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An efficient algorithm for computation of solitary wave solutions to nonlinear differential equations

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Abstract. Nonlinear mathematical problems and their solutions attain much attention in solitary waves. In soliton theory, an efficient tool to attain various types of soliton solutions is the $\exp(-\varphi(\zeta))$-expansion technique. This article is devoted to find exact travelling wave solutions of Drinfeld–Sokolov equation via a reliable mathematical technique. By using the proposed technique, we attain soliton wave solution of various types. It is observed that the technique under discussion is user friendly with minimum computational work, and can be extended for physical problems of different nature in mathematical physics.

Keywords. Exp$(-\varphi(\zeta))$-expansion technique; Drinfeld–Sokolov equation; homogeneous principle; exact and travelling wave solutions.

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1. Introduction

In the last few years, we have observed that soliton theory has attracted a great deal of attention. Solitons have been studied by various mathematicians, physicists and engineers because of their applications in physical phenomena. Soliton waves are first observed by an engineer John Scott Russell. Various phenomena in mathematics and physics are modelled by differential equations. In nonlinear science it is of great importance and interest to explain physical models and attain analytical solutions. In the recent past, large series of chemical, biological, physical singularities are feint by nonlinear partial differential equations. At present, prominent and valuable progress are made in the field of physical sciences. Development of various techniques to search for exact solitary wave solutions of nonlinear differential equations is a great achievement. In nonlinear physical sciences, an essential contribution is of exact solutions and because of this we can study physical behaviours and discus more features of the problem which give direction to more applications.

The search for solutions of nonlinear evolution equations (NLEEs) in the form of travelling wave solutions has attracted lots of attention by scientists in the field of nonlinear science. Obviously, it is very difficult to solve. In addition, there is no general procedure which is applicable for all such equations. So each equation has to be studied by considering it as an individual problem. For this goal, many novel methods for the detection of exact travelling wave solutions of NLEEs have been drawn in huge combinations by a large number of experts. As a result, a lot of work has been done in formulating several convincing and significant techniques. Different researchers apply these techniques such as the homogeneous balance method [1,2], Hirota's bilinear transformation method [3,4], auxiliary equation method [5], trial function method [6,7], tanh-function method [8–10], homotopy perturbation method [11], sine–cosine method [12,13], truncated Painleve expansion method [14], variational iteration method [15–17], exp-function method [18,19], (G'/G)-expansion method [20–25], improved (G'/G)-expansion method [26], $\exp(-\varphi(\zeta))$-expansion method [27–29] etc. on mathematical and physical models. For recent results on exact solutions and integral transform methods, see refs [30–34].

It is very interesting to find soliton-like solutions of nonlinear Drinfeld–Sokolov equation, by applying a novel technique. The applications of nonlinear equation under study are very vast in different areas of physical sciences and engineering. Additionally, such types of equations are found in different physical phenomena related to fluid mechanics, astrophysics, solid-state physics, chemical kinematics, ion-acoustic waves in plasma, control and optimization theory, nonlinear optics etc.

In this work, our elementary incentive is the application of the much reliable and effective technique known in literature as $\exp(-\varphi(\zeta))$-expansion method to attain soliton-like solutions of nonlinear Drinfeld–Sokolov equation. The soliton solution method under study is quite compatible and user friendly for such nonlinear problems. The obtained results are very encouraging. The procedure of this technique is quite simple, explicit, and can easily be extended to all types of NLEEs. The straightforward emphasis of our technique is that the obtained solutions of partial differential equation are expressed in the form of polynomial in $\exp(-\varphi(\zeta))$, where $(\varphi(\zeta))$ must satisfy the ordinary differential equation:

$$\varphi'(\zeta) = \exp(-\varphi(\zeta)) + \mu \exp(\varphi(\zeta)) + \lambda, \tag{1.1}$$

where $\zeta = x + y + z - \omega t$.

The degree of the polynomial is found by the homogeneous principle. By balancing the highest-order derivative with nonlinear term we attain a set of algebraic equations. These algebraic equations are solved to obtain coefficients of the polynomial. This article is divided into different sections. In the next section, we give the analysis of the method used to attain soliton wave solutions. Section 3 is devoted to the application of $\exp(-\varphi(\zeta))$-expansion technique. Results and discussion are given in §4 and §5 draws some conclusions.

2. Analysis of the technique

Nonlinear partial differential equation in general form is

$$P(\eta, \eta_x, \eta_y, \eta_z, \eta_{xx}, \eta_{xy}, \eta_{xz}, \ldots) = 0. \tag{2.1}$$

In the above, $\eta(x, y, z, t)$ is the unknown function, P is the polynomial in $\eta(x, y, z, t)$ and different derivatives of $\eta(x, y, z, t)$ involving nonlinear terms and highest-order differential. We want to attain soliton wave solutions of the given differential equation by $\exp(-\varphi(\zeta))$-expansion technique. For this, we follow steps given below in detail:

Step 1: Assume the transformation:

$$\eta(x, y, z, t) = \eta(\zeta), \quad \zeta = x + y + z - \omega t. \tag{2.2}$$

Here ω represents the wave speed. Using eq. (2.2) in (2.1) we obtained an ordinary differential equation.

$$Q(\eta, \eta', \eta'', \eta''', \ldots) = 0. \tag{2.3}$$

In (2.3) derivative with respect to ζ is symbolized by prime. We integrate the above equation one or more than one times depending on the situation, and ignore constant of integration.

Step 2: Assume that the solution of eq. (2.3) is articulated in the form of polynomial in $\exp(-\varphi(\zeta))$ as

$$\eta(\zeta) = \alpha_n(\exp(-\zeta))^n + \alpha_{n-1}(\exp(-\zeta))^{n-1} + \cdots. \tag{2.4}$$

In the above equation α_n, α_{n-1} are the constants which are to be calculated in such a way that $\alpha_n \neq 0$ and $\varphi(\zeta)$ satisfies (1.1).

Step 3: To obtain the value of integer n we apply the homogeneous principle. By balancing the highest-order linear and the highest-order nonlinear term involved in (2.3), we calculate the value of n. The solution depends on different parameters involved in eq. (1.1).

Case 1: $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$,

$$\varphi(\zeta) = \ln\left\{\frac{1}{2\mu}\left(-\sqrt{\lambda^2 - 4\mu} \times \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\zeta + c_1)\right) - \lambda\right)\right\}, \tag{2.5}$$

where c_1 is a constant of integration.

Case 2: $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$,

$$\varphi(\zeta) = \ln\left\{\frac{1}{2\mu}\left(-\lambda + \sqrt{4\mu - \lambda^2} \times \left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\zeta + c_1)\right)\right)\right\}. \tag{2.6}$$

Case 3: $\lambda \neq 0$ and $\mu = 0$,

$$\varphi(\zeta) = -\ln\left\{\frac{\lambda}{\exp(\lambda(\zeta + c_1)) - 1}\right\}. \tag{2.7}$$

Case 4: $\lambda^2 - 4\mu = 0$ and $\mu \neq 0, \lambda \neq 0$,

$$\varphi(\zeta) = \ln\left\{\frac{2(\lambda(\zeta + c_1)) + 2}{\lambda^2(\zeta + c_1)}\right\}. \tag{2.8}$$

Case 5: $\lambda = 0$ and $\mu = 0$,

$$\varphi(\zeta) = \ln(\zeta + c_1). \tag{2.9}$$

Step 4: By inserting (2.4) into (2.3) and then using (1.1), we convert the left-hand side in the form of a polynomial in $\exp(-\varphi(\zeta))$. We equate each coefficient of this polynomial to zero, and get a set of algebraic equations for $\alpha_n, \ldots, \omega, \lambda$ and μ.

Step 5: In this step by using Maple 13 the algebraic equations are solved. We attain the values for the constants $\alpha_n, \ldots, \omega, \lambda$ and μ.

Substituting the values of constants, the solution of (1.1) into (2.4), we obtain some useful travelling wave solutions of (2.1).

3. Solution procedure

In this section, we apply $\exp(-\varphi(\zeta))$-expansion technique to construct solitary wave solution of nonlinear Drinfeld–Sokolov equation. The obtained results are very efficient and encouraging.

Let us assume the nonlinear Drinfeld–Sokolov equation

$$\eta_t + (v^2)_x = 0, \tag{3.1}$$
$$v_t - a v_{xxx} + 3b\eta_x v + 3k\eta v_x = 0. \tag{3.2}$$

Invoking:

$$\zeta = x - \omega t,$$
$$-\omega\eta' + (v^2)' = 0, \tag{3.3}$$
$$-\omega v' - a v''' + 3b\eta' v + 3k\eta v' = 0. \tag{3.4}$$

Integrating eq. (3.3) and neglecting constant of integration, we obtain

$$\omega\eta = v^2. \tag{3.5}$$

Using eq. (3.5) in (3.4) and then integrating once, we get

$$\omega^2 v(\zeta) - (2b+k)v^3(\zeta) + a\omega v'' = 0. \tag{3.6}$$

Using the homogeneous principle,

$$3M = M + 2,$$
$$M = 1.$$

Equation (2.4) reduces to

$$\eta(\zeta) = \alpha_0 + \alpha_1 e^{-\varphi(\zeta)}. \tag{3.7}$$

Here in eq. (3.7) α_0 and α_1 are the constants which are to be calculated.

We insert (3.7) into (3.6) and gather all those terms which have the like power of $e^{-\varphi(\zeta)}$. As a result, left-hand side is converted into the form of polynomial. By equating all coefficients of this polynomial to zero, we attain algebraic equations for $\alpha_0, \alpha_1, \lambda, \eta$ and ω given below:

$$\omega^2\alpha_1 + 2a\omega\alpha_1\mu + a\omega\alpha_1\lambda^2 - 6b\alpha_0^2\alpha_1 - 3k\alpha_0^2\alpha_1 = 0, \tag{3.8}$$
$$3a\omega\alpha_1\lambda - 6b\alpha_1^2\alpha_0 - 3k\alpha_1^2\alpha_0 = 0, \tag{3.9}$$
$$2a\omega\alpha_1 - 2b\alpha_1^3 - k\alpha_1^3 = 0, \tag{3.10}$$
$$\omega^2\alpha_0 + a\omega\alpha_1\mu\lambda - 2b\alpha_0^3 - k\alpha_0^3 = 0. \tag{3.11}$$

Solving the set of simultaneous algebraic equations with the help of Maple 13, we obtain the following solution set:

$$\omega = -\frac{1}{2}a(4\mu - \lambda^2),$$
$$\alpha_0 = \frac{1}{2}\frac{\sqrt{-(2b+k)(4\mu-\lambda^2)a}\,\lambda}{2b+k},$$
$$\alpha_1 = \frac{\sqrt{-(2b+k)(4\mu-\lambda^2)a}}{2b+k}, \tag{3.12}$$

where λ and μ are arbitrary constants.

By using (3.12) in (3.7), we obtain

$$\eta(\zeta) = \alpha_0 + 2e^{-\varphi(\zeta)}, \tag{3.13}$$

where

$$\zeta = x + y - Vt.$$

Substituting the solutions of (1.1) in (3.13), we get five cases of travelling wave solutions for the Drinfeld–Sokolov equation.

Case 1: When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we obtain the hyperbolic function travelling wave solution

$$\eta_1 = \frac{a}{4b+2k}\left(\lambda + \frac{4\mu}{-\sqrt{\lambda^2-4\mu}\tanh(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\zeta t+c_1)) - \lambda}\right)^2, \tag{3.14}$$
$$v_1 = \frac{a\sqrt{-(2b+k)(4\mu-\lambda^2)}}{4b+2k}\left(\lambda + \frac{4\mu}{-\sqrt{\lambda^2-4\mu}\tanh(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\zeta t+c_1)) - \lambda}\right). \tag{3.15}$$

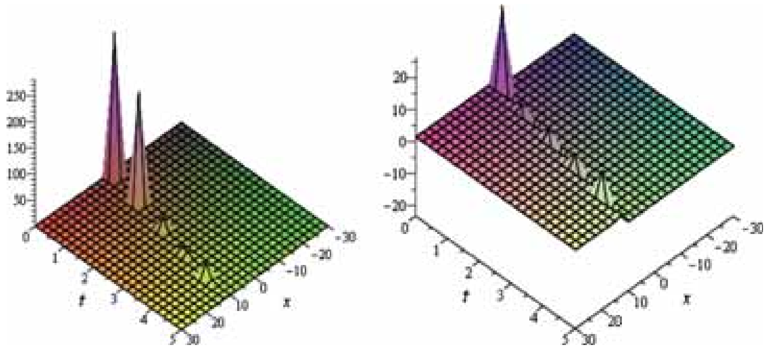

Figure 1. Hyperbolic, trigonometric function travelling wave solution and exponential solution respectively when $\lambda = -1, \mu = -1, c_1 = 1, a = 1, b = 1, k = 1$.

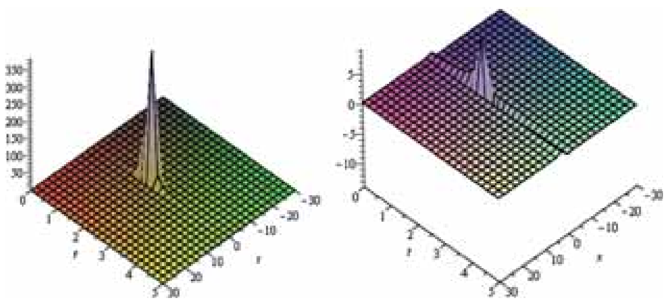

Figure 2. Exponential function travelling wave solution when $\lambda = -1, \mu = -1, c_1 = 1, a = 1, b = 1, k = 1$.

Case 2: When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we obtain the trigonometric solution

$$\eta_2 = \frac{a}{4b+2k}\left(\lambda + \frac{4\mu}{-\lambda + \sqrt{4\mu - \lambda^2}\tanh(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,(\zeta t + c_1)) - \lambda}\right)^2, \tag{3.16}$$

$$v_2 = \frac{a\sqrt{-(2b+k)\,(4\mu - \lambda^2)}}{4b+2k}\left(\lambda + \frac{4\mu}{-\lambda + \sqrt{4\mu - \lambda^2}\tanh(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,(\zeta t + c_1)) - \lambda}\right). \tag{3.17}$$

Case 3: When $\mu = 0$ and $\lambda \neq 0$, we obtain the exponential solution

$$\eta_3 = \frac{a}{4b+2k}\left(\lambda + \frac{2\mu}{e^{\lambda}(\zeta t + c_1) - 1}\right)^2, \tag{3.18}$$

$$v_3 = \frac{a\sqrt{(2b+k)\lambda^2}}{4b+2k}\left(\lambda + \frac{2\mu}{e^{\lambda}(\zeta t + c_1) - 1}\right). \tag{3.19}$$

Case 4: When $\lambda^2 - 4\mu = 0$, $\mu = 0$ and $\lambda \neq 0$, we obtain the rational function solution

$$\eta_4 = \frac{a\lambda^2}{\left(b + \frac{k}{2}\right)(\lambda(\zeta t + c_1) + 2)^2}, \tag{3.20}$$

$$v_4 = \frac{a\lambda^2\sqrt{-(2b+k)\,(4\mu - \lambda^2)}}{\left(b + \frac{k}{2}\right)(\lambda(\zeta t + c_1) + 2)^2}. \tag{3.21}$$

Case 5: When $\mu = 0$ and $\lambda = 0$, we obtain the rational function solution

$$\eta_5 = \frac{2a}{(\zeta + c_1)^2\,(2b+k)}, \tag{3.22}$$

$$v_5 = 0, \tag{3.23}$$

where

$$\zeta = x - \left(-\frac{1}{2}a(4\mu - \lambda^2)\right).$$

4. Results and discussion

From the above figures, we note that soliton is a wave which preserve its shape after it collides with another wave of the same kind. By solving Drinfeld–Sokolov equation, we attain the desired soliton wave solutions. The solitary wave moves towards right if the velocity is positive or towards left if the velocity is negative and

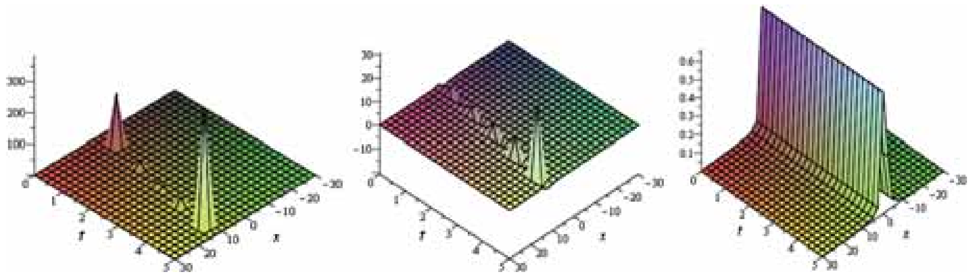

Figure 3. Rational function travelling wave solution when $\lambda = -1, \mu = -1, c_1 = 1, a = 1, b = 1, k = 1$.

the amplitudes and velocities are controlled by various parameters. Solitary waves show more complicated behaviours which are controlled by various parameters. Figure 1 represents the trigonometric, hyperbolic function travelling wave solution and exponential solution respectively when $\lambda = -1, \mu = -1, c_1 = 1, a = 1, b = 1, k = 1$. The exponential function travelling wave solutions are shown in figure 2 when $\lambda = -1, \mu = -1, c_1 = 1, a = 1, b = 1, k = 1$. We also obtain rational function travelling wave solutions of nonlinear Drinfeld–Sokolov equation in figure 3 when $\lambda = -1, \mu = -1, c_1 = 1, a = 1, b = 1, k = 1$. Since the solutions depend on arbitrary functions, we choose different parameters as inputs to our simulations.

5. Conclusion

In this paper, our main focus is to find, test and analyse the novel soliton wave solutions and physical properties of nonlinear partial differential equation. For this, Drinfeld–Sokolov equation is considered and we applied $\exp(-\varphi(\zeta))$-expansion method. We attained desired solutions through rational, trigonometric, exponential, and hyperbolic functions. The accuracy of the attained results is guaranteed by backward substitution into the original equation with Maple 13. The scheming procedure of this method is simple, straight and productive. We observed that the technique under study is more reliable and has minimum computational task, and so widely applicable. We can say that this method is quite competent for evaluating exact solution of NLEEs. Results obtained by this method are very encouraging and reliable for solving any other type of NLEEs. The graphical representations clearly indicate solitary solutions.

References

[1] M Wang, *Phys. Lett. A* **199**, 165 (1995)
[2] E M E Zayed, H A Zedan and K A Gepreel, *Chaos, Solitons and Fractals* **22**, 285 (2004)
[3] R Hirota, *J. Math. Phys*. **14**, 805 (1973)
[4] R Hirota and J Satsuma, *Phys. Lett. A* **85**, 407 (1981)
[5] Sirendaoreji, *Chaos, Solitons and Fractals***19**, 147 (2004)
[6] M Inc and D J Evans, *Int. J. Comput. Math*. **81**, 191 (2004)
[7] A T Ali, *J. Comput. Appl. Math.* **235**, 4117 (2011)
[8] M A Abdou, *Appl. Math. Comput*. **190**, 988 (2007)
[9] E G Fan, *Phys. Lett. A* **277**, 212 (2000)
[10] W Malfliet, *Am. J. Phys*. **60**, 650 (1992)
[11] S T Mohyud-Din, A Yildirim and S Sariaydin, *Int. J. Numer. Method H* **21**, 822 (2011)
[12] A M Wazwaz, *Math. Comp. Model*. **40**, 499 (2004)
[13] S Bibi and S T Mohyud-Din, *J. Assoc. Arab Univ. Basic Appl. Sci.* **15**, 90 (2014)
[14] J Weiss, M Tabor and G Carnevale, *J. Math. Phys.* **24**, 522 (1983)
[15] J H He, *Commun. Nonlin. Sci. Numer. Simulat*. **2**, 235 (1997)
[16] M A Abdou and A A Soliman, *Physica D: Nonlinear Phenomena* **211**, 1 (2005)
[17] S Abbasbandy, *Int. J. Numer. Methods in Engg*. **70**, 876 (2007)
[18] J H He and X H Wu, *Chaos, Solitons and Fractals* **30**, 700 (2006)
[19] M A Akbar and N H M Ali, *World Appl. Sci. J.* **17**, 1603 (2012)
[20] M Wang, X Li and J Zhang, *Phys. Lett. A* **372**, 417 (2008)
[21] M A Akbar, N H M Ali and E M E Zayed, *Comm. Theor. Phys.* **57**, 173 (2012)
[22] M A Akbar, N H M Ali and S T Mohyud-Din, *World Appl. Sci. J*. **16**, 1551 (2012)
[23] E M E Zayed, *J. Appl. Math. Informat*. **28**, 383 (2010)
[24] M A Akbar and N H M Ali, *Int. J. Phys. Sci*. **6**, 7910 (2011)
[25] A R Shehata, *Appl. Math.Comput.* **217**, 1 (2010)
[26] J Zhang, F Jiang and X Zhao, *Int. J. Comput. Math.* **87**, 1716 (2010)
[27] K Khan and M A Akbar, *World Appl. Sci. J.* **24**, 1373 (2013)
[28] W Liping, C Senfa and P Chunping, *Appl. Math. Model*. **33**, 4126 (2009)
[29] F Zhang, J M Qi and W J Yuan, *J. Appl. Math.* **2013**, 523732 (2013)

[30] X J Yang, J A Tenreiro Machado, D Baleanu and C Cattani, *Chaos* **26**, 084312 (2016)
[31] J M Rodrguez, *Comput. Math. Appl*. **73**, 203 (2017)
[32] X J Yang, *Appl. Math. Lett.* **64**, 193 (2017)
[33] X J Yang and G Feng, *Therm. Sci*. **21**, 133 (2017)
[34] X J Yang, *Therm. Sci*. **20**, 639 (2016)