac{k}{k_d}\left(\frac{\omega_{pp}}{\omega_f}\right)^2 \bar{W}_f^{1/2} \ll 1. \tag{18}$$

For fluid, the proton thermal velocity v_{Tp} is much less than the phase velocity v_φ $(=\omega_f/k)$. Using eqs (17) and (18), we have

$$\left|\frac{n_f^p \mathbf{v}_f^p}{n_s^p \mathbf{v}_s^p}\right| \sim \frac{k}{k_d}\left(\frac{\omega_{pp}}{\omega_f}\right)^2 \bar{W}_f^{1/2}\left(\frac{\omega_{pp}}{\omega_f}\bar{W}_f^{1/2}\right)\left(\frac{v_{Tp}}{|\mathbf{v}_s^p|}\right) \sim \frac{v_{Tp}}{v_\varphi}\bar{W}_f\left(\frac{\omega_{pp}}{\omega_f}\right)^2 \ll 1. \tag{19}$$

Equation (9) becomes

$$\frac{\partial}{\partial t}n_s^p + \nabla\cdot\left(n_s^p \mathbf{v}_s^p\right) = 0. \tag{20}$$

For the terms corresponding to anions, from eq. (2) we also find

$$\frac{\partial}{\partial t}n_f^i + \nabla\cdot\left(n_s^i \mathbf{v}_f^i\right) = 0; \tag{21}$$

$$\frac{\partial}{\partial t}n_s^i + \nabla\cdot\left(n_s^i \mathbf{v}_s^i\right) = 0. \tag{22}$$

Similarly, substituting the corresponding fast and slow components into eq. (3), one can obtain

$$\begin{aligned}\frac{\partial \mathbf{v}_s^p}{\partial t} &+ \left(\mathbf{v}_s^p\cdot\nabla\right)\mathbf{v}_s^p + \left\langle\left(\mathbf{v}_f^p\cdot\nabla\right)\mathbf{v}_f^p\right\rangle \\ &= \frac{e}{m_p}\left[\mathbf{E}_s + \frac{\mathbf{v}_s^p}{c}\times\mathbf{B}_s + \left\langle\mathbf{v}_f^p\times\frac{\mathbf{B}_f}{c}\right\rangle\right] \\ &\quad - \frac{\nabla p_s^p}{m_p n_s^p} + \frac{\mu_p\left(\nabla^2\mathbf{v}_s^p + \frac{1}{3}\nabla\left(\nabla\cdot\mathbf{v}_s^p\right)\right)}{m_p n_s}\end{aligned} \tag{23}$$

and

$$\begin{aligned}\frac{\partial \mathbf{v}_f^p}{\partial t} &+ \left(\mathbf{v}_s^p\cdot\nabla\right)\mathbf{v}_f^p + \left(\mathbf{v}_f^p\cdot\nabla\right)\mathbf{v}_s^p \\ &+ \left[\left(\mathbf{v}_f^p\cdot\nabla\right)\mathbf{v}_f^p - \left\langle\left(\mathbf{v}_f^p\cdot\nabla\right)\mathbf{v}_f^p\right\rangle\right] \\ &= \frac{e}{m_p}\left[\mathbf{E}_f + \frac{\mathbf{v}_s^p}{c}\times\mathbf{B}_f \right. \\ &\quad \left. + \mathbf{v}_f^p\times\frac{\mathbf{B}_s}{c} + \frac{\mathbf{v}_f^p}{c}\times\mathbf{B}_f - \left\langle\frac{\mathbf{v}_f^p}{c}\times\mathbf{B}_f\right\rangle\right] \\ &\quad - \frac{\nabla p_f^p}{m_p n_s^p} + \frac{\mu_p\left(\nabla^2\mathbf{v}_f^p + \frac{1}{3}\nabla(\nabla\cdot\mathbf{v}_f^p)\right)}{m_p n_s^p}.\end{aligned} \tag{24}$$

Introducing

$$\alpha_f^2 = \frac{\rho(\partial\mathbf{v}_f^p/\partial t)}{\mu\nabla^2\mathbf{v}_f^p} \sim \frac{\rho\omega_f v_f^p}{\mu k^2 v_f^p} \sim \frac{\rho\omega_f}{\mu k^2} \sim \frac{\omega_f}{\omega_s}\frac{\rho\omega_s}{\mu k_s^2} \equiv \frac{\omega_f}{\omega_s}\alpha^2,$$

with $k_s \sim k$ and

$$\frac{\omega_f}{\omega_s} \sim \sqrt{\frac{\chi n_0 e^2}{n_0 q^2}}\sqrt{m_i/m_p} = \sqrt{\frac{\chi n_0 e^2}{n_0 \chi^2 e^2}}\sqrt{m_i/m_p} = \sqrt{\frac{1}{\chi}}\sqrt{m_i/m_p} = \sqrt{\frac{6.6\times 10^4}{200}} = 18 \gg 1,$$

where n_0 is the equilibrium number density of albumin and α is the Womersley number. In the circulatory system [14], the values of α are listed below: the aorta with $\alpha = 10$, the femoral artery with $\alpha = 3$, the saphenous artery with $\alpha < 1$. Therefore, we can neglect the viscous term in the fast-component eq. (24) except the microcirculation with $\alpha < 0.05$. By comparing the other terms in eq. (24) with $\partial\mathbf{v}_f^p/\partial t$, this yields

$$\frac{\partial \mathbf{v}_f^p}{\partial t} \approx \frac{e}{m_p}\mathbf{E}_f + \frac{e}{m_p c}\mathbf{v}_f^p\times\mathbf{B}_s - \frac{\nabla p_f^p}{m_p n_s^p}. \tag{25}$$

Based on eqs (5) and (25), one has

$$\mathbf{B}_f \approx -\frac{m_p c}{e}\left(\nabla\times\frac{\partial\mathbf{\Phi}_p}{\partial t}\right) + \nabla\times(\mathbf{\Phi}_p\times\mathbf{B}_s), \tag{26}$$

where

$$\mathbf{v}_f^p \equiv \frac{\partial}{\partial t}\mathbf{\Phi}_p.$$

From eq. (26), eq. (23) can be transformed into

$$\begin{aligned}\frac{\partial \mathbf{v}_s^p}{\partial t} &+ (\mathbf{v}_s^p\cdot\nabla)\mathbf{v}_s^p = \frac{e}{m_p}\left[\mathbf{E}_s + \frac{\mathbf{v}_s^p}{c}\times\mathbf{B}_s\right] \\ &- \frac{\nabla p_s^p}{m_p n_s^p} + \frac{\mu_p\left(\nabla^2\mathbf{v}_s^p + \frac{1}{3}\nabla\left(\nabla\cdot\mathbf{v}_s^p\right)\right)}{m_p n_s^p} + \mathbf{F}_p^p,\end{aligned} \tag{27}$$

where F_p^p is the pondermotive force due to the high-frequency oscillations on the slow motion of the proton fluid:

$$\mathbf{F}_p^p = -\frac{1}{2}\nabla\left\langle\left(\mathbf{v}_f^p\right)^2\right\rangle + \frac{e}{m_p c}\left\langle\mathbf{v}_f^p\times\nabla\times(\mathbf{\Phi}_p\times\mathbf{B}_s)\right\rangle. \tag{28}$$

Similarly, the slow-time-scale momentum equation for the ions is derived as

$$\begin{aligned}\frac{\partial \mathbf{v}_s^i}{\partial t} + \left(\mathbf{v}_s^i\cdot\nabla\right)\mathbf{v}_s^i &= \frac{-q}{m_i}\left[\mathbf{E}_s + \frac{\mathbf{v}_s^i}{c}\times\mathbf{B}_s\right] - \frac{\nabla p_s^i}{m_i n_s^i} \\ &+ \frac{\mu_i\left(\nabla^2\mathbf{v}_s^i + \frac{1}{3}\nabla\left(\nabla\cdot\mathbf{v}_s^i\right)\right)}{m_i n_s^i} + \mathbf{F}_p^i\end{aligned} \tag{29}$$

and

$$\mathbf{F}_p^i = -\frac{1}{2}\nabla\left\langle\left(\mathbf{v}_f^i\right)^2\right\rangle - \frac{q}{m_i c}\left\langle\mathbf{v}_f^i\times\nabla\times(\mathbf{\Phi}_i\times\mathbf{B}_s)\right\rangle, \tag{30}$$

with

$$\mathbf{v}_f^i = \frac{\partial}{\partial t}\Phi_i. \tag{31}$$

Combining eqs (6), (16), (25) and (26) yields the transport equation for fast oscillation

$$\nabla\times\nabla\times\dot{\mathbf{\Phi}}_p + \frac{1}{c^2}\dddot{\mathbf{\Phi}}_p + \frac{1}{c^2}\frac{4\pi e^2}{m_p}n_s^p\dot{\mathbf{\Phi}}_p - \frac{1}{c^2}\omega_{Bp}\ddot{\mathbf{\Phi}}_p\times\left(\frac{\mathbf{B}_s}{B_s}\right) - \frac{e}{m_p c}\nabla\times\nabla\times(\mathbf{\Phi}_p\times\mathbf{B}_s) - \frac{\gamma_p v_{Tp}^2}{c^2}\frac{1}{n_s^p}\nabla\left(\nabla\cdot\left(n_s^p\mathbf{v}_f^p\right)\right) = 0, \tag{32}$$

where we have used $\nabla p_f^p = \gamma_p T_p \nabla n_f^p$, $\omega_{Bp} = eB_s/m_p c$. If the magnetic field is absent or is weak enough, then

$$\omega_{pp} \gg \omega_{Bp}.$$

Putting

$$n_s^i \equiv n_s = n_0 + \delta n, \quad |\delta n| \ll n_0 = \text{const}.$$

eq. (32) becomes

$$\nabla\times\nabla\times\mathbf{v}_f^p + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{v}_f^p + \frac{1}{c^2}\omega_{pp}^2\mathbf{v}_f^p - \frac{\gamma_p v_{Tp}^2}{c^2}\nabla\left(\nabla\cdot\mathbf{v}_f^p\right) = -\frac{1}{c^2}\omega_{pp}^2\frac{\delta n}{n_0}\mathbf{v}_f^p. \tag{33}$$

Taking scalar product on both sides of eq. (27) with χm_p and eq. (29) with m_i, we shall derive the following relation by adding them together:

$$\frac{\partial\mathbf{v}_s^i}{\partial t} + \left(\mathbf{v}_s^i\cdot\nabla\right)\mathbf{v}_s^i = -\frac{\gamma_i T_i + \chi\gamma_p T_p}{m_i+\chi m_p}\frac{\nabla\delta n}{n_0} + \frac{\mu_i\left(\nabla^2\mathbf{v}_s^i + \frac{1}{3}\nabla\left(\nabla\cdot\mathbf{v}_s^i\right)\right)}{n_0(m_i+\chi m_p)} - \frac{\chi m_p}{2(m_i+\chi m_p)}\left(\nabla\left\langle\left(\mathbf{v}_f^p\right)^2\right\rangle\right), \tag{34}$$

where we have neglected the pondermotive force corresponding to the ions, because it is far less than that of the protons. The heavy ion can transfer greater momentum, then in general $\mu_p \ll \mu_i$. If there is no magnetic field, the blood plasma motion satisfies the condition: $v_s^e \approx v_s^i$. Linearizing eqs (22) and (34), they respectively yield

$$\frac{\partial}{\partial t}\delta n + n_0\nabla\cdot\mathbf{v}_s^i = 0 \tag{35}$$

$$\frac{\partial\mathbf{v}_s^i}{\partial t} = -c_s^2\frac{\nabla\delta n}{n_0} + \frac{\mu_i\left(\nabla^2\mathbf{v}_s^i + \frac{1}{3}\nabla\left(\nabla\cdot\mathbf{v}_s^i\right)\right)}{n_0(m_i+\chi m_p)} - \frac{\chi m_p}{2(m_i+\chi m_p)}\left(\nabla\left\langle\left(\mathbf{v}_f^p\right)^2\right\rangle\right). \tag{36}$$

Considering $T_p \approx T_i$, the ion acoustic velocity c_s can be written as

$$c_s^2 = \frac{\gamma_i T_i + \chi\gamma_p T_p}{m_i+\chi m_p} \approx \frac{(\gamma_i+\chi\gamma_p)T_p}{m_i}, \tag{37}$$

where γ_i, γ_p are the specific heat ratios for ion and proton, respectively. In view of eqs (35) and (36), we obtain

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2 - \frac{\frac{4}{3}\mu_i(\partial/\partial t)\nabla^2}{n_0 m_i}\right)\frac{\delta n}{n_0} = \nabla^2\left[\frac{\chi m_p}{2m_i}\langle(\mathbf{v}_f^p)^2\rangle\right]. \tag{38}$$

If we write

$$\mathbf{E}_f = \frac{1}{2}[\mathbf{E}(\mathbf{r},t)\,\mathrm{e}^{i\omega t} + \text{c.c.}], \tag{39}$$

$$\mathbf{v}_f^p = \frac{1}{2}[\mathbf{v}(r,t)\,\mathrm{e}^{i\omega t} + \text{c.c.}], \tag{40}$$

where the amplitudes $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$ are slowly varying functions over time and space. According to eq. (11), we shall have

$$\mathbf{v}(\mathbf{r},t) = \frac{e}{im_p\omega}\mathbf{E}(\mathbf{r},t) \tag{41}$$

and

$$\langle(\mathbf{v}_f^p)^2\rangle = \frac{1}{2}|\mathbf{v}(\mathbf{r},t)|^2 = \frac{e^2}{2m_p^2\omega^2}|\mathbf{E}(\mathbf{r},t)|^2. \tag{42}$$

When $\omega \approx \omega_{pp}$ (where is called the critical surface), from eqs (11), (33) and (39), the following results can be obtained:

$$2i\omega_{pp}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) + c^2\nabla\times\nabla\times\mathbf{E}(\mathbf{r},t) - \gamma_p v_{Tp}^2\nabla(\nabla\cdot\mathbf{E}(\mathbf{r},t)) + \omega_{pp}^2\frac{\delta n}{n_0}\mathbf{E}(\mathbf{r},t) = 0. \tag{43}$$

In terms of eq. (42), eq. (38) can be changed into

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2 - \frac{\frac{4}{3}\mu_i(\partial/\partial t)\nabla^2}{n_0 m_i}\right)\frac{\delta n}{n_0} = \nabla^2\left[\frac{|\mathbf{E}(\mathbf{r},t)|^2}{16\pi n_0 m_i}\right]. \tag{44}$$

We define the following dimensionless variables:

$$t' = \frac{2\omega_{pp}\kappa\varepsilon}{\gamma_p}t, \quad \mathbf{r}' = \frac{2k_d\sqrt{\kappa\varepsilon}}{\gamma_p}\mathbf{r},$$

$$\mathbf{E}'(\mathbf{r}',t') = \frac{\sqrt{\gamma_p}\mathbf{E}(\mathbf{r},t)}{8\varepsilon(\pi\kappa n_0 T_p)^{1/2}}, \quad \varepsilon = \gamma_i + \chi\gamma_p,$$

$$n = \frac{\gamma_p}{4\kappa\varepsilon}\frac{\delta n}{n_0}, \quad \beta = \frac{8\mu_i k_d^2}{3n_0 m_i \omega_{pp}\gamma_p},$$

$$\eta = \frac{c^2}{\gamma_p v_{Tp}^2}, \quad \kappa = \frac{m_p}{m_i}.$$

Equations (43) and (44) may be written as

$$i\frac{\partial}{\partial t}\mathbf{E} + \eta\nabla\times\nabla\times\mathbf{E} - \nabla(\nabla\cdot\mathbf{E}) + n\mathbf{E} = 0. \tag{45}$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 - \beta\frac{\partial}{\partial t}\nabla^2\right)n = \nabla^2(|\mathbf{E}|^2). \tag{46}$$

For simplicity, we have neglected the prime in the above two equations. Equations (45) and (46) are the modified Zakharov equations by taking the viscosity into account. Equation (46) is the driven ion-sound equation, the pondermotive force on the right side resulting in density evacuation.

3. Dispersion equation

Rewrite the complex conjugate of eqs (45) and (46) as

$$i\frac{\partial}{\partial t}\mathbf{E} - \eta\nabla\times\nabla\times\mathbf{E} + \nabla(\nabla\cdot\mathbf{E}) - n\mathbf{E} = 0, \tag{47}$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 - \beta\frac{\partial}{\partial t}\nabla^2\right)n = \nabla^2(|\mathbf{E}|^2). \tag{48}$$

We have neglected the conjugate symbol '*' of the field $\mathbf{E}$. Provided the steady state is

$$n_I = 0, \quad \mathbf{E}_I = \mathbf{E}_0 \mathrm{e}^{[i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)]}, \quad \mathbf{k}_0 \parallel \mathbf{E}_0, \tag{49}$$

where $\mathbf{E}_0$, $\mathbf{k}_0$ and ω_0 are the dimensionless amplitude, wave number and frequency of the plane wave, respectively. From eq. (47), we derive

$$\omega_0 = k_0^2. \tag{50}$$

Considering the following perturbations:

$$\delta\mathbf{E} = \Big\{(\mathbf{E}_1 + \mathbf{E}_2)\,\mathrm{e}^{[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]} + \left(\mathbf{E}_1^+ + \mathbf{E}_2^+\right)\mathrm{e}^{[-i(\mathbf{k}\cdot\mathbf{r}-\omega t)]}\Big\} \times \mathrm{e}^{[i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)]}, \tag{51}$$

$$n_{II} = \frac{n}{2}\Big\{\mathrm{e}^{[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]} + \mathrm{e}^{[-i(\mathbf{k}\cdot\mathbf{r}-\omega t)]}\Big\}, \tag{52}$$

where $\mathbf{E}_1$, $\mathbf{E}_1^+$ are longitudinal perturbations, and $\mathbf{E}_2$, $\mathbf{E}_2^+$ are transverse perturbations,

$$\mathbf{E}_1 = E_1\mathbf{e}_1, \quad \mathbf{E}_1^+ = \mathbf{E}_1^*\mathbf{e}_1^+,$$
$$\mathbf{E}_2 = E_2\mathbf{e}_2, \quad \mathbf{E}_2^+ = \mathbf{E}_2^*\mathbf{e}_2^+,$$
$$\mathbf{e}_1 \parallel \mathbf{k}_+, \quad \mathbf{e}_1^+ \parallel \mathbf{k}_-, \quad \mathbf{e}_2 \perp \mathbf{k}_+,$$
$$\mathbf{e}_2^+ \perp \mathbf{k}_-, \quad \mathbf{k}_\pm \equiv \mathbf{k} \pm \mathbf{k}_0, \quad \omega_\pm \equiv \omega \pm \omega_0, \tag{53}$$

where $\mathbf{e}_1$, $\mathbf{e}_1^+$, $\mathbf{e}_2$, $\mathbf{e}_2^+$ are real unit vectors. Substituting eq. (53) into (51), we obtain

$$\delta\mathbf{E} = (E_1\mathbf{e}_1 + E_2\mathbf{e}_2)\,\mathrm{e}^{[i(\mathbf{k}_+\cdot\mathbf{r}-\omega_+ t)]} + \left(E_1^*\mathbf{e}_1^+ + E_2^*\mathbf{e}_2^+\right)\mathrm{e}^{[-i(\mathbf{k}_-\cdot\mathbf{r}-\omega_- t)]}. \tag{54}$$

Linearizing eqs (47) and (48) with respect to the perturbations, we obtain

$$i(\delta\mathbf{E})_t + \nabla(\nabla\cdot\delta\mathbf{E}) - \eta[\nabla\times(\nabla\times\delta\mathbf{E})] - n_{II}\mathbf{E}_I = 0, \tag{55}$$

$$(n_{II})_{tt} - \nabla^2 n_{II} - \beta\frac{\partial}{\partial t}\nabla^2 n_{II} = \nabla^2\left(\mathbf{E}_I\cdot(\delta\mathbf{E})^* + \mathbf{E}_I^*\cdot(\delta\mathbf{E})\right). \tag{56}$$

Substituting eqs (49), (52) and (54) into (55) and into its conjugate equation yield

$$(\omega_+ - k_+^2)E_1\mathbf{e}_1 + (\omega_+ - \eta k_+^2)E_2\mathbf{e}_2 = \frac{n\mathbf{E}_0}{2}, \tag{57}$$

$$-(\omega_- + k_-^2)E_1\mathbf{e}_1^+ - (\omega_- + \eta k_-^2)E_2\mathbf{e}_2^+ = \frac{n\mathbf{E}_0^*}{2}. \tag{58}$$

Similarly, from eq. (56) we can easily obtain

$$-\frac{n}{2}\left(\omega^2 - k^2 + i\omega\beta k^2\right) = -k^2\left[\mathbf{E}_0\cdot\left(E_1\mathbf{e}_1^+ + E_2\mathbf{e}_2^+\right) + \mathbf{E}_0^*\cdot(E_1\mathbf{e}_1 + E_2\mathbf{e}_2)\right]. \tag{59}$$

Combining eqs (50) and (57)–(59), we find that

$$1 + \frac{k^2|E_0|^2}{(\omega^2 - k^2 + i\omega\beta k^2)}\left[\frac{\cos^2\theta_+}{-\omega - k_0^2 + (\mathbf{k}_0 + \mathbf{k})^2} + \frac{\cos^2\theta_-}{\omega - k_0^2 + (\mathbf{k}_0 - \mathbf{k})^2} + \frac{\sin^2\theta_+}{-\omega - k_0^2 + \eta(\mathbf{k}_0 + \mathbf{k})^2} + \frac{\sin^2\theta_-}{\omega - k_0^2 + \eta(\mathbf{k}_0 - \mathbf{k})^2}\right] = 0, \tag{60}$$

where $\theta_\pm$ is the angle between $\mathbf{k}_0$ and $\mathbf{k} \pm \mathbf{k}_0$.

4. Modulation instability

In §3, we have derived the dispersion equation for the growth rate, that is the equation of $\omega(k)$. In order to investigate the solution of eq. (60), we take the following parameters:

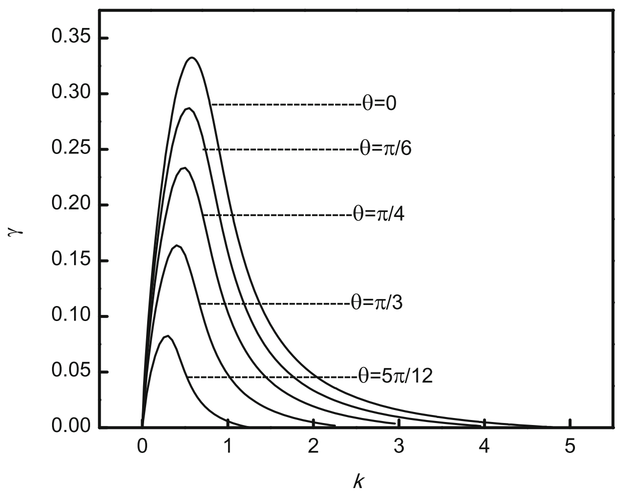

Figure 1. The variation of growth rate with respect to the wave number for different values of θ.

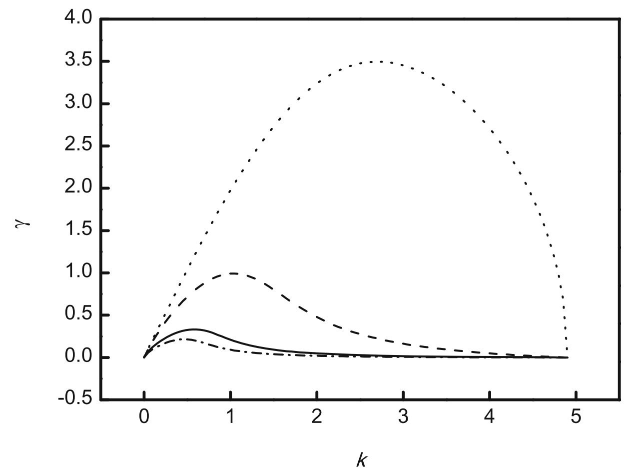

Figure 2. The growth rate with respect to the wave number when $\theta = 0$. The dash-dot, solid, dash and dot lines correspond to $\beta = 250$, $\beta = 104.5$, $\beta = 10$ and $\beta = 0$, respectively.

$|\mathbf{E}_0|^2 = 12, \quad T = 310\,\mathrm{K}, \quad m_i = 1.1039 \times 10^{-22}\,\mathrm{kg}$,
$\chi = 200$, $\rho_{Alb} = 44\,\mathrm{kg \cdot m^{-3}}$,
$\mu_i = 1.12 \times 10^{-3}\,\mathrm{kg \cdot m^{-1} \cdot s^{-1}}$, $\gamma_i = \gamma_p = 3$.

The corresponding β and η are 104.55 and 1.17×10^{10}, respectively. The dispersion eq. (60) is quite complicated, one can consider the special case,

$$|\mathbf{k}| \gg |\mathbf{k}_0|,$$

thus

$$\theta_+ \approx \theta_- \equiv \theta.$$

Equation (60) can be simplified to be

$$(\omega^2 - k^2 + i\omega\beta k^2) + k^2|E_0|^2 \left[\frac{\cos^2\theta}{-\omega + k^2} + \frac{\cos^2\theta}{\omega + k^2} + \frac{\sin^2\theta}{-\omega + \eta k^2} + \frac{\sin^2\theta}{\omega + \eta k^2} \right] = 0. \tag{61}$$

Equation (61) has six complex solutions. One of them has the positive imaginary part, which represents the growth rate of the instability. The variations of temporal growth rate γ with wave number for different θ are shown in figure 1. With increasing wave number, the growth rate first increases and then decreases. There is a maximum at the small wave number (vessel wall). Obviously, the growth rate decreases when the angle increases. When $\theta = 0$, i.e., $\mathbf{k}_0 \| \mathbf{k}$, this is the case of longitudinal perturbation, which has the maximum growth rate. For transverse perturbation, i.e., $\mathbf{k}_0 \perp \mathbf{k}$, $\theta = \pi/2$, the calculation shows $\gamma \to 0$. Figure 2 tells us that the growth rate greatly increases if $\beta = 0$: the viscosity limits to the instability.

5. Discussion and conclusions

The initial pumping wave eq. (49) is the accurate solutions of the modified Zakharov equations. By considering small disturbances as shown in eqs (51) and (52), we obtain the final dispersion equation. The perturbed wavelength is far less than that of the pumping wave when $|\mathbf{k}| \gg |\mathbf{k}_0|$. It is the modulation instability in the Lyapunov sense if the amplitude of the small disturbance amplifies. For perturbation of different directions, the variations of growth rate with perturbation wave number are numerically calculated in figure 1. It seems that the longitudinal perturbation can initiate the instability more easily. To study the influence of viscosity on the modulation instability, the growth rate distributions are curved in figure 2 with different values of β. It is shown that the instability is suppressed by the viscous force. The instability would disappear as long as the viscosity is large enough. With the decrease of the viscosity, the maximum instability shifting towards greater values of wave number k is apparent.

Chandrasekhar has pointed out that the instability of viscous flow is completely different from the inviscid one [15], and the viscosity limits to the instability. These results were also applicable for magnetorotational instability. In the presence of a vertical weak magnetic field, the growth rate in the viscous case deviated more from the ideal case [16]. In addition, we have shown that [17–19], it is just the large viscosity resulting from collapse of self-magnetic fluid with very small scale, to make astrophysical accretion disks the active objects in the Universe: this magnetic flux exerts magnetic viscous force on fluid element with great magnetic viscosity μ_m, which result in a factor of more than 10^8 amplification to usual viscosity [11,20].

Now for blood, as a type of viscous fluid, it is necessary to introduce viscosity when analysing its instability. The non-linear development of modulation instability would induce inhomogeneous distribution of the volume density, which perhaps is related to the thrombus in medical science. In this study, using two-component mechanics and two-scale method, the Zakharov equations by considering viscosity were firstly derived. Then through numerical calculations, we come to the conclusions as follows:

1. The spatial plasmons are unstable, which leads to the modulation of the initially uniform wave. On the one hand, modulation instability would result in local evacuation of density. Both the wave field and the material are inhomogeneously distributed, followed by the formation of blood clot or thrombosis. On the other hand, for the axial flow, the modulation instability often pushes aside particles from the axis to the vessel wall (the region of small wave number) [21]. Therefore, platelets are able to enter the edge flow. The chance of platelet adhesion to the inner membrane is increased, which also favours thrombosis.
2. In this paper, we take the ordinary water viscosity ($\mu = 1.12 \times 10^{-3}\,\text{kg·m}^{-1}\text{·s}^{-1}$) for blood plasma instead of μ_i. The results indicate that the viscous force greatly suppresses the instability. Experiments have shown that the viscosity of turbulence, i.e. chaos, is much greater than the ordinary viscosity [20]. If the blood flow is in the state of chaos or turbulence, the modulation instability may be suppressed completely, then the formation of blood clot will be prevented. The experiment tells us that chaos in the body signals health, while periodic behaviour can foretell disease [22]. Therefore, we analytically illustrate the important nonlinear physiological phenomena, which may challenge the traditional principle of medical science.

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