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The observer-based synchronization and parameter estimation of a class of chaotic system via a single output

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Abstract. Observer-based synchronization and parameter estimation of chaotic systems has been an interesting and important issue in theory and various fields of application. In this paper first we investigate the observer-based synchronization of a class of chaotic systems, and then discuss its parameter estimation via a single output. We assume that only the sum of the first and second state variables is available. By constructing a proper observer, some novel criteria for observer-based synchronization and parameter estimation are proposed via a scalar input. The Lü chaotic system is taken as an example to demonstrate the efficiency of the proposed approach.

Keywords. Chaotic system; observer-based synchronization; parameter estimation; single output.

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1. Introduction

Chaos is a widespread phenomenon occurring in many nonlinear systems, such as communication system, meteorological system etc. Since Pecora and Carroll [1] developed a method to realize the chaos synchronization between two identical chaotic systems with different initial conditions, the synchronization of chaotic systems has attracted considerable attention due to its potential applications in many areas, such as secure communication, information processing, biological systems, and chemical reactions. Several types of synchronizations, such as complete synchronization [1], antisynchronization [2], lag-synchronization [3], anticipating synchronization [4], phase synchronization [5], projective synchronization [6], function projective synchronization [7], combination synchronization [8], equal combination synchronization [9], etc. have been found in chaotic systems. Up to now, a wide variety of synchronization approaches have been developed for chaotic systems, such as adaptive control scheme [10], back-stepping method [11], state-feedback technique [12], sliding control approach [13], and so on.

Most of the theoretical results concerning chaos synchronization mainly focus on systems whose parameters are exactly known in advance. But in many practical situations, due to unmodelled dynamics and structural variations of the systems, the true values of parameters of many systems may not be known entirely. However, for practical applications, it is necessary to know the true value of the unknown parameter. Therefore, how to effectively synchronize and estimate the unknown parameters of chaotic systems is an important problem for the theoretical and practical applications. Many efforts have been devoted to the issue of synchronization of chaotic systems and estimation of the unknown parameter of chaotic systems in the literature. For example, the adaptive-impulsive synchronization and estimation of parameters of chaotic systems only by using discontinuous drive signals are investigated in [14]. Ma *et al* [15] discussed the complete synchronization, phase synchronization and parameter estimation by using adaptive synchronization scheme and Lyapunov stability theory. Banerjee and Chowdhury [16] developed a new method of synchronization between two nonlinear systems based on Lyapunov function and parameter estimation through modulational equations. The parameter estimation for chaotic systems by using the chaotic-search artificial bee colony algorithm (CSABC) is proposed in [17]. Zhao *et al* [18] considered the synchronization and parameter identification of chaotic system with unknown parameters and mixed delays. Based on a special matrix structure, a new approach is proposed for designing controller and update rule of unknown parameters.

It should be noted that the approaches proposed in refs [14–18] were under the same condition that the states of the drive system were completely known. It is well known that the states in some systems are partially or fully unavailable in many practical control problems. Therefore, it is necessary and important to investigate the synchronization of chaotic system with limited output information.

A great many efforts have been made to synchronize chaotic systems by using limited output information. For instance, based on observer design method, the problems of chaotic synchronization for a class of uncertain chaotic systems and chaos-based secure communication are discussed in [19]. Jiang *et al* [20] investigated the observer-based chaos synchronization and chaos-based secure communication problems for a class of uncertain chaotic systems with both parameter uncertainties and channel noise. In [21], the observer-based synchronization of chaotic systems with first-order coder in the presence of information constraints was introduced via adaptive control method. Sharma and Kar [22] proposed an observer-based synchronization scheme for a class of chaotic systems by using contraction theory. The adaptive synchronization problem of the drive–response-type chaotic systems via a scalar transmitted signal is discussed in [23].

It is easy to see that the estimation of unknown parameters has not been considered in refs [19–23]. So far as we know, less attention has been paid to the issue of estimating unknown parameters of chaotic systems via observer-based synchronization method.

Motivated by the above discussion, this paper investigates the observer-based synchronization and parameter estimation of chaotic systems of a class of chaotic systems via a single output. By using the sum of the first and second state variables, some novel criteria for synchronization and parameter estimation are proposed via constructing a proper observer. The Lü chaotic system is taken as an example to show the efficiency of the proposed approach.

The lay-out of the rest of this paper is organized as follows: A brief description of a class of chaotic systems is introduced in §2. Section 3 investigates the design of the observer of the chaotic systems. The parameter estimation scheme of the chaotic systems is proposed in §4. Numerical examples to demonstrate the effectiveness of the proposed approach are shown in §5. Finally, some conclusions are presented in §6.

2. System description

Consider the following chaotic system:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = bx_2 + f(x), \\ \dot{x}_3 = -cx_3 + g(x_1, x_2), \\ x_{\text{out}} = k_1 x_1 + k_2 x_2, \end{cases} \tag{1}$$

where x_1, x_2, x_3 are the state variables of system (1). $f(x)$, $g(x_1, x_2)$ are continuous functions and $g(0, 0) = 0$. $a > 0, c > 0$ are known parameters while b is the unknown parameter which is to be estimated later. x_{out} denotes the output variable, $k_1, k_2\ (\neq 0)$ are the given parameters.

Remark 1. Many chaotic systems can be described by (1), such as the classic Lü system [24], the Chen system [25], the Lorenz system [26], the unified system [27], etc.

3. Observer-based synchronization

In this section, we investigate the synchronization of system (1) via observer-based method. To this end we introduce an observer of system (1) as follows:

$$\begin{cases} \dot{\hat{x}}_1 = \dfrac{a}{k_2}(k_1 x_1 + k_2 x_2) - \dfrac{k_1 + k_2}{k_2} a\hat{x}_1, \\ \hat{x}_2 = w + \dfrac{1}{k_2}(k_1 x_1 + k_2 x_2), \\ \dot{w} = \dfrac{k_1}{k_2} a(\hat{x}_1 - \hat{x}_2), \\ \dot{\hat{x}}_3 = -c\hat{x}_3 + g(\hat{x}_1, \hat{x}_2), \end{cases} \tag{2}$$

where $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ are the estimated values of x_1, x_2 and x_3, respectively.

In order to get our results, we introduce a lemma which will be used in the proof of the following theorem.

Lemma 1 [28]. *If* $\lim_{t\to+\infty} \alpha(t) = 0$, *then the origin of system*

$$\dot{x} = -\lambda x + \alpha(t)$$

is globally asymptotically stable, *where* $\lambda > 0$.

Theorem 1. *If there exist* k_1 *and* k_2 *such that*

$$\begin{cases} k_1 k_2 > 0, \\ c > 0, \end{cases} \tag{3}$$

then system (2) *will synchronize system* (1) *in the sense of* $\lim_{t\to\infty} e_1 = \lim_{t\to\infty} e_2 = \lim_{t\to\infty} e_3 = 0$, *where* $e_1 = \hat{x}_1 - x_1, e_2 = \hat{x}_2 - x_2, e_3 = \hat{x}_3 - x_3$.

Proof. From systems (1) and (2), we have

$$\dot{e}_1 = \dot{\hat{x}}_1 - \dot{x}_1 = \frac{ak_1}{k_2}x_1 + ax_2 - \frac{(k_1+k_2)a}{k_2}\hat{x}_1 - a(x_2 - x_1) = -\frac{(k_1+k_2)a}{k_2}e_1.$$

Since $[(k_1+k_2)a/k_2] > 0$, one gets $\lim_{t\to\infty} e_1 = 0$.

In order to show $\lim_{t\to\infty} e_2 = 0$, subtracting (1) from (2) gives

$$\dot{e}_2 = \dot{\hat{x}}_2 - \dot{x}_2 = \frac{k_1}{k_2}a(\hat{x}_1 - \hat{x}_2) + \frac{k_1}{k_2}a(x_2 - x_1) = -\frac{k_1}{k_2}a(e_2 - e_1).$$

Note that $\lim_{t\to\infty} e_1 = 0$ and $(k_1 a/k_2) > 0$. Thus, by using Lemma 1 one derive $\lim_{t\to\infty} e_2 = 0$.

In addition,

$$\dot{e}_3 = \dot{\hat{x}}_3 - \dot{x}_3 = -c\hat{x}_3 + g(\hat{x}_1, \hat{x}_2) - (-cx_3 + g(x_1, x_2)) = -ce_3 + g(\hat{x}_1, \hat{x}_2) - g(x_1, x_2).$$

In view of $\lim_{t\to\infty} e_1 = \lim_{t\to\infty} e_2 = 0$, we have $\lim_{t\to\infty}(g(\hat{x}_1, \hat{x}_2) - g(x_1, x_2)) = 0$. According to Lemma 1, one gets $\lim_{t\to\infty} e_3 = 0$. The proof of Theorem 1 is completed.

4. The scheme of estimating parameter *b*

Since b is an unknown parameter, in the following we design an approach to estimate parameter b by using x_{out} and $\hat{x}_1, \hat{x}_2, \hat{x}_3$.

From the first and second equations of system (1), we get

$$k_1\dot{x}_1 + k_2\dot{x}_2 = ak_1(x_2 - x_1) + k_2 f(x_1, x_2, x_3) + bk_2x_2. \tag{4}$$

Based on (4), we construct the following equation:

$$\dot{x}_{\text{out}} = ak_1(\hat{x}_2 - \hat{x}_1) + k_2 f(\hat{x}_1, \hat{x}_2, \hat{x}_3) + \hat{b}_1 k_2\hat{x}_2. \tag{5}$$

Note that $\lim_{t\to\infty} \hat{x}_i = \lim_{t\to\infty} x_i, i = 1, 2, 3$. Then from eqs (4) and (5) one can easily conclude that

$$\lim_{t\to\infty} \hat{b}_1 = b. \tag{6}$$

It is noted that $\dot{x}_{\text{out}}$ is implemented in practical. In order to obtain $\hat{b}_1$, multiplying by an integration factor $e^{\lambda t}$ on both sides of (5) yields

$$\frac{d}{dt}(e^{\lambda t}x_{\text{out}}) = e^{\lambda t}(ak_1(\hat{x}_2 - \hat{x}_1) + k_2 f(\hat{x}_1, \hat{x}_2, \hat{x}_3) + \hat{b}_1 k_2\hat{x}_2 + \lambda x_{\text{out}}), \tag{7}$$

where $\lambda(>0)$ is a constant.

Integrating both sides of (7) from 0 to t and dividing by $e^{\lambda t}$ gives

$$x_{\text{out}} - x_{\text{out}}(0)e^{-\lambda t} = \int_0^t (ak_1(\hat{x}_2 - \hat{x}_1) + k_2 f(\hat{x}_1, \hat{x}_2, \hat{x}_3) + \lambda x_{\text{out}})e^{\lambda(\tau - t)}d\tau + \hat{b}_1\int_0^t k_2\hat{x}_2 e^{\lambda(\tau - t)}d\tau, \tag{8}$$

where $x_{\text{out}}(0)$ is the initial value of x_{out} at $t = 0$.

It is obvious that $\lim_{t\to\infty} x_{\text{out}}(0)e^{-\lambda t} = 0$. Based on (8) we introduce $\hat{b}_2$ such that

$$x_{\text{out}} = \int_0^t (ak_1(\hat{x}_2 - \hat{x}_1) + k_2 f(\hat{x}_1, \hat{x}_2, \hat{x}_3) + \lambda x_{\text{out}})e^{\lambda(\tau - t)}d\tau + \hat{b}_2\int_0^t k_2\hat{x}_2 e^{\lambda(\tau - t)}d\tau. \tag{9}$$

Obviously, one gets

$$\lim_{t\to\infty} \hat{b}_2 = \lim_{t\to\infty} \hat{b}_1. \tag{10}$$

From (6) and (10) it is obvious that $\lim_{t\to\infty} \hat{b}_2 = b$. So, in the following we only need to compute $\hat{b}_2$.

Based on (9), it is easy to see that $\hat{b}_2$ can be calculated as follows:

$$\hat{b}_2 = \frac{x_{\text{out}} - \int_0^t (ak_1(\hat{x}_2 - \hat{x}_1) + k_2 f(\hat{x}_1, \hat{x}_2, \hat{x}_3) + \lambda x_{\text{out}})e^{\lambda(\tau - t)}d\tau}{\int_0^t k_2\hat{x}_2 e^{\lambda(\tau - t)}d\tau} = \frac{x_{\text{out}} - \varphi_0}{\varphi_1}, \tag{11}$$

where

$$\varphi_0 = \int_0^t (ak_1(\hat{x}_2 - \hat{x}_1) + k_2 f(\hat{x}_1, \hat{x}_2, \hat{x}_3) + \lambda x_{\text{out}})e^{\lambda(\tau - t)}d\tau, \tag{12}$$

$$\varphi_1 = \int_0^t k_2\hat{x}_2 e^{\lambda(\tau - t)}d\tau. \tag{13}$$

It is noted that φ_1 may be zero at some points of time t. As it is well known that the denominator cannot be zero, we should get rid of the denominator in (11). To this end we recommend a parameter $\hat{b}$ which is defined by the following equation:

$$\dot{\hat{b}} = \mu|\varphi_1|(\hat{b}_2 - \hat{b}), \tag{14}$$

where $\mu > 0$.

According to (14), we easily know that $\lim_{t\to\infty} \hat{b} = \lim_{t\to\infty} \hat{b}_2 = b$. Combining (11) and (14) yields

$$\begin{aligned}\dot{\hat{b}} &= \mu|\varphi_1|\left(\frac{x_{\text{out}} - \varphi_0}{\varphi_1} - \hat{b}\right)\\ &= \mu\,\text{sign}(\varphi_1)(x_{\text{out}} - \varphi_0 - \varphi_1\hat{b}).\end{aligned} \tag{15}$$

The quantities φ_0 and φ_1 are not easily computed by using the explicit forms in (12) and (13). Differentiating (12) and (13) using the Leibnitz' rule, one gets

$$\begin{aligned}\dot{\varphi}_0 &= ak_1(\hat{x}_2 - \hat{x}_1) + k_2 f(\hat{x}_1, \hat{x}_2, \hat{x}_3)\\ &\quad +\lambda x_{\text{out}} - \lambda\varphi_0,\end{aligned} \tag{16}$$

$$\dot{\varphi}_1 = k_2\hat{x}_2 - \lambda\varphi_1, \tag{17}$$

with initial conditions $\varphi_0(0) = \varphi_1(0) = 0$.

Based on the above discussion, we have the following theorem.

Theorem 2. *The unknown parameter b can be estimated by $\hat{b}$ which is defined by*

$$\begin{cases}\dot{\hat{b}} = \mu\,\text{sign}(\varphi_1)(x_{\text{out}} - \varphi_0 - \varphi_1\hat{b}),\\ \dot{\varphi}_0 = ak_1(\hat{x}_2 - \hat{x}_1) + k_2 f(\hat{x}_1, \hat{x}_2, \hat{x}_3)\\ \qquad +\lambda x_{\text{out}} - \lambda\varphi_0,\\ \dot{\varphi}_1 = k_2\hat{x}_2 - \lambda\varphi_1,\end{cases} \tag{18}$$

with initial conditions $\varphi_0(0) = \varphi_1(0) = 0$.

Remark 2. From system (2) it is easy to see that parameters a and c are used in designing the response system. If a and c are unknown parameters, then it may be difficult to establish the response system. So, in order to easily construct the response system we assume that a and c are known parameters. Since system (2) does not contain parameter b, b can be viewed as the unknown parameter. According to Theorem 2, the unknown parameter b can be estimated by using eqs (18).

5. Numerical results

In this section, we apply the synchronization and parameter estimation approaches to Lü system and some simulation results are also given.

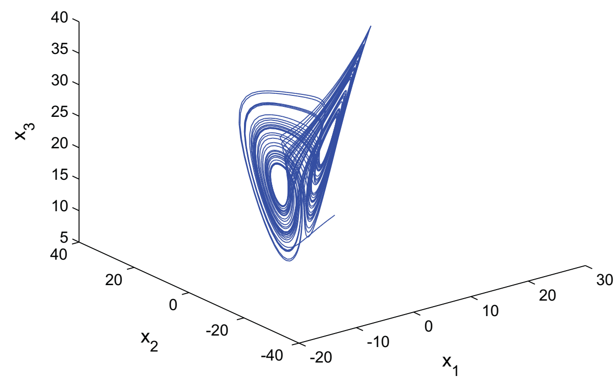

Figure 1. The chaotic attractor of system (19) when $a = 36, b = 20, c = 3$.

The state equations of Lü system [24] with output variable are given by

$$\begin{cases}\dot{x}_1 = a(x_2 - x_1),\\ \dot{x}_2 = -x_1x_3 + bx_2,\\ \dot{z} = x_1x_2 - cx_3,\\ x_{\text{out}} = k_1x_1 + k_2x_2,\end{cases} \tag{19}$$

where $a, b, c \in R_+$. Lü system (19) is chaotic if the parameters satisfy the following conditions: $a = 36, c = 3$ and $12.7 < b < 17.0, 18.0 < b < 22.0, 23.0 < b < 28.5, 28.6 < c < 29.0, 29.2334 < b < 29.345$. The chaotic attractor of system (19) with $a = 36, b = 20, c = 3$ is given in figure 1.

Case 1. *The synchronization of Lü system*

Comparing system (19) with system (1) yields $f(x_1, x_2, x_3) = -x_1x_3, g(x_1, x_2) = x_1x_2$. In order to synchronize system (19), based on system (2) the observer (response system) is given as

$$\begin{cases}\dot{\hat{x}}_1 = \dfrac{a}{k_2}x_{\text{out}} - \dfrac{k_1 + k_2}{k_2}a\hat{x}_1,\\ \hat{x}_2 = w + \dfrac{1}{k_2}x_{\text{out}},\\ \dot{w} = \dfrac{k_1}{k_2}a(\hat{x}_1 - \hat{x}_2),\\ \dot{\hat{x}}_3 = -c\hat{x}_3 + \hat{x}_1\hat{x}_2,\end{cases} \tag{20}$$

where

$$\frac{(k_1 + k_2)a}{k_2} > 0, \quad \frac{k_1 a}{k_2} > 0.$$

According to Theorem 1, the synchronization between systems (19) and (20) will be achieved.

In our simulation process, we take $a = 36, b = 20, c = 8, k_1 = -2, k_2 = -1$. It is easy to see that the conditions of Theorem 1 are satisfied. Based on Theorem 1, the synchronization between systems

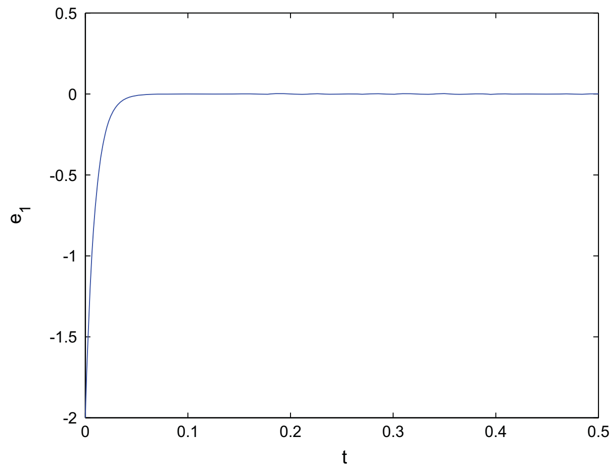

Figure 2. The time evolution of error e_1 between system (19) and observer (20).

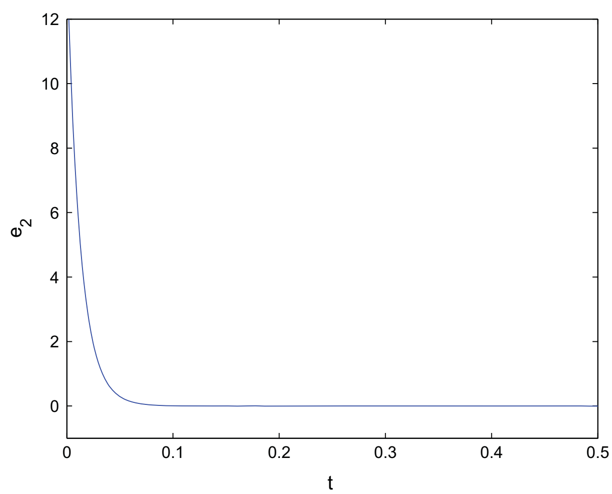

Figure 3. The time evolution of error e_2 between system (19) and observer (20).

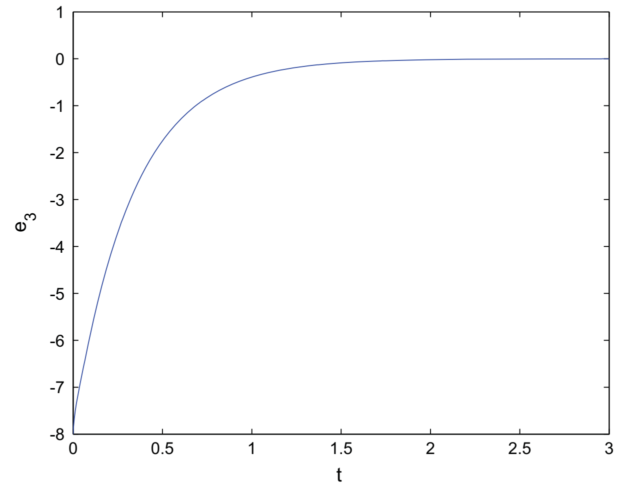

Figure 4. The time evolution of error e_3 between system (19) and observer (20).

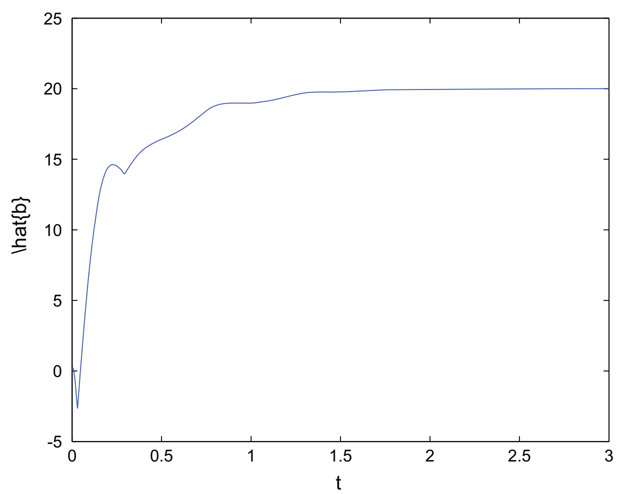

Figure 5. The estimated value of unknown parameter b.

(19) and (20) will be reached. The initial conditions of system (19) and observer (20) are chosen as $(x_1(0), x_2(0), x_3(0)) = (4, -3, 12)$ and $(\hat{x}_1(0), w(0), \hat{x}_3(0)) = (2, 6, 4)$, respectively. Numerical results of the time evolution of errors e_1, e_2, e_3 between system (19) and observer (20) are shown in figures 2, 3, 4 and 5, respectively. From figures 2–5, one can conclude that the synchronization errors converge asymptotically to zero and the synchronization between system (19) and observer (20) is achieved.

Case 2. *The estimation of parameter b*

In this subsection, we give the simulation result of estimating the unknown parameter b. By using Theorem 2, the unknown parameter b can be estimated by $\hat{b}$ which is defined by

$$\begin{cases} \dot{\hat{b}} = \mu \,\mathrm{sign}(\varphi_1)(x_{\mathrm{out}} - \varphi_0 - \varphi_1 \hat{b}), \\ \dot{\varphi}_0 = ak_1(\hat{x}_2 - \hat{x}_1) - k_2\hat{x}_1\hat{x}_3 + \lambda x_{\mathrm{out}} - \lambda\varphi_0, \\ \dot{\varphi}_1 = k_2\hat{x}_2 - \lambda\varphi_1, \end{cases} \quad (21)$$

with initial conditions $\varphi_0(0) = \varphi_1(0) = 0$. In our simulation process, we set $a = 36, b = 20, c = 8, k_1 = -2, k_2 = -1$. The simulation result with $\hat{b}(0) = 0, \varphi_0(0) = \varphi_1(0) = 0$ and $\mu = \lambda = 20$ is presented in figure 5.

In view of figure 5, it is obvious that the unknown parameter b can be estimated precisely by $\hat{b}$ which means that our presented scheme is effective and it is feasible to estimate the unknown parameter.

Remark 3. It is well known that the speed of convergence is an important issue in parameter estimation. Equalities (8) and (14) show that the convergence speed is closely dependent on the constants μ and λ. So, we can accelerate the speed of convergence by increasing μ and λ.

6. Conclusion

This work investigated the observer-based synchronization and parameter estimation of a class of chaotic systems where only the sum of the first and second state variables is available. Some new sufficient conditions for observer-based synchronization and parameter estimation of a class of chaotic systems were proposed by designing the proper observer. The proposed approach was verified by applying it to deal with the observer-based synchronization and parameter estimation of the Lü chaotic system. With the method discussed in this paper, we do not need to use the unavailable states of the chaotic systems, and thus our approach is more realistic and practical than most of the existing ones which are assuming that all the states are available.

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