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Complex scaling and residual flavour symmetry in the neutrino mass matrix

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Abstract. Using the residual symmetry approach, we propose a complex extension of the scaling ansatz on the neutrino Majorana mass matrix M_ν which allows a nonzero mass for each of the three light neutrinos as well as a nonvanishing θ_{13}. Leptonic Dirac CP violation must be maximal while atmospheric neutrino mixing need not be exactly maximal. Each of the two Majorana phases, to be probed by the search for $0\nu\beta\beta$ decay, has to be zero or π and a normal neutrino mass hierarchy is allowed.

Keywords. Neutrinos; residual flavour symmetry; scaling ansatz.

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1. Introduction

In this contribution, we focus on the observed phenomenon of neutrino mixing and not so much on the dynamics of the generation of $\mathcal{O}$(eV) light neutrino neutrino masses. Let us start with the neutrino flavour eigenstate fields ν_l $(l = e, \mu, \tau)$. Then we have three left-chiral light neutrinos with the corresponding mass eigenstate fields ν_i $(i = 1, 2, 3)$, all of which are assumed to be Majorana in nature. The mass term in the neutrino Lagrangian relevant to laboratory energies is

$$-\mathcal{L}^{\nu}_{\text{mass}} = \frac{1}{2}\bar{\nu}^C_l (M_\nu)_{lm}\nu_m + \text{h.c.} \tag{1}$$

with $M^*_\nu \neq M_\nu = M^T_\nu$. Thus, there is a 3×3 unitary matrix U such that

$$U^T M_\nu U = M^d_\nu \equiv \text{diag}(m_1, m_2, m_3), \tag{2}$$

where each m_i is real and positive. We shall assume throughout that all light neutrino masses are nonzero. Experiments tell us that they are nondegenerate. For comparison, we may write the charged lepton mass matrix in our weak basis as

$$M_{\text{ch}} = \text{diag}(m_e, m_\mu, m_\tau). \tag{3}$$

The unitary mixing matrix U of (2) can now be written in the PDG convention [1] as

$$U \equiv U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & e^{i\frac{\alpha}{2}}s_{12}c_{13} & s_{13}e^{-i(\delta-\frac{\beta}{2})} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & e^{i\frac{\alpha}{2}}(c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta}) & c_{13}s_{23}e^{i\frac{\beta}{2}} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & e^{i\frac{\alpha}{2}}(-c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta}) & c_{13}c_{23}e^{i\frac{\beta}{2}} \end{pmatrix} \tag{4}$$

with $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$ and $\theta_{ij} = [0, \pi/2]$ while $\delta, \alpha, \beta = [0, 2\pi]$.

Let us quickly browse the neutrino fact file. The latest 3σ ranges [2] for the relevant neutrino mass parameters, obtained from oscillation data and cosmological observations, are:

Solar: $\Delta m^2_{21} \equiv m^2_2 - m^2_1$: $(7.02–8.09) \times 10^{-5}$ eV^2,

Atmospheric: $|\Delta m^2_{31}| \equiv |m^2_3 - m^2_1|$: $(2.32–2.59) \times 10^{-3}$ eV^2,

Planck: $\Sigma_i m_i < 0.23$ eV.

The 3σ intervals of the mixing angles, introduced in (4), are [2]

θ_{12}: 31.29°–35.91°

θ_{23}: 38.3°–53.3°

θ_{13}: 7.87°–9.11°.

In quoting the above numbers, we have not made the fine distinction between normal ($m_3 > m_2 > m_1$) and inverted ($m_2 > m_1 > m_3$) types of mass ordering of the neutrinos, but we carefully do so in our numerical work reported later.

2. The problematic simple real scaling

Simple real scaling (SRS) was first proposed [3] as an ansatz and posited the relation (within our sign convention)

$$\frac{\left(M_\nu^{\rm SRS}\right)_{e\mu}}{\left(-M_\nu^{\rm SRS}\right)_{e\tau}} = \frac{\left(M_\nu^{\rm SRS}\right)_{\mu\mu}}{\left(-M_\nu^{\rm SRS}\right)_{\mu\tau}} = \frac{\left(M_\nu^{\rm SRS}\right)_{\tau\mu}}{\left(-M_\nu^{\rm SRS}\right)_{\tau\tau}} = k \tag{5}$$

k being a real, positive, dimensionless scaling factor. Given (5), the form of $M_\nu^{\rm SRS}$ follows immediately to be

$$M_\nu^{\rm SRS} = \begin{pmatrix} x & -Yk & Y \\ -Yk & Zk^2 & -Zk \\ Y & -Zk & Z \end{pmatrix}, \tag{6}$$

where x, Y, Z *a priori* are unknown mass-dimensional quantities; x can be chosen to be real by absorbing an overall phase in the neutrino fields, but Y and Z are in general complex. The case $k = 1$ corresponds to a $\mu\tau$ interchange symmetric M_ν with the additional constraint $M^\nu_{\mu\mu} = -M^\nu_{\mu\tau}$. Since $\det M_\nu^{\rm SRS} = 0$, one null eigenvalue of $M_\nu^{\rm SRS}$ is implied. The corresponding eigenvector can be deduced to be the third column of $U_{\rm PMNS}$, i.e.

$$C_3^{\rm SRS} = \begin{pmatrix} 0 \\ (1+k^2)^{-\frac{1}{2}} {\rm e}^{i\frac{\beta}{2}} \\ k(1+k^2)^{-\frac{1}{2}} {\rm e}^{i\frac{\beta}{2}} \end{pmatrix} \tag{7}$$

implying that $m_3^{\rm SRS} = 0 = \theta_{13}^{\rm SRS}$ and $s_{23} = (1+k^2)^{-1/2}$, $c_{23} = k(1+k^2)^{-1/2}$ so that $\tan\theta_{23} = k^{-1}$ while $\theta_{23}^{\rm SRS}$ is left undetermined (whereas other results change in an extension that we shall propose, the $\tan\theta_{23} = k^{-1}$ relation will be seen to survive). The earlier mentioned unitary matrix is now given by

$$U^{\rm SRS} = \begin{pmatrix} c_{12} & s_{12}{\rm e}^{i\frac{\alpha}{2}} & 0 \\ -k(1+k^2)^{-\frac{1}{2}} s_{12} & k(1+k^2)^{-\frac{1}{2}} c_{12}{\rm e}^{i\frac{\alpha}{2}} & (1+k^2)^{-\frac{1}{2}}{\rm e}^{i\frac{\beta}{2}} \\ k(1+k^2)^{-\frac{1}{2}} s_{12} & -k(1+k^2)^{-\frac{1}{2}} c_{12}{\rm e}^{i\frac{\alpha}{2}} & k(1+k^2)^{-\frac{1}{2}}{\rm e}^{i\frac{\beta}{2}} \end{pmatrix}. \tag{8}$$

This SRS proposal got knocked out by the experimental exclusion of a vanishing θ_{13} at the 10σ level [4].

3. Residual flavour symmetry and generalized real scaling

Any flavour symmetry of M_ν, implemented by the unitary transformation G and operating as $\nu_{L\alpha} \to G_{\alpha\beta}\nu_{L\beta}$, implies the following relations:

$$G^T M_\nu G = M_\nu,$$
$$U^\dagger G U = d,$$
$$d_{\alpha\beta} = \pm\delta_{\alpha\beta}. \tag{9}$$

Of the eight possible diagonal matrices d, $d = \pm I$ are trivial while the remaining six can be split into $\{d_a\}$ and $\{-d_a\}$ with $a = 1, 2, 3$. Each d_a corresponds to a G_i but there is a relation $G_a = \epsilon_{abc}G_bG_c$ which leaves only two independent Gs. We choose them to be $G_{2,3}$ with

$$U^\dagger G_2 U = d_2 \equiv {\rm diag}(-1, 1, -1),$$
$$U^\dagger G_3 U = d_3 \equiv {\rm diag}(-1, -1, 1). \tag{10}$$

There is thus a residual $\mathbb{Z}_2 \times \mathbb{Z}_2'$ flavour symmetry [5] in M_ν corresponding to $G_{2,3}$:

$$\mathbb{Z}_2 : G_2^T M_\nu G_2 = M_\nu,$$
$$\mathbb{Z}_2' : G_3^T M_\nu G_3 = M_\nu. \tag{11}$$

We identify $\mathbb{Z}_2'$ as with $\mathbb{Z}_2^{\rm scaling}$ and equate $G_3^{\rm scaling}$ with $U^{\rm SRS} d_3 U^{\rm SRS\dagger}$, obtaining

$$G_3^{\rm scaling} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \dfrac{1-k^2}{1+k^2} & \dfrac{2k}{1+k^2} \\ 0 & \dfrac{2k}{1+k^2} & -\dfrac{1-k^2}{1+k^2} \end{pmatrix}. \tag{12}$$

Moreover, equating G_2^k with $U^{\rm SRS} d_2 U^{\rm SRS\dagger}$, we obtain

$$G_2^k = \begin{pmatrix} -\cos 2\theta_{12} & \dfrac{k\sin 2\theta_{12}}{1+k^2} & -\dfrac{\sin 2\theta_{12}}{1+k^2} \\ \dfrac{k\sin 2\theta_{12}}{1+k^2} & -\dfrac{(1-\cos 2\theta_{12})}{k^2(1+k^2)} & -\dfrac{(1+\cos 2\theta_{12})}{k(1+k^2)} \\ -\dfrac{\sin 2\theta_{12}}{1+k^2} & -\dfrac{(1+\cos 2\theta_{12})}{k(1+k^2)} & -\dfrac{(k^2-\cos 2\theta_{12})}{1+k^2} \end{pmatrix}. \tag{13}$$

The most general form of M_ν, obeying the invariance $(G_3^{\text{scaling}})^T M_\nu G_3^{\text{scaling}} = M_\nu$ is found to be not M_ν^{SRS} but its generalized real scaling (GRS) form

$$M_\nu^{\text{GRS}} = \begin{pmatrix} x & -Yk & Y \\ -Yk & Z - Wk^{-1}(k^2-1) & W \\ Y & W & Z \end{pmatrix} \quad (14)$$

with an additional complex mass-dimensional element W. The latter needs to equal $-Zk$ as a special case to yield M_ν^{SRS}. This M_ν^{GRS} can accommodate a nonzero m_3; nevertheless, it is phenomenologically unacceptable. The column C_3 of (7) is still an eigenvector of the matrix (14) now with a nonzero m_3. In consequence, the former is the third column of the corresponding unitary mixing matrix leading to a vanishing θ_{13} and ruling out M_ν^{GRS} as a realistic neutrino mass matrix.

4. Complex extension of scaling: Our proposal

We propose [6] a complex extension of the scaling ansatz through a nonstandard CP and flavour transformation $\nu_{L\alpha} \to i(G_3)_{\alpha\beta}\gamma^0\nu_{L\beta}^C$ leading to

$$G_3^{\text{scaling}} M_\nu G_3^{\text{scaling}} = M_\nu^*. \quad (15)$$

The most general complex extended scaling (CES) invariant form of M_ν, satisfying (15), is

$$M_\nu^{\text{CES}} = \begin{pmatrix} x & -y_1k + i\dfrac{y_2}{k} & y_1 + iy_2 \\ -y_1k + i\dfrac{y_2}{k} & z_1 - w\dfrac{k^2-1}{k} - iz_2 & w - i\dfrac{k^2-1}{2k}z_2 \\ y_1 + iy_2 & w - i\dfrac{k^2-1}{2k}z_2 & z_1 + iz_2 \end{pmatrix}. \quad (16)$$

The mass matrix (16) is characterized by six real mass dimensional parameters x, $y_{1,2}$, $z_{1,2}$ and w apart from the scaling factor k. It follows that [6]

$$G_3U^* = U\tilde{d},$$
$$\tilde{d} = \text{diag}(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3),$$
$$\tilde{d}_a = \pm 1 \quad \text{for } a = 1, 2, 3. \quad (17)$$

By substituting G_3^{scaling} from (12) and qualifying U by U^{CES}, we explicitly obtain from the first equation of (17) that

$$\begin{pmatrix} -(U_{e1}^{\text{CES}})^* & -(U_{e2}^{\text{CES}})^* & -(U_{e3}^{\text{CES}})^* \\ \dfrac{1-k^2}{1+k^2}(U_{\mu1}^{\text{CES}})^* + \dfrac{2k}{1+k^2}(U_{\tau1}^{\text{CES}})^* & \dfrac{1-k^2}{1+k^2}(U_{\mu2}^{\text{CES}})^* + \dfrac{2k}{1+k^2}(U_{\tau2}^{\text{CES}})^* & \dfrac{1-k^2}{1+k^2}(U_{\mu3}^{\text{CES}})^* + \dfrac{2k}{1+k^2}(U_{\tau3}^{\text{CES}})^* \\ \dfrac{2k}{1+k^2}(U_{\mu1}^{\text{CES}})^* - \dfrac{1-k^2}{1+k^2}(U_{\tau1}^{\text{CES}})^* & \dfrac{2k}{1+k^2}(U_{\mu2}^{\text{CES}})^* - \dfrac{1-k^2}{1+k^2}(U_{\tau2}^{\text{CES}})^* & \dfrac{2k}{1+k^2}(U_{\mu3}^{\text{CES}})^* - \dfrac{1-k^2}{1+k^2}(U_{\tau3}^{\text{CES}})^* \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{d}_1U_{e1}^{\text{CES}} & \tilde{d}_2U_{e2}^{\text{CES}} & \tilde{d}_3U_{e3}^{\text{CES}} \\ \tilde{d}_1U_{\mu1}^{\text{CES}} & \tilde{d}_2U_{\mu2}^{\text{CES}} & \tilde{d}_3U_{\mu3}^{\text{CES}} \\ \tilde{d}_1U_{\tau1}^{\text{CES}} & \tilde{d}_2U_{\tau2}^{\text{CES}} & \tilde{d}_3U_{\tau3}^{\text{CES}} \end{pmatrix}. \quad (18)$$

From (18) one sees that, in the identification of U^{CES} with U_{PMNS}, any combination with $\tilde{d}_1 = 1$ yields an imaginary $c_{12}c_{13}$ in U_{PMNS} and is hence ruled out. Four other nontrivial possibilities remain, with

$$\tilde{d}_1 = -1,$$
$$\tilde{d}_2 = \eta,$$
$$\tilde{d}_3 = \xi \quad (19)$$

and $\eta_{a,b} = 1$, $\eta_{c,d} = -1$, $\xi_{a,c} = 1$ and $\xi_{b,d} = -1$. Now a detailed comparison between the columns of U^{CES} and U_{PMNS} of (4) leads to the results

$$e^{-i\alpha} = -\eta,$$
$$e^{i(2\delta-\beta)} = -\xi, \quad (20)$$

so that we obtain

$$\alpha = \pi, 0 \quad \text{for } \eta = +1, -1,$$
$$2\delta - \beta = \pi, 0 \quad \text{for } \xi = +1, -1. \quad (21)$$

On matching the remaining six elements of U_{PMNS}, one obtains six independent linear constraint conditions, as tabulated in table 1.

The lowermost condition in table 1, given the observed fact that $c_{12} \neq 0$, yields the relations

Table 1. Constraint equations on elements of the mixing matrix.

Element of U^{CES}	Constraint condition
$\mu 1$	$2kU^{\text{CES}}_{\mu 1} = (1-k^2)U^{\text{CES}}_{\tau 1} - (1+k^2)(U^{\text{CES}}_{\tau 1})^*$
$\tau 1$	$2kU^{\text{CES}}_{\tau 1} = -(1-k^2)U^{\text{CES}}_{\mu 1} - (1+k^2)(U^{\text{CES}}_{\mu 1})^*$
$\mu 2$	$2kU^{\text{CES}}_{\mu 2} = (1-k^2)U^{\text{CES}}_{\tau 2} + \eta(1+k^2)(U^{\text{CES}}_{\tau 2})^*$
$\tau 2$	$2kU^{\text{CES}}_{\tau 2} = -(1-k^2)U^{\text{CES}}_{\mu 2} + \eta(1+k^2)(U^{\text{CES}}_{\mu 2})^*$
$\mu 3$	$2kU^{\text{CES}}_{\mu 3} = (1-k^2)U^{\text{CES}}_{\tau 3} + \xi(1+k^2)(U^{\text{CES}}_{\tau 3})^*$
$\tau 3$	$2kU^{\text{CES}}_{\tau 3} = -(1-k^2)U^{\text{CES}}_{\mu 3} + \eta(1+k^2)(U^{\text{CES}}_{\mu 3})^*$

Table 2. Output values obtained for normal mass ordering.

x (eV)	y_1 (eV)	y_2 (eV)	z_1 (eV)	z_2 (eV)	w (eV)
−0.20–+0.21	−0.12–+0.11	−0.05–+0.05	−0.17–+0.17	−0.18–+0.17	−0.16–+0.15

m_1 (eV)	m_2 (eV)	m_3 (eV)
9.2×10^{-5}–0.071	0.01–0.077	0.051–0.082

Table 3. Output values obtained for inverted mass ordering.

x (eV)	y_1 (eV)	y_2 (eV)	z_1 (eV)	z_2 (eV)	w (eV)
−0.44–+0.46	−0.16–+0.16	−0.14–+0.14	−0.01–+0.01	−0.01–+0.01	−0.05–+0.06

m_1 (eV)	m_2 (eV)	m_3 (eV)
0.051–0.085	0.049–0.079	8.2×10^{-5}–0.068

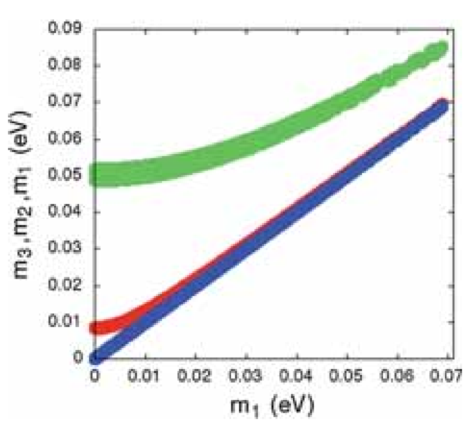

Figure 1. Variations of the light neutrino masses with the lightest mass m_1 (normal ordering).

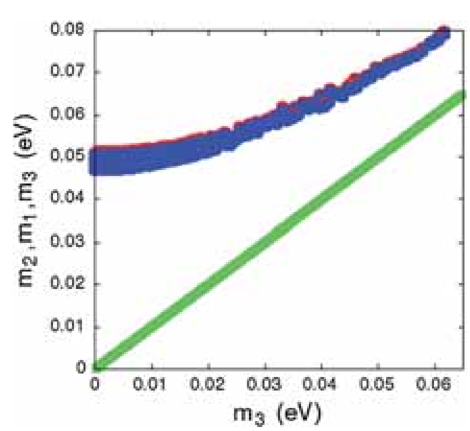

Figure 2. Variations of the light neutrino masses with the lightest mass m_3 (inverted ordering).

$$2kc_{23}\cos\frac{\beta}{2} = [k^2(1+\xi) - 1 + \xi]s_{23}\cos\frac{\beta}{2},$$
$$2kc_{23}\sin\frac{\beta}{2} = [k^2(1-\xi) - 1 - \xi]s_{23}\sin\frac{\beta}{2}. \quad (22)$$

Since $\xi^2 = 1$, a multiplication of the two sides of the above equations implies that $\sin\beta = 0$, i.e.

$$\beta = 0 \text{ or } \pi. \quad (23)$$

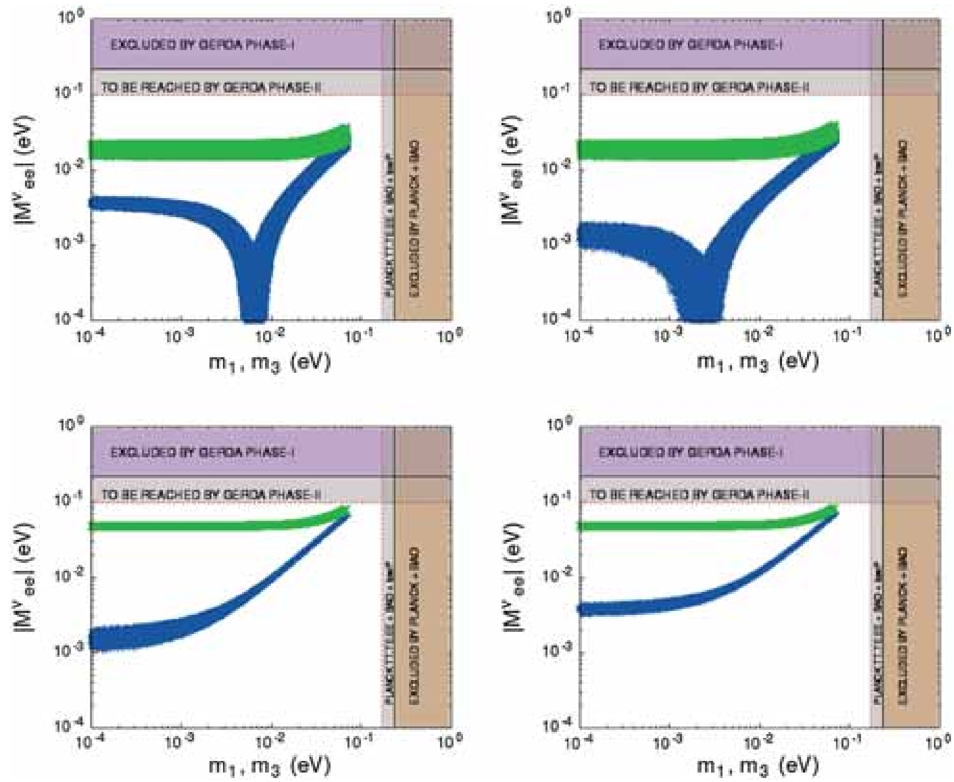

Figure 3. Plot of $|M^{\nu}_{ee}|$ vs. the lightest neutrino mass: the top two figures are for $\tilde{d}_a$ and $\tilde{d}_b$ while the last two figures are for $\tilde{d}_c$ (left) and $\tilde{d}_d$ (right).

From (21) and (23) we see that each of the two Majorana phases takes the value zero or π, i.e., there is no Majorana CP violation in this model. Continuing further, we can identify four possibilities for β, ξ and $\tan\theta_{23}$:

$$\begin{aligned}
&\beta = 0, \xi = 1 \Rightarrow \tan\theta_{23} = k^{-1}\\
&\beta = \pi, \xi = -1 \Rightarrow \tan\theta_{23} = k^{-1},\\
&\beta = 0, \xi = -1 \Rightarrow \tan\theta_{23} = -k,\\
&\beta = \pi, \xi = 1 \Rightarrow \tan\theta_{23} = -k.
\end{aligned} \tag{24}$$

The last two possibilities can be excluded because when used on the fourth relation from the top listed in table 1, these lead to the conclusion that $c_{12} = 0$ – in contradiction with observation. Thus, we now unequivocally have

$$\tan\theta_{23} = k^{-1} \tag{25}$$

and also from (20) and (21)

$$\cos\delta = 0, \quad \text{i.e.} \quad \delta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}. \tag{26}$$

In other words, Dirac CP violation is maximal.

5. Phenomenological discussion

We have tried to determine the six mass-dimensional unknown parameters x, $y_{1,2}$, $z_{1,2}$ and w of our model by inputting the observed 3σ ranges of Δm^2_{21}, $|\Delta m^2_{31}|$, θ_{12}, θ_{23} and θ_{13} as well as the cosmological upper bound on $\Sigma_i m_i$ quoted in the neutrino fact file of §1. The output values depend on the type of mass ordering of the three neutrinos that is assumed. These are given in tables 2 and 3 for normal ($m_3 > m_2 > m_1$) and inverted ($m_1 > m_2 > m_3$) ordering respectively. The corresponding allowed mass bands for m_3, m_2 and m_1 are shown in figures 1 and 2 against the central value of the lightest neutrino mass as the latter is varied continuously as a parameter.

We next turn our attention to neutrinoless double beta ($0\nu\beta\beta$) decay. The relevant constant here is the *ee* element of M_ν. The latter is given in our model by

$$(M_\nu)_{ee} = c^2_{12}c^2_{13}m_1 + s^2_{12}c^2_{13}m_2 e^{i\alpha} + s^2_{13}m_3 e^{i(\beta-2\delta)}. \tag{27}$$

There are four possibilities for the diagonal matrix $\tilde{d}$ that the complex-extended scaling ansatz embodies: (1) $\tilde{d}_a = \text{diag}(-1, +1, +1)$, (2) $\tilde{d}_b = \text{diag}(-1, +1, -1)$, (3) $\tilde{d}_c = \text{diag}(-1, -1, +1)$, (4) $\tilde{d}_d = \text{diag}(-1, -1,$

-1). The corresponding mass bands are for $|M_\nu|_{ee}$ as shown in figure 3. The upper and lower bands are for a normal (plotted against m_1) and for an inverted (plotted against m_3) mass ordering respectively. Only the extreme right corners of these plots will be accessible to forthcoming experiments such GERDA.

6. Conclusion

- We have proposed a complex extended scaling ansatz on M_ν (see eq. (16)).
- We have obtained a 6-parameter form of M_ν (see eq. (16)).
- We have derived that Dirac (Majorana) CP violation should be maximal (absent).
- We have both normal and inverted types of mass ordering as allowed possibilities.
- Our neutrinos are mass hierarchical for most allowed values of the mass sum $\Sigma_i m_i$ and approach quasi-degeneracy only when the latter is near the cosmological upper bound 0.23 eV.
- We have interesting predictions on $0\nu\beta\beta$ decay.

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