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Involution symmetries and the PMNS matrix

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Abstract. C S Lam has suggested that the PMNS matrix (or at least some of its elements) can be predicted by embedding the residual symmetry of the leptonic mass terms into a bigger symmetry. We analyse the possibility that the residual symmetries consist of involution generators only and explore how Lam's idea can be implemented.

Keywords. Neutrino mixing; residual symmetry; Coxeter groups.

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1. Introduction

The quark mixing matrix is very close to the identity matrix. In other words, all off-diagonal elements are quite small. Hence, one might expect that the structure of the quark mixing matrix can be described by some very small perturbations over the weak basis. This is not the case for the lepton mixing matrix, which is called the PMNS matrix. Here, some (though not all) of the off-diagonal elements are quite large, comparable to the diagonal elements. Therefore, some kind of non-trivial symmetry may be required to understand the structure of the PMNS matrix.

The PMNS matrix comes out of diagonalization of the mass matrices of charged leptons and neutrinos. There are two ways to talk about symmetries of the mass matrices. In one way, one starts from a high energy theory which dictates some symmetries, and sees what part of the symmetries survive at low energies. In the other approach, advocated first by Lam [1], one starts by looking at the symmetries of the low-energy Lagrangian, and tries to see which group can contain these symmetries. The bigger symmetry might then determine the PMNS matrix, or at least some information about its elements. In this paper, we are going to discuss some investigations in this second approach.

2. PMNS matrix and symmetry generators

The charged current interaction of leptons is governed by the following term in the Lagrangian:

$$\mathscr{L}_{\rm cc} = \frac{g}{\sqrt{2}} \sum_{\ell,\alpha} \bar{\ell} U_{\ell\alpha} \gamma^\mu L \nu_\alpha W_\mu^+ + \text{h.c.}, \tag{1}$$

and the mass terms are

$$\mathscr{L}_{\rm mass} = -\sum_{\ell} M_\ell \bar{\ell}\ell - \frac{1}{2} \sum_{\alpha} m_\alpha \nu_\alpha^\top C \nu_\alpha\,, \tag{2}$$

assuming Majorana neutrinos. Note that the mass terms admit the following symmetries:

$$\nu_\alpha \to \eta_\alpha \nu_\alpha, \quad \eta_\alpha = \pm 1; \tag{3}$$

$$\ell \to \exp(i\varphi_\ell)\ell. \tag{4}$$

The kinetic terms are also invariant under these transformations.

The symmetry transformations imply that the charged current interaction is invariant under the change

$$U_{\ell\alpha} \to U_{\ell\alpha} \mathrm{e}^{-i\varphi_\ell} \eta_\alpha. \tag{5}$$

Hence the value of $U_{\ell\alpha}$ is not physical, but the absolute value is.

According to the basic idea proposed by Lam, we can find $|U_{\ell\alpha}|$ from the symmetry, without going through the Lagrangian, if the symmetries of the mass terms are seen as remnant symmetries after all symmetry breaking. To implement this idea, we start with a more general symmetry:

$$\nu \to S\nu, \quad \ell \to T\ell, \tag{6}$$

where ν and ℓ, without any index, represent columns of all three eigenstates of the particles.

In the flavour basis in which the mass terms of the charged leptons are diagonal, let the neutrino fields be represented by a column

$$\tilde{\nu} = U\nu, \tag{7}$$

where U is the PMNS matrix. In this basis, the neutrino part of the symmetry of eq (6) would take the form

$$\tilde{\nu} \to USU^\dagger \tilde{\nu} \equiv S'\tilde{\nu}. \tag{8}$$

We assume T and S' symmetries as part of a bigger symmetry. This symmetry has information about U, through

$$S' = USU^\dagger. \tag{9}$$

To proceed, we make some simplifying assumptions.

1. The symmetry T is discrete.
2. Determinant of all transformations is equal to 1.
3. All generators of the discrete group are involutions, i.e., elements of order 2.

There is a huge body of work in the literature [2] where only the first two assumptions have been used. In a recent paper [3] we take the third assumption in addition, and try to find symmetries consistent with this extra one. The purpose of this talk is to summarize essential results of that paper.

The possible T generators, subject to the assumptions taken above, are as follows:

$$T_e = \mathrm{diag}(1, -1, -1), \quad T_\mu = \mathrm{diag}(-1, 1, -1),$$
$$T_\tau = \mathrm{diag}(-1, -1, 1) \tag{10}$$

and the possible S generators are

$$S_1 = \mathrm{diag}(1, -1, -1), \quad S_2 = \mathrm{diag}(-1, 1, -1),$$
$$S_3 = \mathrm{diag}(-1, -1, 1), \tag{11}$$

with S' defined through eq (9). Note that all the generators shown here are not independent: there exist the relations

$$T_e T_\mu T_\tau = 1, \quad S_1 S_2 S_3 = 1. \tag{12}$$

Let us now define

$$a_{\ell\alpha} = \mathrm{Tr}(T_\ell S'_\alpha) = \mathrm{Tr}(T_\ell U S_\alpha U^\dagger). \tag{13}$$

It is easily seen, by evaluating the trace, that there is a very simple relation between these $a_{\ell\alpha}$'s and the modulus of elements of the PMNS matrix:

$$|U_{\ell\alpha}|^2 = \frac{1}{4}(1 + a_{\ell\alpha}). \tag{14}$$

On the one hand, this relation shows that each $a_{\ell\alpha}$ is real. Secondly, it tells us that we can compute $a_{\ell\alpha}$ in order to find $|U_{\ell\alpha}|^2$.

How many of these $|U_{\ell\alpha}|^2$ values can be calculated? It depends on how many T-type and how many S'-type generators are there in our symmetry group. Equation (12) tells us that the maximum number of independent generators of each type is two. Thus, only a few possibilities can arise. We present these in a tabular form in table 1.

Table 1. Number of generators of the group vs. the number of elements of the PMNS matrix whose modulus can be determined from symmetry.

No. of generators		From	Using
T-type	S-type	eq. (14)	unitarity
1	1	1	
2	1	2 in a column	Full column
1	2	2 in a row	Full row
2	2	4 in a block	Full matrix

Note that unitarity of the PMNS matrix can be used to determine the modulus of one element if the others in the same row or the same column are known. This property has been used in writing the final column of the table. Our next task is to examine whether these values can be consistent with experimental bounds on the PMNS matrix.

Suppose the order of the group element $T_\ell S'_\alpha$ is $p_{\ell\alpha}$. Eigenvalues are of the form $\exp(2\pi i k/p_{\ell\alpha})$, for integer k. The trace is the sum of three such eigenvalues. Given that the determinant has been assumed to be 1 and that trace is real, the only possible solution is of the form

$$a_{\ell\alpha} = 1 + \left[\exp\left(2\pi i \frac{k_{\ell\alpha}}{p_{\ell\alpha}}\right) + \exp\left(-2\pi i \frac{k_{\ell\alpha}}{p_{\ell\alpha}}\right)\right]. \tag{15}$$

Then

$$|U_{\ell\alpha}|^2 = \frac{1}{2}\left[1 + \cos\left(2\pi \frac{k_{\ell\alpha}}{p_{\ell\alpha}}\right)\right] = \cos^2\left(\pi \frac{k_{\ell\alpha}}{p_{\ell\alpha}}\right). \tag{16}$$

For values of $p_{\ell\alpha}$ up to 5, we tabulate the combinations of $k_{\ell\alpha}$ and $p_{\ell\alpha}$ that give values which are consistent with experimental bounds. The result is shown in table 2, where we have taken

$$0 \le k_{\ell\alpha} \le \frac{1}{2} p_{\ell\alpha}, \quad \gcd(k_{\ell\alpha}, p_{\ell\alpha}) = 1, \tag{17}$$

Table 2. Possible values of the matrix elements consistent with experimental limits.

$p_{\ell\alpha}$	$k_{\ell\alpha}$	$\|U_{\ell\alpha}\|^2$	$\|U_{\ell\alpha}\|^2$ in range
Any	0	1	None
2	1	0	None
3	1	$\frac{1}{4}$	$e2, \mu 1, \mu 2, \tau 1, \tau 2$
4	1	$\frac{1}{2}$	$\mu 2, \mu 3, \tau 2, \tau 3$
5	1	$\frac{1}{8}(3+\sqrt{5})$	$e1$
5	2	$\frac{1}{8}(3-\sqrt{5})$	$\mu 1, \tau 1$

Table 3. Solutions of eq. (19). The order of the three numbers can be arbitrary.

N	$\{n_1, n_2, n_3\}$	Values of $\lvert U_{\ell\alpha}\rvert^2$
12	$\{3, 4, 4\}$	$\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$
15	$\{3, 5, 6\}$	$\{\frac{1}{8}(3+\sqrt{5}), \frac{1}{4}, \frac{1}{8}(3-\sqrt{5})\}$

because violating any of these conditions will not give any new value for $|U_{\ell\alpha}|$.

If the group has more than two generators, there are extra constraints coming from unitarity. The reason is the following. In this case, there must be more than one generator of one kind – either S'-type or T-type, or both. If there are two S' generators, say S'_1 and S'_2, then S'_3 is also a group element because it is the product of S'_1 and S'_2 through eq. (12). Thus, all three elements of one row (corresponding to the T generator) will have the form described in eq. (16). Unitarity would require

$$\sum_{\alpha=1,2,3} \cos^2\left(\pi \frac{k_{\ell\alpha}}{p_{\ell\alpha}}\right) = 1. \tag{18}$$

Two T-type generators would imply a similar relation for elements of a column. Both kinds of equations can be written in the form

$$\cos^2\left(\frac{\pi n_1}{N}\right) + \cos^2\left(\frac{\pi n_2}{N}\right) + \cos^2\left(\frac{\pi n_3}{N}\right) = 1. \tag{19}$$

This is therefore an extra condition that needs to be satisfied, viz., we need to find three rational fractional multiples of π whose cosine-squared values would add up to unity. As it turns out, there are only two types of solutions to this equation, which are presented in table 3.

3. Searching finite Coxeter groups

With this background, let us first see whether we can get information about the PMNS matrix from finite Coxeter groups. A Coxeter group is a group that has only involution generators, and is completely determined by specifying the order of binary products of the generators. All finite Coxeter groups are known, and those with four or fewer generators are shown in figure 1.

In showing these groups, we have used a graphical notation that is known as Coxeter diagrams. Here, each involution generator is represented by a blob. The blobs corresponding to two generators r_1 and r_2 are not connected by a link if $r_1 r_2$ has order 2, i.e., $(r_1 r_2)^2 = 1$. If the order of $r_1 r_2$ is bigger than 2, there is a line connecting the two blobs. If the order of $r_1 r_2$ is larger than 3, the order is written next to the link. If nothing is written next to the line, it implies that the order of the product is 3. Note that we have not shown some groups which do

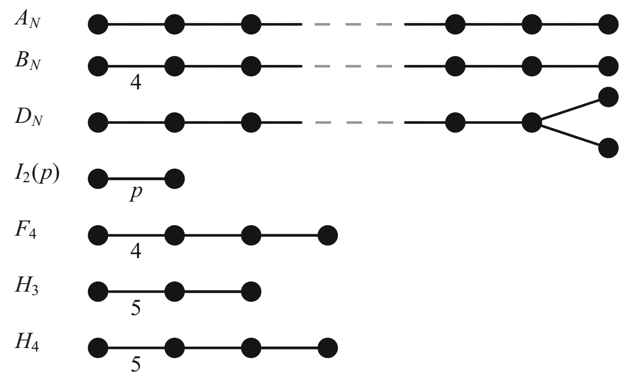

Figure 1. Coxeter diagrams for finite Coxeter groups. The notation has been explained in the text. We have not drawn groups with more than four generators, as well as groups which are products of the groups shown here.

not occur in chains of arbitrary length and which have more than four generators. Also, note that there are some groups which have been counted more than once in this figure, viz.,

$$I_2(3) = D_2 = A_2, \quad I_2(4) = B_2, \quad D_3 = A_3. \tag{20}$$

While looking for viable groups from the diagram, we need to follow a few general guidelines:

- Any two S' generators commute. So two S' generators cannot be connected by a line.
- Same for two T generators.
- The group must have a three-dimensional irreducible representation. Otherwise, the PMNS matrix will turn out to be block-diagonal, with zeroes in some off-diagonal places, which is not allowed by experimental data.

Consider first the groups with two generators. By eq. (20), all these groups are of the I_2 type. These groups do not have any three-dimensional irreducible representation, and are therefore not allowed.

It turns out that there is also no solution for groups with four generators. The reason is the following. Looking at figure 1, we see that if we assign any blob to a T-type generator and the other T-type generator to an unlinked blob, we cannot put the two S'-type generators at blobs which will be connected to both T-blobs. In other words, there will always be at least one TS' pair that is unconnected by a line. According to our notation, that pair will have $p = 2$. But with $p = 2$, there is no acceptable solution, as seen from table 2.

With three generators, however, there are acceptable solutions, and these are shown in table 4. As commented earlier, with three generators one can predict the absolute values of at most one row or one column of the PMNS matrix.

Table 4. Acceptable solutions with 3-generator Coxeter groups. The third column shows the value of N given in table 3 which has been used, and the fourth column shows which row (R) or column (C) is determined as a result of the exercise.

Group	Generators	N	Determined	Values
A_3	$\{S'_1, T_\mu, S'_2\}$	12	R2	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{S'_2, T_\mu, S'_1\}$	12	R2	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{S'_1, T_\tau, S'_2\}$	12	R3	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{S'_2, T_\tau, S'_1\}$	12	R3	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{T_e, S'_2, T_\mu\}$	12	C2	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{T_\mu, S'_2, T_e\}$	12	C2	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{T_e, S'_2, T_\tau\}$	12	C2	$\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$
	$\{T_\tau, S'_2, T_e\}$	12	C2	$\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$
B_3	$\{T_\mu, S'_2, T_\tau\}$	12	C2	$\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$
	$\{T_\tau, S'_2, T_\mu\}$	12	C2	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{S'_3, T_\mu, S'_1\}$	12	R2	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{S'_3, T_\mu, S'_2\}$	12	R2	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{S'_3, T_\tau, S'_1\}$	12	R3	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{S'_3, T_\tau, S'_2\}$	12	R3	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
	$\{T_\mu, S'_2, T_e\}$	12	C2	$\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$
	$\{T_\tau, S'_2, T_e\}$	12	C2	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
H_3	$\{T_e, S'_1, T_\mu\}$	15	C1	$\frac{3+\sqrt{5}}{8}, \frac{1}{4}, \frac{3-\sqrt{5}}{8}$
	$\{T_e, S'_1, T_\tau\}$	15	C1	$\frac{3+\sqrt{5}}{8}, \frac{3-\sqrt{5}}{8}, \frac{1}{4}$
	$\{T_\mu, S'_1, T_\tau\}$	15	C1	$\frac{3+\sqrt{5}}{8}, \frac{3-\sqrt{5}}{8}, \frac{1}{4}$
	$\{T_\tau, S'_1, T_\mu\}$	15	C1	$\frac{3+\sqrt{5}}{8}, \frac{1}{4}, \frac{3-\sqrt{5}}{8}$

4. Other groups generated by involutions

It would be nice if we can have a group that can predict the absolute values of all elements of the PMNS matrix. With finite Coxeter groups, we saw that it is not possible. Therefore, we now go beyond these groups. After all, the machinery developed in §2 applies for any group which can be generated only by involutions, and can be applied to groups other than finite Coxeter groups. So we now search for 4-generator groups outside the realm of finite Coxeter groups to see whether any group can give us the full PMNS matrix. Such groups might even be infinite, in which case we try to see whether there is a finite subgroup of it that can fit the bill.

We make a search using the GAP repository of finite groups [4]. We start with a presentation with four involution generators, along with arbitrary integer parameters that denote the order of the products of various elements, including products of three or more elements. Then we take values of these parameters consistent with the constraints mentioned in table 3, and with k/p values given by the allowed entries of table 2. Since only the value of the ratio k/p is important, we extend the search to ratios of the form ka/pa, with values of a from 1 to 3. In each case, we tried to see whether the resulting group has a subgroup with cardinality less than 3000. The results are summarized in table 5. As one sees, we find only one solution that has three-dimensional irreducible representation.

Table 5. Search for solutions outside finite Coxeter groups. The cardinality and the serial Id specifies a finite group uniquely in the GAP repository.

Type of	Finite subgroup		3-d irrep?
PMNS matrix	Cardinality	Serial Id	
12 only	12	4	No
	108	17	No
	576	8654	No
15 only	(No solution)		
12 & 15	1080	260	Yes

5. Comments and warnings

We thus see that it is very difficult to assign involution generators to flavour symmetries and as a result, obtain the entire PMNS matrix. This statement can be interpreted in two different ways. In one way, one can say that the success rate is very small, and the search is useless. In another way, one can say that, if we assume that the involution generators correspond to flavour symmetries, the search can be narrowed down so that we have only very few alternatives left, which means that we are close to the end of the tunnel.

In case of a negative outcome, there is also a warning worth remembering. If we discard a certain combination of generators from being the generator of a group that governs the PMNS matrix, it does not mean that the group itself is discarded. For a given group, the generators can be chosen in many ways. We have discussed only involution generators. For any group that we have discarded, it is perfectly possible that the same group, with some other choice of generators which are not involutions, gives acceptable entries for the PMNS matrix.

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