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Spatiotemporal soliton clusters in the (3+1)-dimensional nonlinear Schrödinger equation with spatially modulated nonlinearity

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Abstract. From a generic transformation, a (3+1)-dimensional nonlinear Schrödinger equation with spatially modulated nonlinearity is studied and exact spatiotemporal soliton cluster solutions are derived. When the azimuthal parameter $m = 0$, Gaussian solitons are constructed. For the modulation depth $q = 1$ and the azimuthal parameter $m \neq 0$, vortex solitons are obtained. In the limit $q = 0$ and $m \neq 0$, and for some specific values of the parameters (m, l), multipole solitons are presented.

Keywords. Nonlinear Schrödinger equation; spatially modulated nonlinearity; Gaussian soliton; vortex soliton; multipole soliton.

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1. Introduction

Solitons, localized in space and stable and sustainable in time [1,2], exist in many real systems of engineering, physics and biology [3–5]. They exhibit various self-sustained modes such as spatial soliton [6], spatiotemporal soliton [7,8], rogue wave [9,10] and semifoldon [11]. In recent years, spatiotemporal localizations in highly nonlocal nonlinear media have also been studied. Necklace soliton [12], multipole soliton [13], Laguerre–Gaussian soliton cluster [14] and Hermite–Gaussian soliton [15] have been reported in highly nonlocal nonlinear media. However, these discussions on localizations are based on linear Schrödinger equation without considering local nonlinearities.

In local Kerr media, the governing equation for solitons is the nonlinear Schrödinger equation (NLSE). Based on NLSE, does there exist these localized structures which are similarly constructed in highly nonlocal nonlinear media in refs [12–15]? Recently, using self-focussing spatiotemporally modulated nonlinearity, the infinite number of exact (1+1)-dimensional solitons, such as resonant and breathing ones were constructed [16]. Exact (1+1)-dimensional soliton solutions of the NLSE with spatiotemporally modulated nonlinearity have also been studied [17]. More recently, it is reported that spatially modulated nonlinearity makes two-dimensional vortex soliton stable [18]. A question arises: Does spatially modulated nonlinearity play a role in the formation of three-dimensional spatiotemporal localizations in local Kerr nonlinearity? Our present work provides a detailed answer to these questions. Based on spatiotemporal soliton solutions of a (3+1)-dimensional NLSE with spatially modulation, we study the evolutional behaviours of Gaussian soliton, vortex soliton and multipole soliton.

2. Exact spatiotemporal soliton solutions

Upon standard rescaling, the respective NLSE takes the form of

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\nabla^2\psi + g(r)|\psi|^2\psi + V(r)\psi, \tag{1}$$

where ψ is the macroscopic wave function of the condensate, the 3D Laplacian

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},$$

and the coefficient of the inhomogeneous nonlinearity $g(r)$, as well as the external potential $V(r)$, are assumed to be functions of radial coordinate r.

Assuming the stationary wave function as

$$\psi(r,\theta,\varphi,t) = A(r)Y(\theta,\varphi)\exp(-i\mu t), \tag{2}$$

where μ is the chemical potential, and inserting eq. (2) into eq. (1), one obtains

$$\frac{r^2}{A}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right) + 2[\mu - V(r)]A - 2g(r)A^3\right\} = l(l+1), \tag{3}$$

$$-\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = l(l+1), \tag{4}$$

with the self-consistency condition $|Y(\theta,\varphi)|^2 = 1$. Equation (4) admits solution in spherical harmonics $Y(\theta,\varphi) = C\phi_m(\varphi)P_l^m(\cos\theta)$, where $C = \sqrt{[(2l+1)(l-m)!]/[2\pi(l+m)!]}$ and $P_l^m(\cos\theta)$ are the associated Legendre polynomials with $l \geq m \geq 0$. The azimuthal part of the solution is of the form $\phi_m(\varphi) = (1/\sqrt{1+q^2})[\cos(m\varphi) + \mathrm{i}q\sin(m\varphi)]$, with an azimuthal parameter m and the modulation depth q $(0 \leq q \leq 1)$. Here we use similar approximate procedure for $\phi_m(\varphi)$ by the angular averaging in ref. [19]. This approximation is relevant for weak nonlinearity and for large q, close enough to 1.

Defining $A(r) \equiv \rho(r)U[R(r)], g(r) \equiv g_0 r^{-4}\rho(r)^{-6}$, with $R(r) \equiv \int_0^r [\rho(s)s]^{-2}\mathrm{d}s$, one can find that $\rho(r)$ and $U(R)$ obey the following equations:

$$\rho'' + \frac{2}{r}\rho' + \left[2\mu - 2V(r) - \frac{l(l+1)}{r^2}\right]\rho = \frac{E}{r^4\rho^3}, \tag{5}$$

$$-\frac{\mathrm{d}^2U}{\mathrm{d}R^2} + g_0U^3 = EU, \tag{6}$$

where E and g_0 are constants. The reduction of NLSE to eq. (6) helps one to find exact solutions, as the latter equation is solvable in terms of Jacobi elliptic functions. Then, if a solution to eq. (5) is known, one can construct exact solutions to the underlying NLSE.

We begin constructing exact vortex soliton solutions in the case of attractive nonlinearity ($g_0 < 0$) when $E = 0$, so that eq. (5) is solvable. With the harmonic potential $V = \frac{1}{2}\omega^2r^2$, ρ can be found in terms of the Whittaker's M and W functions [20] as

$$\rho(r) = r^{-3/2}\left[c_1\mathrm{M}\left(\frac{\mu}{2\omega}, \frac{l}{2}+\frac{1}{4}, \omega r^2\right) + c_2\mathrm{W}\left(\frac{\mu}{2\omega}, \frac{l}{2}+\frac{1}{4}, \omega r^2\right)\right], \tag{7}$$

where the restrictions on ρ require $\mu < \mu_0 = 2(1 + |S|)\sqrt{k}$, and $c_1c_2 > 0$. Without the harmonic trap ($\omega = 0$), ρ degenerates to

$$\rho(r) = r^{-1/2}[c_3B_J(\sqrt{-2\mu}r) + c_4B_Y(\sqrt{-2\mu}r)], \tag{8}$$

with $\mu < \mu_0 = 0$, B_J and B_Y being, respectively, the Bessel functions of the first and second kinds, and constants satisfying $c_3c_4 > 0$. To meet the boundary conditions $U(0) = U(\infty) = 0$, the respective exact solution to eq. (6) is chosen as

$$U(R) = (n\eta/\sqrt{-g_0})\,\mathrm{cn}\,(n\eta R - K(\sqrt{2}/2), \sqrt{2}/2), \tag{9}$$

where the radial quantum number $n = 2, 4, 6, \ldots$, $\eta \equiv K(\sqrt{2}/2)/R(\infty)$, and $K(\sqrt{2}/2)$ is the first-kind complete elliptic integral.

3. Dynamics of spatiotemporal solitons

When $m = 0$, one obtains Gaussian solitons. For $l = 0$ the soliton forms a sphere (see figure 1a). For $q = 1$ and $m \neq 0$, one obtains vortex solitons (see figure 2). In the limit $q = 0$ and $m \neq 0$, and for some specific values of the parameters (m, l), we observe necklace (also called multipole) solitons (see figure 3). Similar to the case in highly nonlocal nonlinear media in ref. [15], if $m = 0$, Gaussian solitons can be obtained in figures 1a–1c. Especially, if $l = 0$, a sphere forms. By increasing l, the sphere in figure 1a changes into a torus-shaped structure in figure 1b, and then a pair of drip-shaped structures appear above and below the torus-shaped structure in the middle (see figure 1c).

When $q = 1$ and $m \neq 0$, vortex solitons are as shown in figure 2. These vortex solitons display pair-like structures. For $m = 1$, by increasing l, a pair of drop-shaped ellipsoids shown in figure 2a turns into a pair of torus-shaped structures in figure 2b, and then a pair of drop-shaped structures appear above and below the pair of torus-shaped structures in the middle (see figure 2d). If $m = l$, the pair of drop-shaped ellipsoids can be constructed. When $m = l$, the ellipsoid becomes thinner, and the peak at one end becomes shaper (see figures 2a, 2c and 2f). Similarly, if $l - m$ is the same, vortex solitons possess similar structures as in figures 2b and 2e. Compare figure 2b with figure 2e, the peak at one end becomes shaper by adding values of m and l.

When $q = 0$ and $m \neq 0$, and for some specific values of the parameters (m, l), multipole solitons can be constructed. Some symmetric multipole soliton patterns around the point $(x, y) = (0, 0)$ are shown in figure 3. When $m = l = 1$, a two-layer 'gamopetalous' multipole soliton is formed as in figure 3a. If the value of m is fixed as $m = 1$ and l increases to $l = 2$, the two-layer 'gamopetalous' multipole soliton splits into four separated solitons (see figure 3b). If the value of l further increases, another two-layer 'gamopetalous' multipole

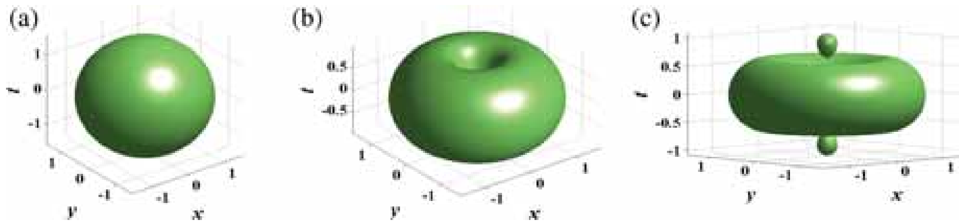

Figure 1. Gaussian solitons for $n = 2, m = 0$ with (**a**) $l = 0$, (**b**) $l = 1$ and (**c**) $l = 2$.

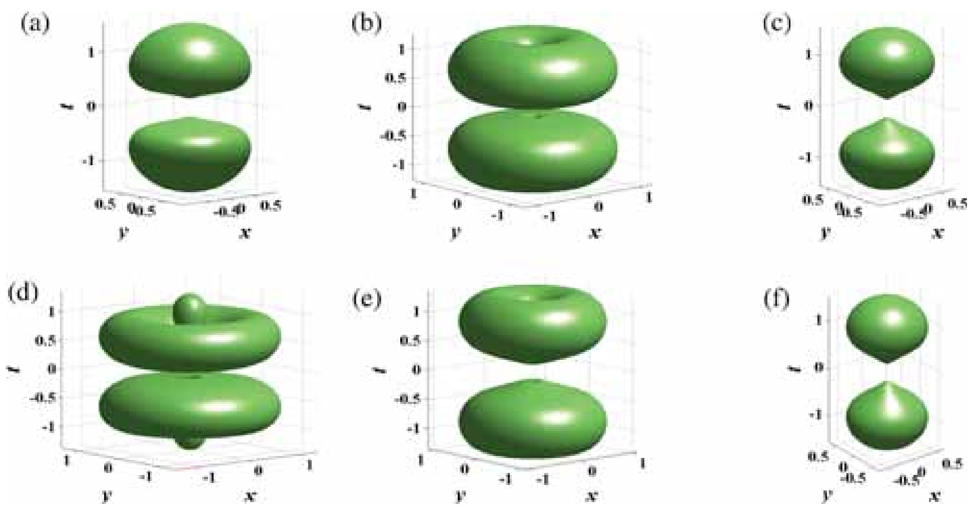

Figure 2. Vortex solitons for $n = 2, q = 1$ with (**a**) $m = l = 1$, (**b**) $m = 1, l = 2$, (**c**) $m = 2, l = 2$, (**d**) $m = 1, l = 3$, (**e**) $m = 2, l = 3$ and (**f**) $m = 3, l = 3$.

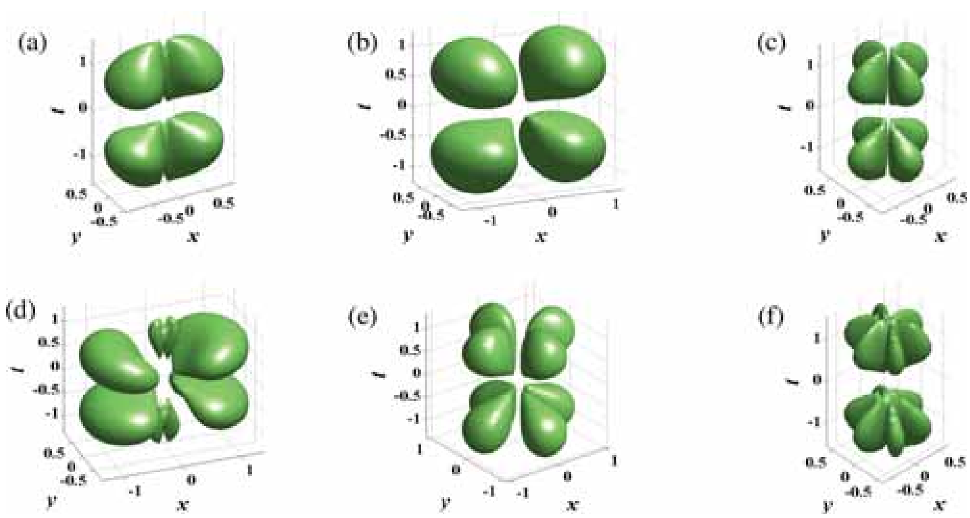

Figure 3. Multipole solitons for $n = 2, q = 0$ with (**a**) $m = l = 1$, (**b**) $m = 1, l = 2$, (**c**) $m = 2, l = 2$, (**d**) $m = 1, l = 3$, (**e**) $m = 2, l = 3$ and (**f**) $m = 3, l = 3$.

soliton appears in the middle of four separated solitons (see figure 3d). If $m = l$, multipole solitons exhibit two-layer 'gamopetalous' structures as in figures 3a, 3c and 3f, and the 'petal' number of 'gamopetalous' structures is determined by $2m$ or $2l$. If $m = 2$, by increasing l, the two-layer 'gamopetalous' multipole soliton with $l = 2$ in figure 3c splits into eight drop-shaped solitons with $l = 3$ in figure 3e.

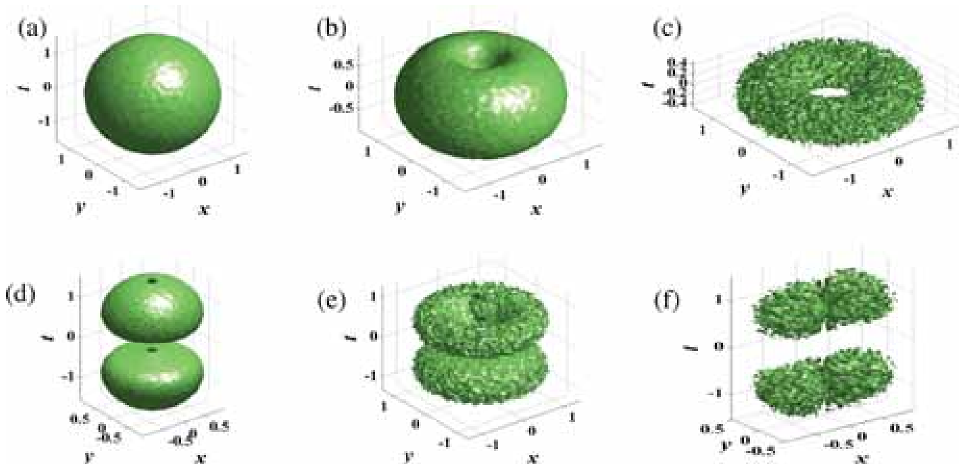

Figure 4. (**a**)–(**f**) Stable and unstable spatiotemporal soliton at $z = 100$ corresponding to figures 1a, 1b, 1c, 2a, 2d and 3a, respectively. An added 5% white noise is added to the initial values. The parameters are the same as those in the corresponding analytical plots.

In real physical contexts, analytical solutions cannot exactly describe real problems. Thus, it is crucial to discuss the stability of analytical solutions against finite perturbations with a 5% white noise, which simulates the influence of fluctuated environment on analytical solutions.

Although some authors reported that the spatiotemporal pulses spread out or collapse [21,22], other authors verified that it is possible to obtain stable spatiotemporal soliton solutions by modulating diffraction/dispersion and the nonlinearity [23,24]. In this paper, we find that the spatially modulated Kerr nonlinearity stabilizes some soliton structures in $(3+1)$ dimensions. We study analytical solutions evolving along distance when they are disturbed from their analytically given forms for eq. (1). We perform a direct numerical simulation by the split-step Fourier method with initial white noise for solution (2) of eq. (1) in some cases. Six examples of such behaviours are displayed in figure 4.

Figures 4a–4c display the numerical rerun of Gaussian solitons corresponding to figure 1 at $z = 100$. As can be seen in figures 4a and 4b, the numerical results do not give any visible instability except for some small oscillations on the surfaces. However, when the value of l increases, Gaussian solitons cannot preserve their original shapes, break up and ultimately decay into noise. Figure 4c shows the unstable evolution of Gaussian soliton with $l = 2$. For vortex solitons, only the lowest-order soliton with $l = m = 1$ is stable (see figure 4d), and all the other vortex solitons are unstable. Figure 4e shows unstable evolution of vortex soliton with $m = 1, l = 3$. For multipole solitons, no stable structures are found. Even the lowest-order multipole soliton with $l = m = 1$ is also unstable in figure 4f, and it distorts and finally collapses into noise.

4. Conclusions

In conclusion, we answer two issues presented in the Introduction, that is,

(i) Spatiotemporal soliton clusters are constructed.

Similar to localized structures in highly nonlocal nonlinear media, we can construct spatiotemporal soliton clusters in local Kerr media based on NLSE. By means of a generic transformation, a $(3+1)$-dimensional NLSE with spatially modulated nonlinearity is studied and exact spatiotemporal soliton solutions are obtained. When the azimuthal parameter $m = 0$, one obtains Gaussian solitons. For the modulation depth $q = 1$ and the azimuthal parameter $m \neq 0$, one obtains vortex solitons. In the limit $q = 0$ and $m \neq 0$, and for some specific values of the parameters (m, l), we observe multipole solitons.

(ii) Spatially modulated nonlinearity plays a role in the formation of spatiotemporal solitons.

The expressions above eq. (5) indicates that spatially modulated nonlinearity $g(r)$ is related to $\rho(r)$, which determines the form factors of spatiotemporal solitons such as amplitude and width.

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