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Charge and/or spin limits for black holes at a non-commutative scale

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Abstract. In the commutative geometrical background, one finds the total charge (Q) and/or the total angular momentum (J) of a generalized black hole of mass M to be bounded by the condition $Q^2 + (J/M)^2 \leq M^2$, whereas the inclusion of the concept of non-commutativity in geometry leads to a much more richer result. It predicts that the upper limit to Q and/or J is not fixed but depends on the mass/length scale of black holes; it (the upper limit to Q and/or J) goes towards a 'commutative limit' when $M \gg \sqrt{\vartheta}$ ($\sqrt{\vartheta}$ characterizes the minimal length scale) and rapidly diminishes towards zero with M decreasing in the strongly non-commutative regime, until approaching a perfect zero value for $M \simeq 1.904\sqrt{\vartheta}$. We have performed separate calculations for a pure Kerr or a pure Reissner–Nordström black hole, and briefly done it for a generalized black hole.

Keywords. Quantum gravity; non-commutative black hole; extremity.

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1. Introduction

A black hole's status at the extreme state of its existence in the classical commutative space–time background is determined by virtue of a hypothesis called the cosmic censorship conjecture. All physically reasonable space–times are globally hyperbolic, i.e. apart from a possible initial singularity (such as the 'big-bang' singularity) no singularity is ever 'visible' to any observer. This cosmic censorship hypothesis was introduced by Penrose in 1969 [1]; a precise formulation of the hypothesis can be found in [2]. A related argument, namely the third law of black-hole thermodynamics, holds that it is impossible to achieve $\kappa = 0$ by a physical process; κ being the surface gravity of the black hole. The basic black-hole thermodynamics may be found in [3] and in references like [2,4] also.

For non-commutative geometry-inspired black holes, there appears no real singularity in space–time (as has been shown by Modesto and Nicolini [5]) and also the space–time is not globally hyperbolic. Qualitatively similar result follows from the other approaches to quantum gravity as well (see Modesto [6–8]). Therefore, cosmic censorship does not run in the usual way. However the cosmic censorship is not merely about a global hyperbolicity or singularity of space–time, but it may also be thought as a statement concerned with the indestructibility of black holes. A black hole survives till there exists an event-horizon in gravity space–time. Therefore, either we may reinterpret the 'cosmic censorship conjecture' of Penrose as to assuring a de-Sitter origin covered always by the horizon(s), or we may easily work with the third law of BH-mechanics in order to have the extremal status of black holes at a seriously non-commutative length scale. Indeed, if we regard the third law of black-hole thermodynamics to be true, then there should never be a situation where the horizons disappear to show up the naked de-Sitter origin; i.e., the surface gravity (κ) would never become a perfect zero. It happens naturally in a commutative geometrical theory of black hole that the third law of black-hole mechanics or in other words the cosmic censorship sets up the limit for maximum allowable angular momentum and/or charge of the black hole [9]. In this paper the same idea will be extended to black holes at non-commutative length scale.

Non-commutative geometry is originally employed to implement the 'fuzziness' of space–time by means of $[x^i, x^j] = i\vartheta^{ij}$, where, in the simplest case, ϑ^{ij} is an antisymmetric, real, 4×4 (4 is the dimension of space–time) matrix, which determines the fundamental cell discretization of space–time much in the same way as the Planck constant $\hbar$ discretizes the phase space. As a consequence, the resulting geometry is 'pointless', because the notion of point is no longer meaningful because of the uncertainty

$$\Delta x^i \Delta x^j \geq \frac{1}{2}|\vartheta^{ij}|$$

induced by the non-commutative behaviour of coordinates. It is to be noted that in a commutative theory of space–time, we assume the distribution of matter–energy to have well-defined values at points in space–time. If we instead believe in the non-commutative framework, then we are required to accept non-exactness of the above-mentioned distribution. Non-commutativity and its spirit in weak gravity [10] inspire one to decide that in the deepest of the length scales, matter distribution does not vary smoothly; instead a significant Gaussian fluctuation around the expected average value is encountered. In the three-dimensional non-commutative manifold, $[x^i, x^j] = i\vartheta\epsilon^{ij}$ (of 4D space–time) matter-distribution is perceived like [11]

$$\rho_m(\vartheta) = \frac{m_0}{(4\pi\vartheta)^{3/2}} \mathrm{e}^{-r^2/4\vartheta}.$$

Similarly, the charge distribution assumes a minimal spread Gaussian profile given by

$$\rho_q(\vartheta) = \frac{Q}{(4\pi\vartheta)^{3/2}} \mathrm{e}^{-r^2/4\vartheta}.$$

In these equations, ϑ is the 'non-commutative parameter' with dimension of a length squared that encodes a minimal length in the manifold (there is a general consensus about the appearance of non-commutativity phenomenology at intermediate energies between the scale of the Standard Model of particle physics and the Planck scale). In effect, the smeared density profiles modify the energy–momentum tensor carrying term in Einstein's equation while not affecting the curvature part at all, and generates new line elements for the non-commutative black holes. This kind of approach to non-commutativity is called the coordinate coherent state formalism [12]. For rigorous details on non-commutative inspired black holes, one may follow refs [5,13–22].

Our principal focus will be on black holes in the non-commutative length scale. It will be observed that the non-commutative nature of space–time causes the 'specific angular momentum' $[J/M]$ of an extremal Kerr black hole to diminish from 1 to 0. The situation is similar for the Reissner–Nordström (RN) black holes; they lose their 'specific charge-carrying capacity', i.e. $[Q/M]$ drops down to 0 with the decreasing mass scale of holes. It seems that as one approaches the characteristic energy or length scale of non-commutativity, the difference between the Schwarzschild space–time and the Kerr or the RN space–time gets less pronounced. At the end, when one reaches the last surviving stage of a black hole, only the non-commutative nature of space–time geometry remains visible in the Schwarzschild structure. A black hole is to become more and more spherically symmetric and electrically neutral as the black hole size enters deeper into the length scale. In this article, we aim to estimate the maximal charge or angular momentum-carrying capacity of a black hole as a function of its mass.

2. Generalized static space–time for non-commutative spinning and charged black hole

The line element for a non-commutativity-inspired generalized black hole (as has been shown by Modesto and Nicolini [5]) runs as

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{\Delta - a^2\sin^2\theta}{\Sigma}\mathrm{d}t^2 - \frac{\Sigma}{\Delta}\mathrm{d}r^2 - \Sigma\mathrm{d}\theta^2 \\ & + 2a\sin^2\theta\left(1 - \frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)\mathrm{d}t\mathrm{d}\phi \\ & - \sin^2\theta\left[\Sigma + a^2\sin^2\theta\left(2 - \frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)\right]\mathrm{d}\phi^2, \end{aligned} \tag{1}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{2}$$

$$\Delta = r^2 - 2m(r)r + q(r)^2 + a^2 \tag{3}$$

with $a \equiv (J/M)$ being the angular momentum per unit mass of the black hole. The quantities $m(r)$ and $q(r)$ are given by

$$m(r) = M\frac{\gamma(3/2, r^2/4\vartheta)}{\Gamma(3/2)}, \tag{4}$$

$$\begin{aligned} q(r)^2 = {} & \frac{Q^2}{\pi}\left[\left\{\gamma^2(1/2, r^2/4\vartheta) - \frac{r}{\sqrt{2\vartheta}}\gamma(1/2, r^2/2\vartheta)\right\}\right. \\ & \left. + r\sqrt{\frac{2}{\vartheta}}\gamma(3/2, r^2/4\vartheta)\right]. \end{aligned} \tag{5}$$

This metric as given above (eq. (1)) reproduces exactly the ordinary Kerr or Reissner–Nordström solution at large distance, while at short distance it reduces to the de Sitter space–time rather than giving a singularity in curvature (Modesto and Nicolini [5]). In context we note: Non-commutativity has no role in affecting the Einstein tensor part of the field equations while introducing a modified energy–momentum tensor (associating smeared, Gaussian energy and/or charge density profiles as mentioned in Introduction) as a source. Metric field is a geometrical structure defined over an underlying manifold. Curvature measures the strength of the metric field, i.e. it is the response to the presence of a mass–energy distribution. What we know for sure, is that non-commutativity is an intrinsic property of the manifold itself, rather than a superimposed geometrical structure. In this respect, it affects gravity in a subtle, indirect way [5,14,16].

Now, it is necessary to specify what exactly M or Q means in the non-commutative framework. Whereas Q is simply the so-called total electrical charge of the black hole, M is the total mass–energy of a black hole measured by an asymptotically distant observer. An asymptotically distant observer can only measure the total mass–energy in which he cannot distinguish anymore between gravitational and electrical contributions. So the total mass–energy is now defined as

$$M = \oint_{\Sigma} \left(T^0_{\mu}|_{\text{matt.}} + T^0_{\mu}|_{\text{elec.}} \right) \mathrm{d}\sigma^{\mu},$$

where Σ is a 't = const.', closed 3-surface at infinity and T^{ν}_{μ} is the familiar energy–momentum tensor of Einstein's field equations.

Incidentally, $m_0 = \oint_{\Sigma} T^0_{\mu}|_{\text{matt.}} \mathrm{d}\sigma^{\mu}$ is called the 'bare-mass' term. An observer close to the origin sees only the bare-mass stripped from charge dressing (as a consequence of the linear behaviour of electric field at the minimal length). As we can see, m_0 is only a part of the total mass–energy of the system. The Coulomb energy stored in the electric field is a second contributor to the total mass–energy sourcing the gravitational field.

In this article, we shall analyse the charged and the spinning holes separately. For a pure Kerr black hole $M = m_0$; so the interpretation of the black-hole mass gets simplified. The RN holes, though, retain the complexity. The most beautiful thing to note is that while the black holes are maximally spinning or charged, M can be determined only through the radius of the single degenerate horizon surviving there.

3. How to determine the maximal charge and/or spin of a black hole

The third law of black-hole thermodynamics asserts that no black hole can ever reach the state of perfectly zero surface gravity, it can only approach very close. The same argument may also be advanced from the reinterpreted 'cosmic censorship conjecture' of Penrose. Note that black holes can carry only one degenerate event-horizon when they are approximately at this extreme condition. Therefore, for any 'extremal hole' (black holes that are maximally rotating and/or charged, i.e. at the extreme state of existence) with a single degenerate horizon at $r = r_{\mathrm{H}}$, the following two criteria must be satisfied simultaneously:

$$\Delta(r_{\mathrm{H}}) \to 0 \quad \text{and} \quad \kappa(r_{\mathrm{H}}) \to 0 \tag{6}$$

κ being the familiar surface gravity term for the black holes. The above argument is going to be used extensively in analysing the examples to follow, which accordingly determine their maximal charge holding and/or spinning capabilities. In the graphical analysis, these conditions would be imposed by hand. We now introduce a useful coordinate transformation, namely, $r \to 2M\tilde{r}$, by which $m(r)$ gets transformed in the following way:

$$\frac{m(r)}{r} \longrightarrow \frac{1}{\tilde{r}\sqrt{\pi}}\,\gamma\left(\frac{3}{2}, \frac{M^2\tilde{r}^2}{\vartheta}\right) = \frac{1}{\tilde{r}\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{\tilde{r}^2}{4\tilde{\vartheta}}\right).$$

Similarly,

$$\left(\frac{q(r)}{r}\right)^2 \longrightarrow \frac{\eta}{4\pi\tilde{r}^2}\left[\left\{\gamma^2(1/2, \tilde{r}^2/4\tilde{\vartheta}) - \frac{\tilde{r}}{\sqrt{2\tilde{\vartheta}}}\gamma(1/2, \tilde{r}^2/2\tilde{\vartheta})\right\} + \tilde{r}\sqrt{\frac{2}{\tilde{\vartheta}}}\,\gamma(3/2, \tilde{r}^2/4\tilde{\vartheta})\right].$$

Here $\sqrt{\tilde{\vartheta}} = (\sqrt{\vartheta}/2M)$. So in $2M = 1$ unit $\sqrt{\tilde{\vartheta}} \equiv \sqrt{\vartheta}$ and $r \equiv \tilde{r}$. Both $\sqrt{\tilde{\vartheta}}$ and $\tilde{r}$ are dimensionless variables; also note that $\eta = (Q/M)^2$.

4. Non-commutative Kerr black hole

For such a black hole, space–time no longer remains spherically symmetric (the space–time is now axially symmetric, i.e. a $\mathrm{d}r = \mathrm{d}t = 0$ hypersurface is non-spherical, but has axial symmetry), though that is nothing to do with the constancy of spatial radius of static event-horizon (and with its characteristic 'one-way property'). Referring to the previous section and noting that $\Delta = [r^2 - 2m(r)r + a^2]$, the upper bound to 'specific angular momentum' ($a_{\max}$) for Kerr black holes in non-commutative length scale is obtained by recognizing

$$\left[1 - \frac{2m(r)}{r} + \frac{a^2_{\max}}{r^2}\right]_{r_{\mathrm{H}}} \to 0, \tag{7}$$

while elsewhere $[\Delta(r)/r^2] > 0$. Such characteristics are amenable to graphical analysis. Before going ahead, here we introduce an abbreviation namely, $(a/M)^2 = \zeta$.

Table 1. Data for $\zeta_{\max}(\sqrt{\tilde{\vartheta}})$ of Kerr black holes at strongly non-commutative length scale.

$\sqrt{\tilde{\vartheta}}$	$\zeta_{\max}$	$\sqrt{\tilde{\vartheta}}$	$\zeta_{\max}$	$\sqrt{\tilde{\vartheta}}$	$\zeta_{\max}$	$\sqrt{\tilde{\vartheta}}$	$\zeta_{\max}$	$\sqrt{\tilde{\vartheta}}$	$\zeta_{\max}$	$\sqrt{\tilde{\vartheta}}$	$\zeta_{\max}$
0.05	0.99999	0.1	0.991	0.11	0.978	0.125	0.9455	0.14	0.895	0.15	0.853
0.16	0.804	0.17	0.748	0.18	0.686	0.19	0.619	0.20	0.546	0.21	0.468
0.22	0.387	0.23	0.301	0.24	0.211	0.25	0.1195	0.26	0.025	0.2626	0.000

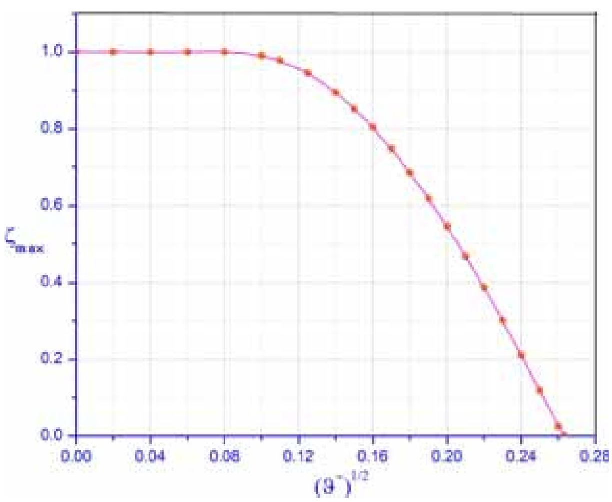

Figure 1. Variation of $(a/M)_{\max}$ as a function of $\sqrt{\tilde{\vartheta}}$ for Kerr black holes having their mass/length scale in relevantly non-commutative regime. $(\sqrt{\tilde{\vartheta}})_{\max} \simeq 0.2626$ corresponds to the minimally energized hole with $M \simeq 1.904\sqrt{\vartheta}$.

With that, we rewrite the argument above in the following form:

$$\left[1-\frac{2m(\tilde{r})}{\tilde{r}}+\frac{1}{4}\frac{\zeta_{\max}}{\tilde{r}^2}\right]_{\tilde{r}=\tilde{r}_{\rm H}}=0;\quad \left[\Delta(\tilde{r})/\tilde{r}^2\right]_{\tilde{r}\neq\tilde{r}_{\rm H}}>0. \tag{8}$$

Thus, based on a graphical analysis (in which both $\kappa(\tilde{r}_{\rm H})=0$ and $\Delta(\tilde{r}_{\rm H})=0$ are simultaneously satisfied), we have the desired data (table 1) for extremal Kerr black holes.

Note that $\sqrt{\vartheta}$ characterizes the 'black-hole mass'. We plot these data in figure 1 to show how the maximal angular momentum-carrying capacity of Kerr black holes in the non-commutative mass–energy regime evolves. The capacity suffers a steep loss with M decreasing at a non-commutative scale, and eventually a black hole loses its ability to drag space–time for $M \simeq 1.904\sqrt{\vartheta}$. Keeping this result in mind after a numerical start we shall now move onto an algebraic analysis.

4.1 *Maximal specific angular momentum for non-commutative Kerr black holes*

This is to be evaluated by equating the surface gravity of black hole to zero. Hence, first of all we note that with regard to a non-commutative Kerr black hole, the surface gravity for a general function $\Delta(r)=[r^2-2m(r)r+a^2]$ is given by (see [5,21])

$$\kappa(r_+)=\frac{\Delta'(r_+)}{2(r_+^2+a^2)}=\frac{r_+}{2(r_+^2+a^2)}\times\left[1-\frac{a^2}{r_+^2}-\frac{(r_+^2+a^2)r_+}{4\vartheta^{3/2}}\frac{{\rm e}^{-r_+^2/4\vartheta}}{\gamma(3/2,r_+^2/4\vartheta)}\right]. \tag{9}$$

Here r_+ stands for the upper horizon. Extremal black holes correspond to

$$\frac{\Delta'(r)|_{r_{\rm H}}}{2(r_{\rm H}^2+a_{\max}^2)}=0, \tag{10}$$

where $r_{\rm H}$ is the only radial position satisfying $\Delta(r)=0$. So, we have for a pure Kerr black hole in the extremal state

$$\left\{\frac{a_{\max}}{r_{\rm H}}\right\}^2+\frac{(r_{\rm H}^2+a_{\max}^2)r_{\rm H}}{4\vartheta^{3/2}}\frac{{\rm e}^{-r_{\rm H}^2/4\vartheta}}{\gamma(3/2,r_{\rm H}^2/4\vartheta)}=1. \tag{11}$$

By rearranging the terms, we may cast the equation in the following form:

$$\left[\frac{a_{\max}}{r_{\rm H}}\right]^2=\left[\frac{1-\frac{r_{\rm H}^3}{4\vartheta^{3/2}}\frac{{\rm e}^{-r_{\rm H}^2/4\vartheta}}{\gamma(3/2,r_{\rm H}^2/4\vartheta)}}{1+\frac{r_{\rm H}^3}{4\vartheta^{3/2}}\frac{{\rm e}^{-r_{\rm H}^2/4\vartheta}}{\gamma(3/2,r_{\rm H}^2/4\vartheta)}}\right]. \tag{12}$$

We now introduce a quantity $f(r_{\rm H})$ given by

$$f(r_{\rm H})=\frac{r_{\rm H}^3}{4\vartheta^{3/2}}\frac{{\rm e}^{-r_{\rm H}^2/4\vartheta}}{\gamma(3/2,r_{\rm H}^2/4\vartheta)}, \tag{13}$$

and rewrite eq. (12) as

$$\left[\frac{a_{\max}}{r_{\rm H}}\right]=\left[\frac{1-f(r_{\rm H})}{1+f(r_{\rm H})}\right]^{1/2}. \tag{14}$$

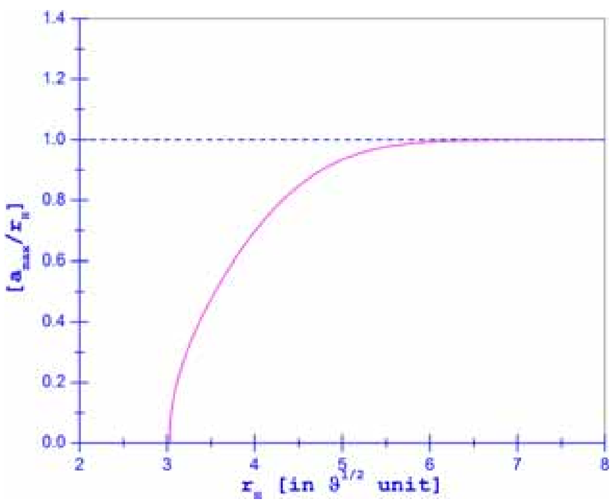

Figure 2. Variation of the ratio $(a_{\max}/r_H)$ with r_H is observed in here; 'r_H' stands as a representative for black holes with their mass-energies lying in different possible scales. The entity $(a_{\max}/r_H)$ shrinks fast from a pure classical value 'one' to 'zero' in the non-commutatively active space–time regime.

The left-hand-side of eq. (14) is plotted against r_H in figure 2.

$a_{\max}$ *to M ratio*: We start by calculating the ratio (r_H/M) as a function of the radius of the black-hole horizon. In doing this, we recall that for an extremal black hole, the horizon r_H always satisfies the conditions $\Delta = 0$ and $\kappa = 0$ simultaneously. Therefore, the equation determining r_H is

$$r_H^2 - 2m(r_H)r_H + \frac{1 - f(r_H)}{1 + f(r_H)} r_H^2 = 0, \tag{15}$$

which gives in turn

$$r_H = m(r_H)\{1 + f(r_H)\}, \tag{16}$$

i.e.

$$\left\{\frac{r_H}{M}\right\} = \frac{\gamma(3/2, r_H^2/4\vartheta)}{\Gamma(3/2)}\{1 + f(r_H)\}. \tag{17}$$

So we have on the one hand the relation eq. (14) and on the other hand an expression for (r_H/M) given by eq. (17). These two together can provide us $(a_{\max}/M)$ as a function of r_H;

$$\left\{\frac{a}{M}\right\} = \frac{\gamma(3/2, r_H^2/4\vartheta)}{\Gamma(3/2)}\{1 - f^2(r_H)\}^{1/2}, \tag{18}$$

where the subscript of a has been removed. See figure 3 (graph at right). So we have found the analytic expression(s) (18) and/or (14), giving us the maximal specific angular momentum $a(=J/M)$ for black holes in every possible mass/length scale. The importance of this result is very clear. In commutative theory of black holes or in other words for black holes at commutative mass scale, the quantity (a/M) is always equal to 1, independent of the mass/length scale of the black holes; and this simple characteristic breaks down near the minimal (non-commutative) length scale where the black-hole mass M becomes comparable to $\sqrt{\vartheta}$.

Thus, for any physical Kerr black hole spin is bound by

$$0 < a \leq M\frac{\gamma(3/2, r_H^2/4\vartheta)}{\Gamma(3/2)}\{1 - f^2(r_H)\}^{1/2}, \tag{19}$$

where r_H is related to mass by a transcendental equation, namely, eq. (17).

4.2 *Spinning rate of extremal non-commutative Kerr black holes*

It is the light rays forming the surface of event horizon of a Kerr black hole that actually rotates and we call their angular velocity, the angular velocity of the hole. At the null three surface of the horizon there, one can associate with each point a unique null vector, namely, $\chi = (1, 0, 0, \Omega_H)$ so as to obtain

$$\chi \cdot \chi = 0; \quad \Omega_H = \frac{a}{(r_+^2 + a^2)}, \tag{20}$$

where r_+ ranges over the horizons (which are not necessarily of extremal nature) at different length scales; this expression is very familiar [2,5,23]. On using the expression for $a_{\max}/r_H$, in the form of eq. (14), one obtains the following formula for extremal Kerr holes:

$$\Omega_H = \frac{1}{2r_H}\{1 - f^2(r_H)\}^{1/2}. \tag{21}$$

Graphical plot for $\Omega_H(r_H)$ (see figure 4) shows a maximum in the corresponding curve at $r_H \approx 4\sqrt{\vartheta}$ and then a rapid decrease to the value 'zero' at $r_H \cong 3.02\sqrt{\vartheta}$. It shows that an extremal hole with the lowest possible mass cannot rotate (as expected, because $a_{\max} = 0$ there).

In context, note that for a classical commutative black-hole space–time geometry, Ω_H grows continuously towards infinity. This is the obvious consequence of the fact that classically there exists no lower limit to the mass of a black hole; and also that it is always possible to have a specific angular momentum for a commutative extremal hole as $a_{\max} = r_H = M$, at every length scale. However, the probable non-commutative nature of space–time near the minimal length scale removes the singularity that otherwise arises in the angular velocity (spinning rate) for an extremal Kerr black hole.

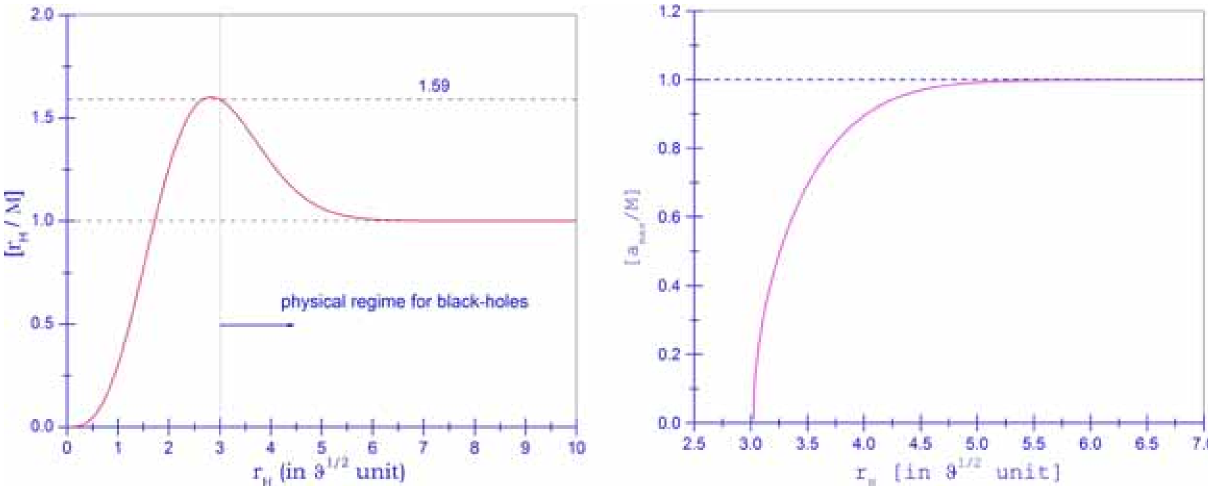

Figure 3. Characteristic effect of space–time non-commutativity on the associated ratios. Physical regime means the regime where black holes can exist in accordance with the 'cosmic censorship conjecture'. Horizon with radius $r_{\rm H} < 3.02\sqrt{\vartheta}$ does not exist.

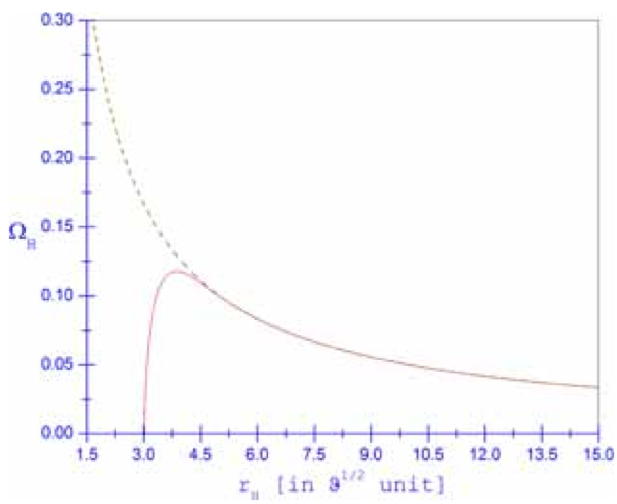

Figure 4. Variation of angular velocity $\Omega(r_{\rm H})$ of extremal Kerr black holes with different mass–energy/length scale regime. While the dotted, deep yellow line represents the case for commutative black holes, solid one stands for the non-commutative holes. The probable non-commutative nature of space–time forces the spinning rate to die down abruptly at length scale $r_{\rm H} \sim (4\sqrt{\vartheta} \to 3\sqrt{\vartheta})$.

5. Non-commutative Reissner–Nordström black hole

Space–time is now spherically symmetric having, $\Delta(r) = r^2 - 2m(r)r + q(r)^2$. We recall that the criterion for survival of a black hole is simply the survival of its event horizon; there must be no naked de-Sitter origin ('cosmic censorship conjecture'). So the limiting condition is that there would be only one radius at which

$$\left[1 - \frac{2m(r)}{r} + \frac{q(r)^2}{r^2}\right]_{r_{\rm H}} \to 0 \tag{22}$$

and elsewhere it is $[\Delta(r)/r^2]>0$. This criterion is graphically analysable. With the definition $\eta = (Q/M)^2$, we rewrite the argument above in following form:

$$\left[1 - \frac{2m(\tilde{r})}{\tilde{r}} + \frac{\eta_{\rm max}}{4\pi\tilde{r}^2}\left[\left\{\gamma^2(1/2, \tilde{r}^2/4\tilde{\vartheta}) - \frac{\tilde{r}}{\sqrt{2\tilde{\vartheta}}}\gamma(1/2, \tilde{r}^2/2\tilde{\vartheta})\right\} + \tilde{r}\sqrt{\frac{2}{\tilde{\vartheta}}}\,\gamma(3/2, \tilde{r}^2/4\tilde{\vartheta})\right]\right]_{\tilde{r}_{\rm H}} = 0, \tag{23}$$

while $[\Delta(\tilde{r})/\tilde{r}^2]_{\tilde{r}\neq\tilde{r}_{\rm H}} > 0$. The graphical analysis yields the numerical data (table 2) as provided below. These data (table 2) are presented in figure 5. Like the former case of extremal Kerr black holes, from here onwards we move on to analyse the extremal RN holes using algebra and calculus.

5.1 *Maximal charge-carrying capacity of a pure RN black hole*

For a pure RN black hole [5,15], i.e. with respect to the metric (1) associating '$a = 0$', the surface gravity κ is familiarly obtained as

$$\kappa(r_+) \equiv \frac{\Delta'(r_+)}{2r_+^2} = \frac{1}{2r_+}\left[1 - \frac{[q^2(r_+)]}{r_+^2} + \frac{[q^2(r)]'|_{r_+}}{r_+} - \frac{\{r_+^2 + [q^2(r_+)]\}r_+}{4\vartheta^{3/2}}\frac{{\rm e}^{-r_+^2/4\vartheta}}{\gamma(3/2, r_+^2/4\vartheta)}\right]. \tag{24}$$

Table 2. Data for $\eta_{\max}(\sqrt{\tilde{\vartheta}})$ of strongly non-commutative extremal Reissner–Nordström black holes.

$\sqrt{\tilde{\vartheta}}$	$\eta_{\max}$	$\sqrt{\tilde{\vartheta}}$	$\eta_{\max}$	$\sqrt{\tilde{\vartheta}}$	$\eta_{\max}$	$\sqrt{\tilde{\vartheta}}$	$\eta_{\max}$	$\sqrt{\tilde{\vartheta}}$	$\eta_{\max}$	$\sqrt{\tilde{\vartheta}}$	$\eta_{\max}$
0.05	1.000	0.10	1.000	0.125	0.9902	0.14	0.9664	0.15	0.9405	0.16	0.9061
0.17	0.8626	0.18	0.8097	0.19	0.7472	0.20	0.6755	0.21	0.5933	0.22	0.5016
0.23	0.4002	0.24	0.2887	0.25	0.1675	0.255	0.1030	0.26	0.036	0.2632	0.000

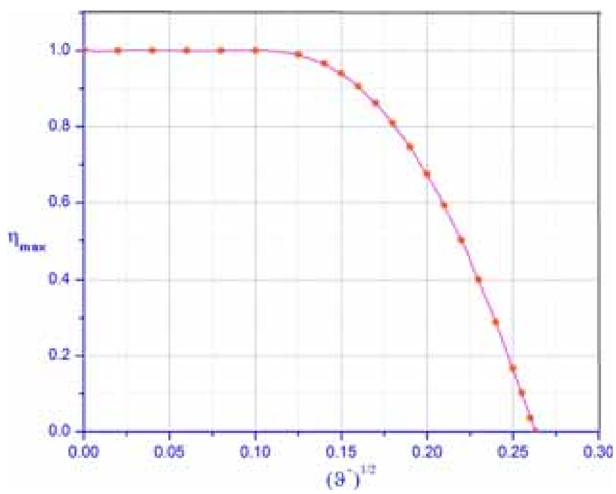

Figure 5. Plot of the maximal charge-carrying capacity of Reissner–Nordström black holes in the non-commutative mass–energy range; $\sqrt{\tilde{\vartheta}} \propto (1/M)$ (where M is considered in $\sqrt{\vartheta}$ units). Note that for a physical hole, the minimum possible mass is $M \simeq 1.904\sqrt{\vartheta}$. Therefore, the maximum possible value of $\sqrt{\tilde{\vartheta}}$ is expected to be $\sqrt{\tilde{\vartheta}}_{\max} \simeq 0.2626$, for which the black hole loses its charge-carrying capability to acquire a perfect Schwarzschild shape.

Here r_+ denotes the upper horizon of non-commutative RN black holes. The spatial extremity of horizon for such a black hole at any length scale necessarily determines its maximal charge holding capacity. The condition for extremity is simply $\kappa \to 0$. Hence, this yields

$$\frac{[q^2(r_{\mathrm{H}})]}{r_{\mathrm{H}}^2} - \frac{[q^2(r)]'|_{r_{\mathrm{H}}}}{r_{\mathrm{H}}} + \frac{\{r_{\mathrm{H}}^2 + [q^2(r_{\mathrm{H}})]\}r_{\mathrm{H}}}{4\vartheta^{3/2}} \times \frac{\mathrm{e}^{-r_{\mathrm{H}}^2/4\vartheta}}{\gamma(3/2, r_{\mathrm{H}}^2/4\vartheta)} \cong 1; \tag{25}$$

where r_{H} stands for 'extremal horizon' (marginally existing single degenerate horizon for black holes with a maximally charged and/or spinning state). The form of $q(r)$ is as given in eq. (5) and it leads us to have

$$\left\{\frac{Q}{r_{\mathrm{H}}}\right\}^2 = \frac{1}{g(r_{\mathrm{H}})}\{1 - f(r_{\mathrm{H}})\}. \tag{26}$$

This expression is fundamental in determining the maximal charge capacity of RN black holes, according to their radii of extremal horizons. The associated figure is shown as figure 6. Unlike the commutative theory, r_{H} is here a non-linear function of charge (or mass–energy) of a maximally charged black hole. Additionally, we must mention the following things: The abbreviation, viz. $f(r_{\mathrm{H}})$ has already been introduced much earlier in the article (see eq. (13)); and further it is to be noted that

$$g(r_{\mathrm{H}}) = \frac{1}{\pi}\left[\gamma^2\left(\frac{1}{2}, \frac{r_{\mathrm{H}}^2}{4\vartheta}\right) - \frac{2r_{\mathrm{H}}}{\sqrt{\vartheta}}\gamma\left(\frac{1}{2}, \frac{r_{\mathrm{H}}^2}{4\vartheta}\right)\mathrm{e}^{-r_{\mathrm{H}}^2/4\vartheta} + \frac{r_{\mathrm{H}}^2}{\vartheta}\mathrm{e}^{-r_{\mathrm{H}}^2/2\vartheta}\right] + \frac{f(r_{\mathrm{H}})}{\pi}\left[\gamma^2\left(\frac{1}{2}, \frac{r_{\mathrm{H}}^2}{4\vartheta}\right) - \frac{r_{\mathrm{H}}}{\sqrt{2\vartheta}}\gamma\left(\frac{1}{2}, \frac{r_{\mathrm{H}}^2}{2\vartheta}\right)\right].$$

It gets further simplified to the following form, if one uses the identity: $[\gamma(3/2, r^2/4\vartheta) = (1/2)\gamma(1/2, r^2/4\vartheta) - (r/2\sqrt{\vartheta})\exp(-r^2/4\vartheta)]$. Thus

$$g(r_{\mathrm{H}}) = \frac{1}{\pi}\left[4\gamma^2(3/2, r^2/4\vartheta) + f(r_{\mathrm{H}})\left\{\gamma^2\left(\frac{1}{2}, \frac{r_{\mathrm{H}}^2}{4\vartheta}\right) - \frac{r_{\mathrm{H}}}{\sqrt{2\vartheta}}\gamma\left(\frac{1}{2}, \frac{r_{\mathrm{H}}^2}{2\vartheta}\right)\right\}\right]. \tag{27}$$

(Let us here for a moment mention a mathematical expression that has been used in deriving eq. (26). Hopefully this will help in understanding the contention

$$[q^2(r)]'_{r=r_{\mathrm{H}}} = \frac{Q^2}{\pi r_{\mathrm{H}}}\left[\frac{2r_{\mathrm{H}}}{\sqrt{\vartheta}}\gamma(1/2, r_{\mathrm{H}}^2/4\vartheta)\mathrm{e}^{-r_{\mathrm{H}}^2/4\vartheta} - \frac{r_{\mathrm{H}}^2}{\vartheta}\mathrm{e}^{-r_{\mathrm{H}}^2/2\vartheta} + \frac{r_{\mathrm{H}}^4}{2\sqrt{2}\vartheta^2}\mathrm{e}^{-r_{\mathrm{H}}^2/4\vartheta}\right] + \frac{Q^2}{\pi r_{\mathrm{H}}}\left[-\frac{r_{\mathrm{H}}}{\sqrt{2\vartheta}}\gamma(1/2, r_{\mathrm{H}}^2/2\vartheta) + r_{\mathrm{H}}\sqrt{\frac{2}{\vartheta}}\gamma(3/2, r_{\mathrm{H}}^2/4\vartheta)\right].$$

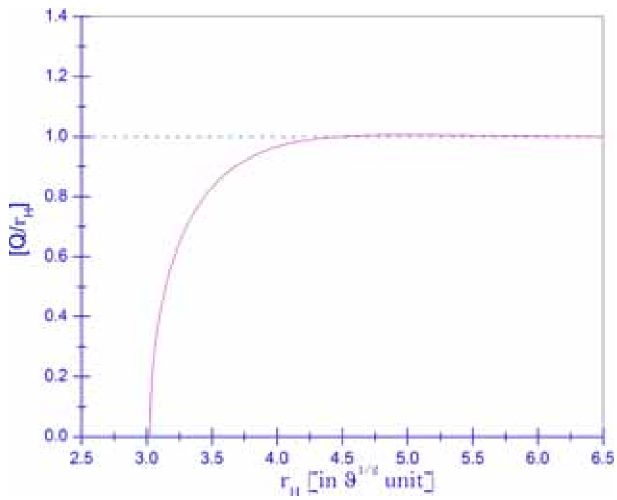

Figure 6. Variation of the ratio $(a_{\max}/r_{\rm H})$ with $r_{\rm H}$, which characterizes the black holes in different mass/length scales. $(a_{\max}/r_{\rm H})$ shrinks fast from 'one' to 'zero' in the non-commutatively active space–time regime.

As before, the prime ' $'$ ' denotes a differentiation with respect to r.)

To proceed further in the direction of algebraic analysis, it is essential to know the ratio of $r_{\rm H}$ to M. Simultaneous use of the conditions: $\kappa(r_{\rm H}) = 0$ and $\Delta(r_{\rm H}) = 0$, i.e. by using eq. (26) accordingly in $\Delta(r_{\rm H}) = 0$ one gets

$$r_{\rm H}^2 - 2m(r_{\rm H})r_{\rm H} + \frac{1-f(r_{\rm H})}{\pi g(r_{\rm H})}\left[\left\{\gamma^2\left(\frac{1}{2},\frac{r_{\rm H}^2}{4\vartheta}\right) - \frac{r_{\rm H}}{\sqrt{2\vartheta}}\gamma\left(\frac{1}{2},\frac{r_{\rm H}^2}{2\vartheta}\right)\right\} + r_{\rm H}\sqrt{\frac{2}{\vartheta}}\gamma\left(\frac{3}{2},\frac{r_{\rm H}^2}{4\vartheta}\right)\right] r_{\rm H}^2 = 0. \tag{28}$$

That naturally means

$$\left\{\frac{r_{\rm H}}{M}\right\} = 2\frac{\gamma(3/2, r_{\rm H}^2/4\vartheta)}{\Gamma(3/2)}h^{-1}(r_{\rm H}), \tag{29}$$

where

$$h(r_{\rm H}) = 1 + \frac{1-f(r_{\rm H})}{\pi g(r_{\rm H})}\left[\left\{\gamma^2\left(\frac{1}{2},\frac{r_{\rm H}^2}{4\vartheta}\right) - \frac{r_{\rm H}}{\sqrt{2\vartheta}}\gamma\left(\frac{1}{2},\frac{r_{\rm H}^2}{2\vartheta}\right)\right\} + r_{\rm H}\sqrt{\frac{2}{\vartheta}}\gamma\left(\frac{3}{2},\frac{r_{\rm H}^2}{4\vartheta}\right)\right].$$

So, ultimately we gain

$$\left\{\frac{Q}{M}\right\} = 4\frac{\gamma(3/2, r_{\rm H}^2/4\vartheta)}{\sqrt{\pi}h(r_{\rm H})} \times \left[\frac{1}{g(r_{\rm H})}\{1-f(r_{\rm H})\}\right]^{1/2} \tag{30}$$

as the grand analytic expression that governs the (Q/M) ratio for the maximally charged Reissner–Nordström black holes in the non-commutative scale. The figures corresponding to eqs (29) and (30) are given in figure 7.

Thus, the charge holding capacity of a physical Reissner–Nordström black hole is limited by

$$0 < a \le 4M\frac{\gamma(3/2, r_{\rm H}^2/4\vartheta)}{\sqrt{\pi}h(r_{\rm H})} \times \left[\frac{1}{g(r_{\rm H})}\{1-f(r_{\rm H})\}\right]^{1/2}, \tag{31}$$

where $r_{\rm H}$ is related to mass by a transcendental equation namely, eq. (29).

6. A brief note on the generalized extremal black holes

Maximal angular momentum per unit mass a of a generalized black hole, and at the same time the maximal charge Q on that hole is to be determined, as usual, by setting the surface gravity, $\kappa \to 0$. Recalling that $\kappa(r_+) = \Delta\acute{\not{g}}(r_+)/[2(r_+^2+a^2)]$, $[\Delta(r) = r^2 - 2m(r)r + q^2(r) + a^2]$, we have

$$\kappa(r_+) = \frac{r_+}{2(r_+^2+a^2)} \times \left[1 - \frac{a^2}{r_+^2} - \frac{[q^2(r_+)]}{r_+^2} + \frac{[q^2(r)]'|_{r_+}}{r_+} - \frac{\{r_+^2 + a^2 + [q^2(r_+)]\}r_+}{4\vartheta^{3/2}} \times \frac{{\rm e}^{-r_+^2/4\vartheta}}{\gamma(3/2, r_+^2/4\vartheta)}\right], \tag{32}$$

and so for the extremal class black hole, it is

$$\frac{a^2}{r_{\rm H}^2} + \frac{[q^2(r_{\rm H})]}{r_{\rm H}^2} - \frac{[q^2(r)]'_{r_{\rm H}}}{r_{\rm H}} + \frac{(r_{\rm H}^2 + a^2 + [q^2(r_{\rm H})])r_{\rm H}}{4\vartheta^{3/2}} \times \frac{{\rm e}^{-r_{\rm H}^2/4\vartheta}}{\gamma(3/2, r_{\rm H}^2/4\vartheta)} = 1, \tag{33}$$

which necessarily corresponds to $\kappa \to 0$ limit. Hence

$$\left\{\frac{a}{r_{\rm H}}\right\}^2 \frac{1+f(r_{\rm H})}{1-f(r_{\rm H})} + \left\{\frac{Q}{r_{\rm H}}\right\}^2 \frac{g(r_{\rm H})}{1-f(r_{\rm H})} = 1. \tag{34}$$

In a more compact form, we may write it as

$$\alpha(r_{\rm H})\left\{\frac{a}{r_{\rm H}}\right\}^2 + \beta(r_{\rm H})\left\{\frac{Q}{r_{\rm H}}\right\}^2 = 1. \tag{35}$$

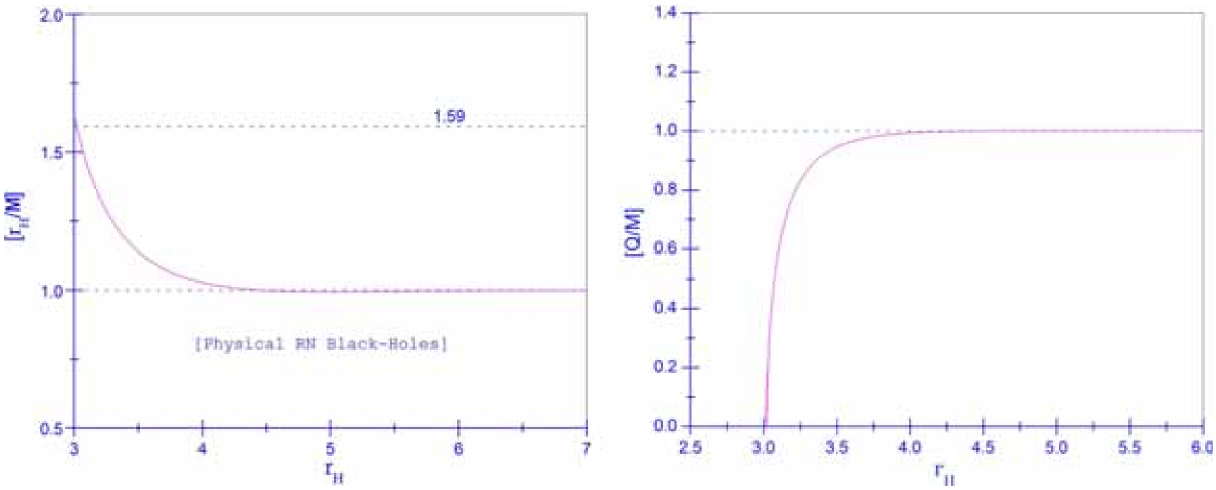

Figure 7. Characteristic effects of space–time non-commutativity on the ratios, $r_{\rm H}/M$ and Q/M, for the extremal RN black holes, specially over a non-commutative scale.

This is a general expression to determine the extremal specific angular momentum and/or charge holding capacity for non-commutative black holes over every length scales.

The dependence of $r_{\rm H}$ on M, Q and a, has to be found by numerically solving the condition $\Delta = 0$, i.e. by solving

$$r_{\rm H}^2 - 2r_{\rm H}m(r_{\rm H}) + a^2 + q^2(r_{\rm H}) = 0. \quad (36)$$

Among M, Q and a, only two are independent. By eq. (33) we may eliminate one. It is to be noted that

$$\frac{1}{r_{\rm H}^2}\left[a^2 + q^2(r_{\rm H})\right] = \frac{1}{1+f(r_{\rm H})}\left\{1 + \left[\frac{[q^2(r)]'_{r_{\rm H}}}{r_{\rm H}} - f(r_{\rm H})\right]\right\}. \quad (37)$$

It has been earlier noted that $[q^2(r)]'_{r_{\rm H}}$ is a function of Q and $r_{\rm H}$. Therefore, in case of a generalized black hole $r_{\rm H}$ may be determined by knowing any of the two quantities among M, Q and a. The simplified form of eq. (36) that should be solved numerically is

$$1 + \frac{2M\gamma(3/2, r_{\rm H}^2/4\vartheta)}{\Gamma(3/2)r_{\rm H}} + \frac{1}{1+f(r_{\rm H})} \times \left\{1 + \left[\frac{[q^2(r)]'_{r_{\rm H}}}{r_{\rm H}} - f(r_{\rm H})\right]\right\} = 0. \quad (38)$$

As one expects, $\alpha(r_{\rm H}), \beta(r_{\rm H}) \to 1$ at the 'commutative length scale regime' ($r_{\rm H} >> \sqrt{\vartheta}$), and returns us the desired result of the singular source-inspired commutative black holes. On the other hand, at a significantly non-commutative scale (where $r_{\rm H}$ is comparable to $\sqrt{\vartheta}$) a distinct change in the behaviours of α and β can easily be detected. For black holes surviving with the lowest possible mass, i.e. for $r_{\rm H} \to 3.02\sqrt{\vartheta}$ both $\alpha(r_{\rm H}),\ \beta(r_{\rm H}) \to \infty$, which ensures 'zero' value of both $[a/r_{\rm H}]$ and $[Q/r_{\rm H}]$. Once again we would like to point out that the value of $r_{\rm H}$ characterizes black holes, with their mass–energy in the entire possible range. We reiterate that eq. (35), inspired by the 'third law of black-hole mechanics', and arising in the non-commutative domain of geometry, is the key to understand an important class of black holes.

7. Concluding remarks

The maximal specific angular momentum and/or charge-holding capacity of black holes at different length/mass scales has definitely got the observational significance apart from being just theoretically appealing. Indeed, the maximal spinning capacity of rotating astrophysical black holes has always been under active observations. For the physically allowed black holes, the dimensionless non-commutative parameter $\sqrt{\tilde{\vartheta}}$, lies within the range $0 \le \sqrt{\tilde{\vartheta}} \le 0.2626$. The maximal 'specific angular momentum' for pure Kerr holes or the maximal 'charge-holding capacity' for Reissner–Nordström holes drops down to 0 as one approaches the last surviving stage. Black holes of a generalized class also agree in this respect. This effect of diminishing upper bounds of a and/or Q may be testable if mini-black holes gets produced by the ultra-high-energy particle collision at the 'LHC'. It is a fact that examining the generalized black holes with respect to their spinning and/or charge holding capacities at a non-commutative scale of length may always be helpful in verifying quantum theory of gravity.

The upper bounds, as found on the spinning and/or charge-holding capacity of a black hole, are essential for examining whether a black hole of non-commutative space–time background is destructible or not around a characteristic scale of length of non-commutativity. A commutative geometry-inspired black hole is regarded as indestructible by external means. The cosmic censorship holds for a classical singular black hole [9,24,25]. As we may say: A reinterpreted version of cosmic censorship dealing solely with the indestructibility of a black hole is analysable. This topic can be very crucial in respect of understanding the Universe. The very end stage of any surviving black hole is Schwarzschild in class which has a degenerate, marginally stable space–time event-horizon. If this horizon gets destructible by any physical means, then that would impact in eliminating the possibility of having the mini 'primordial black holes' (PBHs) as a candidate for cold stable dark matter [11], which incidentally constitutes a considerable part of the matter in the Universe [26,27]. The black holes, those lay only slightly above the minimal mass scale, are weakly rotating since they fall into a strongly non-commutative space–time regime. Indestructible nature of such black holes will pronounce the possibility of PBH relics surviving in the present-day Universe as the physically silent dark matter sector.

The diminishing space–time dragging ability of a black hole at a non-commutative scale is likely to influence the strength of Penrose process resulting in the release of rotational energy. As for tiny black holes, the status of Penrose scheme (an idea on this scheme may be found in [23,28]) may be a relevant subject deserving visit in terms of a consideration of the new bound of black-hole spin analysed in this paper. Similarly, a necessity of revisiting the Blandford–Znajek mechanism (see [28]) by which rotational energy is extracted electromagnetically, may find its motivation from the noticeable modulation observed in a non-commutativity-inspired black hole's spinning capacity.

The basic qualitative idea obtained over the extremity of a non-commutativity-inspired black hole remains valid for other proposed models of smeared energy density-inspired black holes [29–32]. This is to say that the maximal charge holding and/or spinning capacity of a black hole will fall down heavily towards zero as we approach deeper into a quantum gravity scale.

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