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Combination synchronization of time-delay chaotic system via robust adaptive sliding mode control

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Abstract. In this paper, the methodology to achieve combination synchronization of time-delay chaotic system via robust adaptive sliding mode control is introduced. The methodology is implemented by taking identical time-delay Lorenz chaotic system. The selection of switching surface and the design of control law is also discussed, which is an important issue. By utilizing rigorous mathematical theory, sufficient condition is drawn for the stability of error dynamics based on Lyapunov stability theory. Theoretical results are supported with the numerical simulations. The complexity of this methodology is useful to strengthen the security of communication. The hidden message can be partitioned into several parts loaded in two master systems to improve the accuracy of communication.

Keywords. Time-delay chaotic system; combination synchronization; robust adaptive sliding mode control; Lyapunov stability theory.

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1. Introduction

Time-delay systems have attracted a considerable measure of attention in recent years, because of the way that multistability, i.e., the coexistence of multiple attractors, is a common occurrence when the delays are large – typically, much larger than the response time of the system [1]. Enthusiasm in multistability emerges since multistable system play a key role in pattern recognition processes [2] and memory storage devices. A single nonlinear deterministic delay differential equation (DDE) with a fixed time delay T is actually an infinite-dimensional system, first examined by Farmar [3] long ago. Large T usually implies a higher-dimensional chaotic attractor. In fact, dynamical systems generated by a scalar DDE are hyperchaotic with more than one positive Lyapunov exponent. The limit of calculations of speed, memory effects, finite transmission velocity etc. lead unavoidable time delays in various fields such as engineering [1], neural network [4], physics [5], biology [6] etc. In addition, different practical models, for example a single vehicle induced by traffic light and speedup [7], broadband bandpass electro-optic oscillator [8], road traffic [9], food web systems [10], etc. can be described more accurately by using time-delay systems. Chaotic time-delay systems are much useful in secure communication and encryption schemes. Due to finite signal transmission times, switching speeds and memory effects with both signal and multiple times delays are omnipresent in nature. Stability and stabilization of nonlinear dynamical systems which include time delays in their physical models are recurring problems because the existence of delays often induces instability and/or undesired performance [11,12]. Many stability criteria and performance measures are studied in the literature. Bellman and Cooke [13] have very throughly studied the distribution of characteristic roots for differential difference equations including retarded, neutral and advanced systems. Kolmanovskii and Nosov [14] have a wide overview of various methods of stability analysis including both frequency-domain and time-domain methods. It also covers stochastic systems. Delay systems thus is an interesting topic in synchronization and so far not much work has been done.

Due to the tremendous practical applications of chaotic dynamical systems in fields stated above, numerous researches have been done theoretically as well as experimentally on controlling chaos and synchronization. In 1990, Pecora and Carroll [15] gave the synchronization of chaotic systems using the concept of master and slave systems. Also, in 1990 Ott *et al.* [16] introduced the OGY method for controlling chaos.

Looking for better techniques for chaos control and synchronization, distinctive strategies have been created for controlling chaos and synchronization of non-identical and identical systems, for instance, linear feedback [17], optimal control [18], adaptive control [19,20], active control [21,22], sliding control [23], backstepping control [24], robust adaptive sliding mode control [25], optimal control [26] etc. It is vital to know the values of system's parameters for the derivation of the controller. In practical situations, these parameters are unknown. Therefore, the derivation of an adaptive controller for the control and synchronization of chaotic systems in the presence of unknown system parameters is an important issue [27,28]. In robust control systems, the sliding mode control method is often adopted due to its inherent advantages of easy realization, fast response and good transient performance as well as its insensitivity to parameter uncertainties and external disturbances. In this manuscript, we derive results based on the robust adaptive sliding mode control for the global chaos synchronization of identical time-delay chaotic systems.

As a result of greater interest in chaos control and synchronization, various synchronization types and schemes have been proposed and reported, for instance, generalized synchronization [29], projective synchronization [30], modified projective synchronization [31], function projective synchronization [32], modified-function projective synchronization [33], and hybrid synchronization [34]. It is noted that most of the researches are mainly focussed on the previous master–slave synchronization scheme within one master and one slave system. Just a couple of papers have been published on combination synchronization where three or four chaotic systems were taken into account [35,36]. Combination synchronization scheme used in this paper has been generalized in such a manner that other forms of synchronization scheme can be achieved from it. As a result, combination synchronization scheme is more flexible and applicable to the real-world systems. In addition, the combination synchronization also gives better insight into the complex synchronization and several pattern formations that take place in real-world systems because synchronization in real-world systems are complex.

Motivated by the above discussions, in this paper we have introduced the methodology for combination synchronization of time-delay chaotic system via robust adaptive sliding mode control. A very few or no researcher investigated this result till now. So, this is the novelty of this manuscript. The methodology introduced is then implemented by considering identical modified chaotic time-delay Lorenz system.

This manuscript is categorized as follows: In §2 methodology for combination synchronization of time-delay chaotic system via robust adaptive sliding mode control is introduced. In §3 modified chaotic time-delay Lorenz system with two time delays is described. In §4 combination synchronization of time-delay chaotic system via robust adaptive sliding mode control is attained. Finally in §5 concluding remarks are given.

2. Methodology for combination synchronization of time-delay chaotic system via robust adaptive sliding mode control

In this section, the scheme of combination synchronization of time-delay chaotic system using robust adaptive sliding mode control is proposed. For the purpose of combination synchronization we define the two master systems as follows:

$$\dot{u}(t) = f_1(u, u_{T_1}, u_{T_2}, \ldots, u_{T_m}, t) + g_1(u,t)\theta_1 + \sum_{i=1}^{m} h_{1i}(u_{T_i}, t)\lambda_{1i} \quad (1)$$

$$\dot{v}(t) = f_2(v, v_{T_1}, v_{T_2}, \ldots, v_{T_m}, t) + g_2(v,t)\theta_2 + \sum_{i=1}^{m} h_{2i}(v_{T_i}, t)\lambda_{2i}, \quad (2)$$

where $u(t), v(t) \in \mathbf{R}^n$ are the state variables of the master systems. $f_1, f_2, h_{1i}, h_{2i}\colon \mathbf{R}^n \to \mathbf{R}^n$ are the nonlinear functions of its arguments. $\theta_1 = (\theta_{11}, \theta_{12}, \ldots, \theta_{1p})^T \in \mathbf{R}^p$, $g_1 : \mathbf{R}^n \to \mathbf{R}^{n\times p}$, $\theta_2 = (\theta_{21}, \theta_{22}, \ldots, \theta_{2q})^T \in \mathbf{R}^q$, $g_1 : \mathbf{R}^n \to \mathbf{R}^{n\times q}$, $\theta_{1i}, \lambda_{1j}, i = 1(1)p,\ j = 1(1)m$, $\theta_{2i}, \lambda_{2j}, i = 1(1)q,\ j = 1(1)m$ are the uncertain parameters of the system. $T_i,\ i = 1(1)m$ are the constant time delays of the system, where $u_{T_i} = u(t - T_i), i = 1(1)m$ and $v_{T_i} = v(t - T_i), i = 1(1)m$.

The slave system is defined as follows:

$$\dot{w}(t) = F(w, w_{T_1}, w_{T_2}, \ldots, w_{T_m}, t) + G(w,t)\Theta + \sum_{i=1}^{m} H_i(w_{T_i}, t)\Lambda_i + \eta(t), \quad (3)$$

where $w(t) \in \mathbf{R}^n$ are the state variables of the slave systems. $F,\ H_i\colon \mathbf{R}^n \to \mathbf{R}^n$ are the nonlinear functions of its arguments. $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_r)^T \in \mathbf{R}^r$, $G\colon \mathbf{R}^n \to \mathbf{R}^{n\times r}$ are also the nonlinear functions of its arguments. $\Theta_i,\ \Lambda_j, i = 1(1)r,\ j = 1(1)m$ are the uncertain parameters of the system. $T_i,\ i = 1(1)m$ are the constant time delays of the system, where $w_{T_i} = w(t-T_i), i = 1(1)m$. $\eta(t)$ is the controller to be determined.

DEFINITION 1

Combination synchronization of the master systems (1) and (2) and the slave system (3) is said to be achieved, if there exists three non-zero matrices $P,\ Q,\ R \in \mathbf{R}^{n\times n}$ such that

$$\lim_{t\to\infty} \|Pw - Qu - Rv\| = 0,$$

where $\|\cdot\|$ denotes the Euclidean norm.

In the ensuing discussions, the three constant matrices $P,\ Q,\ R \in \mathbf{R}^{n\times n}$ are chosen to be $P = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$, $Q = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $R = \text{diag}(\xi_1, \xi_2, \ldots, \xi_n)$.

The error system is defined as

$$e = Pw - Qu - Rv. \tag{4}$$

The corresponding error dynamics is defined as

$$\begin{aligned}\dot{e} =&\ P\dot{w} - Q\dot{u} - R\dot{v}\\ =&\ P\Big[F(w, w_{T_1}, w_{T_2}, \ldots, w_{T_m}, t) + G(w,t)\Theta\\ &+\sum_{i=1}^{m} H_i(w_{T_i}, t)\Lambda_i + \eta(t)\Big]\\ &-Q\Big[f_1(u, u_{T_1}, u_{T_2}, \ldots, u_{T_m}, t) + g_1(u,t)\theta_1\\ &+\sum_{i=1}^{m} h_{1i}(u_{T_i}, t)\lambda_{1i}\Big]\\ &-R\Big[f_2(v, v_{T_1}, v_{T_2}, \ldots, v_{T_m}, t) + g_2(v,t)\theta_2\\ &+\sum_{i=1}^{m} h_{2i}(v_{T_i}, t)\lambda_{2i}\Big].\end{aligned} \tag{5}$$

To design sliding mode controller, there are two basic steps: (1) select an appropriate switching surface and (2) establish a control law which guarantees stability of the sliding surface.

2.1 *Sliding surface design*

The sliding surface, in general, defined as

$$S(t) = Ae(t), \tag{6}$$

where $A = \text{diag}(s_1, s_2, \ldots, s_n) \in \mathbf{R}^{n\times n}$. The necessary condition for any state trajectories to stay on the switching surface $S(t) = 0$ is

$$\dot{S}(t) = 0. \tag{7}$$

In the sliding mode, we must have

$$S(t) = 0, \quad \dot{S}(t) = 0.$$

We design a control $\eta(t)$ which guarantees that the error system trajectories reach on the sliding surface $S(t) = 0$ and stay on it for all subsequent time.

2.2 *Adaptive sliding mode control design and parameter adaptation laws*

Assuming that the constant rate reaching law is applied, the law can be chosen as

$$\dot{S}(t) = -q\ \text{sgn}\ S(t),$$

where $q > 0$. Using (5), (6) and (7), it follows that

$$\begin{aligned} 0 &= \dot{S}(t)\\ &= A\dot{e}\\ &= A\Big[P\Big[F(w, w_{T_1}, w_{T_2}, \ldots, w_{T_m}, t) + G(w,t)\Theta\\ &\quad+\sum_{i=1}^{m} H_i(w_{T_i}, t)\Lambda_i + \eta(t)\Big]\\ &\quad- Q\Big[f_1(u, u_{T_1}, u_{T_2}, \ldots, u_{T_m}, t) + g_1(u,t)\theta_1\\ &\quad+\sum_{i=1}^{m} h_{1i}(u_{T_i}, t)\lambda_{1i}\Big]\\ &\quad- R\Big[f_2(v, v_{T_1}, v_{T_2}, \ldots, v_{T_m}, t) + g_2(v,t)\theta_2\\ &\quad+\sum_{i=1}^{m} h_{2i}(v_{T_i}, t)\lambda_{2i}\Big]\Big].\end{aligned} \tag{8}$$

Equations (7) and (8) are identical. The following adaptive sliding mode control laws (9) and parameter update laws (10) are proposed for synchronizing the time-delay systems.

2.2.1 *Parameter update laws.*

$$\begin{aligned} P\eta(t) = &-PF - PG\hat{\Theta} - P\sum_{i=1}^{m} H_i\hat{\Lambda}_i + Qf_1 + Qg_1\hat{\theta}_1\\ &+ Q\sum_{i=1}^{n} h_{1i}(u_{T_i}, t)\hat{\lambda}_{1i} + Rf_2 + Rg_2\hat{\theta}_2\\ &+ R\sum_{i=1}^{m} h_{2i}(v_{T_i}, t)\hat{\lambda}_{2i} - \sum_{i=1}^{m} L_i(t)e(t - T_i)\\ &- qA^{-1}\text{sgn}\ S(t) - Ke(t),\end{aligned} \tag{9}$$

where

$$K = \text{diag}(K_1, K_2, \ldots, K_n) \in \mathbf{R}^n \times \mathbf{R}^n$$

is a constant matrix, K is chosen as positive definite matrix and L_i; $i = 1(1)n$ are the delayed time-varying state feedback matrices. Furthermore, $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\Theta}$ are the estimations of the uncertain parameter vectors θ_1, θ_2 and Θ respectively. $\hat{\lambda}_{1i}, \hat{\lambda}_{2i}$ and $\hat{\Lambda}_i$; $1(1)m$ denote the estimations of the uncertain parameters $\lambda_{1i}, \lambda_{2i}$ and Λ respectively. Also, $\text{sgn}(\cdot)$ denotes the signum function, $q > 0$ is a constant gain which is so determined that sliding condition is satisfied and sliding mode motion will occur.

2.2.2 *Adaptation laws.*

$$\begin{cases} \hat{\lambda}_{1i}(t) = Q[Ah_{1i} + B_i e(t-T_i)^T Ae - K_{\lambda_{1i}}\bar{\lambda}_{1i}] \\ \hat{\lambda}_{2i}(t) = R[Ah_{2i} + C_i e(t-T_i)^T Ae - K_{\lambda_{2i}}\bar{\lambda}_{2i}] \\ \hat{\Lambda}_i(t) = -P[AH_i + D_i e(t-T_i)^T Ae - K_{\Lambda_i}\bar{\Lambda}_i] \\ \hat{\theta}_{1i}(t) = Q[Ag_1^T Ae - K_{\theta_1}\bar{\theta}_1] \\ \hat{\theta}_{2i}(t) = R[Ag_2^T Ae - K_{\theta_2}\bar{\theta}_2] \\ \hat{\Theta}_i(t) = -P[AG^T Ae - K_{\Theta}\bar{\Theta}], \end{cases} \tag{10}$$

where $K_{\theta_1} = \text{diag}(K_{\theta_{11}}, K_{\theta_{12}}, \ldots, K_{\theta_{1n}}) \in \mathbf{R}^p \times \mathbf{R}^p$, $K_{\theta_2} = \text{diag}(K_{\theta_{21}}, K_{\theta_{22}}, \ldots, K_{\theta_{2n}}) \in \mathbf{R}^q \times \mathbf{R}^q$ and $K_\Theta = (K_{\Theta_1}, K_{\Theta_2}, \ldots, K_{\Theta_n}) \in \mathbf{R}^r \times \mathbf{R}^r$, $K_{\lambda_{1i}}, K_{\lambda_{2i}}$ and $K_\Lambda \in \mathbf{R}$; $i = 1(1)m$ are all control gains and $\bar{\lambda}_{1i} = \hat{\lambda}_{1i} - \lambda_{1i}$, $\bar{\lambda}_{2i} = \hat{\lambda}_{2i} - \lambda_{2i}$ and $\bar{\Lambda}_i = \hat{\Lambda}_i - \Lambda_i$; $1(1)m$, $\bar{\theta}_1 = \hat{\theta}_1 - \theta_1$, $\bar{\theta}_2 = \hat{\theta}_2 - \theta_2$ and $\bar{\Theta} = \hat{\Theta} - \Theta$. $B, C, D \in \mathbf{R}^n \times \mathbf{R}^n\ \forall i$.

The introduction of matrices B, C and D into the adaptation laws, decision of which is totally in our hand, is where the uniqueness of the technique lies. With their introduction, the method gets the flexibility of determining the feedback terms without using any lemmas in such a way that controllability is obtained.

2.3 *Stability analysis*

Theorem. *If the error dynamics* (5) *is controlled by* $\eta(t)$ *given by* (9) *and* $L_i(t)$ *is determined by equation*

$$L_i(t) = A^{-1}(\bar{\lambda}_{1i}B + \bar{\lambda}_{2i}C + \bar{\Lambda}_i D); \quad i = 1(1)m \tag{11}$$

together with the parameter update laws given by (10), *then the state trajectories will converge to sliding surface* $S(t) = 0$.

Proof. To prove this, let us define the following Lyapunov functional $V(t)$ as

$$\begin{aligned} V(t) &= \frac{1}{2}S(t)^T S(t) + \frac{1}{2}\bar{\theta}_1^T\bar{\theta}_1 + \frac{1}{2}\bar{\theta}_2^T\bar{\theta}_2 + \frac{1}{2}\bar{\Theta}^T\bar{\Theta} \\ &\quad + \frac{1}{2}\sum_{i=1}^m \bar{\lambda}_{1i}^2 + \frac{1}{2}\sum_{i=1}^m \bar{\lambda}_{2i}^2 + \frac{1}{2}\sum_{i=1}^m \bar{\Lambda}_i^2 \\ &\geq 0. \end{aligned} \tag{12}$$

Obviously, $V(t) > 0$. The time derivative of $V(t)$ along the trajectories of the error system (4) is

$$\begin{aligned} \dot{V} &= S(t)^T\dot{S}(t) + \bar{\theta}_1^T\dot{\bar{\theta}}_1 + \bar{\theta}_2^T\dot{\bar{\theta}}_2 + \bar{\Theta}^T\dot{\bar{\Theta}} \\ &\quad + \sum_{i=1}^m \bar{\lambda}_{1i}\dot{\bar{\lambda}}_{1i} + \sum_{i=1}^m \bar{\lambda}_{2i}\dot{\bar{\lambda}}_{2i} + \sum_{i=1}^m \bar{\Lambda}_i\dot{\bar{\Lambda}}_i. \end{aligned} \tag{13}$$

It can be shown using eqs (10), (11) and (13) that

$$\begin{aligned} \dot{V} &= -e^T A^2 K e - qS(t)^T \text{sgn} S(t) - \bar{\theta}_1^T K_{\theta_1}\bar{\theta}_1 \\ &\quad - \bar{\theta}_2^T K_{\theta_2}\bar{\theta}_2 - \bar{\Theta}^T K_\Theta \bar{\Theta} - \sum_{i=1}^m K_{\lambda_{1i}}\bar{\lambda}_{1i}^2 \\ &\quad - \sum_{i=1}^m K_{\lambda_{2i}}\bar{\lambda}_{2i}^2 + \sum_{i=1}^m K_{\Lambda_i}\bar{\Lambda}_i^2 \\ &< 0. \end{aligned} \tag{14}$$

Since $\dot{V} \leq 0$, according to Lyapunov theorem we know $e_i \to 0$ $(i = 1, 2, 3)$ as $t \to \infty$ which means that the required combination synchronization is achieved. □

3. System description

The modified time-delay Lorenz chaotic system is given by

$$\begin{cases} \dot{u}_1 = \sigma(u_2 - u_1), \\ \dot{u}_2 = \rho u_1 - u_1 u_3 - u_2(t - T_1), \\ \dot{u}_3 = u_1 u_2 - \beta u_3(t - T_2), \end{cases} \tag{15}$$

where u_1, u_2, u_3 are the state variables, σ, ρ, β are the parameters and T_1, T_2 are the time delays.

For the parameter values $\sigma = 0.9$, $\rho = 2.5$, $\beta = 0.1$, and time delays $T_1 = 1$, $T_2 = 2$, the system shows chaotic behaviour. The phase portrait of the system is shown in figure 1.

4. Combination synchronization of identical time-delay Lorenz chaotic system via robust adaptive sliding mode control

In this section, illustrative example of the proposed method is shown.

4.1 *Main results*

For combination synchronization, the two identical time-delay Lorenz chaotic system taken as master system is given as

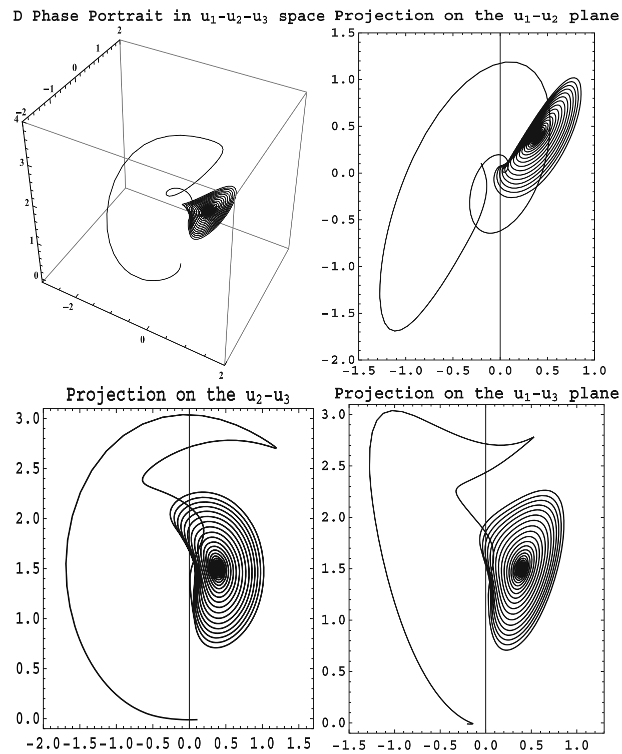

Figure 1. Phase portraits of modified time-delay Lorenz chaotic system.

$$\begin{cases} \dot{u}_1 = \sigma(u_2 - u_1), \\ \dot{u}_2 = \rho u_1 - u_1 u_3 - u_2(t - T_1), \\ \dot{u}_3 = u_1 u_2 - \beta u_3(t - T_2), \end{cases} \tag{16}$$

$$\begin{cases} \dot{v}_1 = \sigma(v_2 - v_1), \\ \dot{v}_2 = \rho v_1 - v_1 v_3 - v_2(t - T_1), \\ \dot{v}_3 = v_1 v_2 - \beta v_3(t - T_2), \end{cases} \tag{17}$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad f_1 = \begin{bmatrix} 0 \\ -u_1 u_3 - u_2(t - T_1) \\ u_1 u_2 \end{bmatrix},$$

$$g = \begin{bmatrix} u_2 - u_1 & 0 \\ 0 & u_1 \\ 0 & 0 \end{bmatrix},$$

$$\theta_1 = \begin{bmatrix} \sigma \\ \rho \end{bmatrix}, \qquad h_{13} = \begin{bmatrix} 0 \\ 0 \\ -u_3(t - T_2) \end{bmatrix}$$

$$h_{11} = h_{12} = 0_{3\times 1}, \qquad \lambda_{13} = \beta, \qquad \lambda_{11} = \lambda_{12} = 0.$$

$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \qquad f_2 = \begin{bmatrix} 0 \\ -v_1 v_3 - v_2(t - T_1) \\ v_1 v_2 \end{bmatrix},$$

$$g_2 = \begin{bmatrix} v_2 - v_1 & 0 \\ 0 & v_1 \\ 0 & 0 \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} \sigma \\ \rho \end{bmatrix},$$

$$h_{23} = \begin{bmatrix} 0 \\ 0 \\ -v_3(t - T_2) \end{bmatrix}$$

$h_{21} = h_{22} = 0_{3\times 1}, \lambda_{23} = \beta, \lambda_{21} = \lambda_{22} = 0$ and the time-delay Lorenz chaotic system taken as slave system is defined as

$$\begin{cases} \dot{w}_1 = \sigma(w_2 - w_1) + \eta_1, \\ \dot{w}_2 = \rho w_1 - w_1 w_3 - w_2(t - T_1) + \eta_2, \\ \dot{w}_3 = w_1 w_2 - \beta w_3(t - T_2) + \eta_3, \end{cases} \tag{18}$$

where

$$W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ -w_1 w_3 - w_2(t - T_1) \\ w_1 w_2 \end{bmatrix},$$

$$G = \begin{bmatrix} w_2 - w_1 & 0 \\ 0 & w_1 \\ 0 & 0 \end{bmatrix}, \qquad \Theta_3 = \begin{bmatrix} \sigma \\ \rho \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 0 \\ 0 \\ -w_3(t - T_2) \end{bmatrix}$$

$H_1 = H_2 = 0_{3\times 1}, \Lambda_3 = \beta, \Lambda_1 = \Lambda_2 = 0$ and η_1, η_2, η_3 are the controllers to be determined.

Define the error system as follows:

$$e = Pw - Qu - Rv, \tag{19}$$

where

$$P = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n),$$
$$Q = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

and

$$R = \text{diag}(\xi_1, \xi_2, \ldots, \xi_n).$$

The error dynamics is obtained as follows:

$$\begin{cases} \dot{e}_1 = \gamma_1 \sigma(w_2 - w_1) - \alpha_1 \sigma(u_2 - u_1) \\ \qquad - \xi_1 \sigma(v_2 - v_1) + \gamma_1 \eta_1 \\ \dot{e}_2 = \gamma_2(\rho w_1 - w_1 w_3 - w_2(t - T_2) + \eta_2) \\ \qquad - \alpha_2(\rho u_1 - u_1 u_3 - u_2(t - T_2)) \\ \qquad - \xi_2(\rho v_1 - v_1 v_3 - v_2(t - T_2)) \\ \dot{e}_3 = \gamma_3(w_1 w_2 - \beta w_3(t - T_2) + \eta_3) \\ \qquad - \alpha_3(u_1 u_2 - \beta u_3(t - T_2)) \\ \qquad - \xi_3(v_1 v_2 - \beta v_3(t - T_2)). \end{cases} \tag{20}$$

Now choose the required matrices as $A = I_3$ and $q = 1$, then $S(t) = (e_1, e_2, e_3)^T$.

So, the corresponding reaching law is obtained as follows: $\dot{S}(t) = (\text{sgn}(e_1), \text{sgn}(e_2), \text{sgn}(e_3))^T$.

Now choose $K_{\theta_1} = K_{\theta_2} = K_{\Theta} = I_2, K = I_3, K_{\lambda_{1i}} = K_{\lambda_{2i}} = K_{\Lambda_i} = 1; i = 1(1)3$ and choose the constant matrices as follows:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The corresponding L_i, $i(1)3$ functions are obtained as

$$L_1 = 0, \quad L_2 = 0,$$
$$L_3 = \begin{bmatrix} 3(\hat{\beta}-\beta) & 0 & (\hat{\beta}-\beta) \\ 0 & 3(\hat{\beta}-\beta) & 0 \\ (\hat{\beta}-\beta) & (\hat{\beta}-\beta) & 3(\hat{\beta}-\beta) \end{bmatrix}.$$

So, the adaptive controller obtained by using sliding mode is given by

$$\begin{cases} \gamma_1\eta_1 = -\gamma_1\hat{\sigma}(w_2 - w_1) + \alpha_1\hat{\sigma}(u_2 - u_1) \\ \qquad + \xi_1\hat{\sigma}(v_2 - v_1) - 3(\hat{\beta}-\beta)e_1(t - T_2) \\ \qquad - (\hat{\beta}-\beta)e_3(t - T_2) - e_1 \\ \gamma_1\eta_2 = -\gamma_2(\hat{\rho}w_1 - w_1w_3 - w_2(t - T_1)) \\ \qquad + \alpha_2(\hat{\rho}u_1 - u_1u_3 - u_2(t - T_1)) \\ \qquad + \xi_2(\hat{\rho}v_1 - v_1v_3 - v_2(t - T_1)) \\ \qquad - 3(\hat{\beta}-\beta)e_2(t - T_2) - e_2 \\ \gamma_1\eta_3 = -\gamma_3(w_1w_2 - \hat{\beta}w_3(t - T_2)) + \alpha_3(u_1u_2 \\ \qquad - \hat{\beta}u_3(t - T_2)) + \xi_3(v_1v_2 - \hat{\beta}v_3(t - T_2)) \\ \qquad - (\hat{\beta}-\beta)e_1(t - T_2) - (\hat{\beta}-\beta)e_2(t - T_2) \\ \qquad - 3(\hat{\beta}-\beta)e_3(t - T_2) - e_3. \end{cases} \tag{21}$$

The parameter update laws are then,

$$\begin{cases} \dot{\hat{\sigma}} = \gamma_1e_1(w_2 - w_1) - \alpha_1e_1(u_2 - u_1) \\ \qquad - \xi_1(v_2 - v_1)e_1 - (\hat{\sigma} - \sigma) \\ \dot{\hat{\rho}} = \gamma_2w_1e_2 - \alpha_2e_2u_1 - \xi_2e_2v_1 - (\hat{\rho} - \rho) \\ \dot{\hat{\beta}} = 3e_1(t - T_2)e_1 + e_3(t - T_2)e_1 + 3e_2(t - T_2)e_2 \\ \qquad - \gamma_3w_3(t - T_2)e_3 + \alpha_3u_3(t - T_2)e_3 \\ \qquad + \xi_3v_3(t - T_2)e_3 + e_1(t - T_2)e_3 + e_2(t - T_2)e_3 \\ \qquad + 3e_3(t - T_2)e_3 - (\hat{\beta} - \beta). \end{cases} \tag{22}$$

For stability analysis, consider the Lyapunov function as follows:

$$V(t) = \frac{1}{2}S(t)^T S(t) + \frac{1}{2}(\hat{\alpha} - \alpha)^2 + \frac{1}{2}(\hat{\beta} - \beta)^2 + \frac{1}{2}(\hat{\rho} - \rho)^2.$$

Obviously, $V(t) > 0$. The time derivative of $V(t)$ along the trajectories of the error system (19) is

$$\begin{aligned} \dot{V}(t) &= S(t)\dot{S}(t) + (\hat{\alpha} - \alpha)\dot{\hat{\alpha}} + (\hat{\beta} - \beta)\dot{\hat{\beta}} + (\hat{\rho} - \rho)\dot{\hat{\rho}} \\ &= -e_1^2 - e_2^2 - e_3^2 - (\hat{\alpha} - \alpha)^2 \\ &\quad -(\hat{\beta} - \beta)^2 - (\hat{\rho} - \rho)^2 \\ &< 0. \end{aligned} \tag{23}$$

This establishes the stability of error dynamics which means that the required synchronization is achieved.

The following Corollaries can easily be obtained from Theorem 1, and the proofs of these Corollaries are similar to Theorem 1. So, the proofs are omitted.

COROLLARY 1

For $\xi_1 = \xi_2 = \xi_3 = 0$, $\gamma_1 = \gamma_2 = \gamma_3 = 1$ *and controllers*:

$$\begin{cases} \gamma_1\eta_1 = -\hat{\sigma}(w_2 - w_1) + \alpha_1\hat{\sigma}(u_2 - u_1) \\ \qquad - 3(\hat{\beta}-\beta)e_1(t - T_2) \\ \qquad - (\hat{\beta}-\beta)e_3(t - T_2) - e_1 \\ \gamma_1\eta_2 = -(\hat{\rho}w_1 - w_1w_3 - w_2(t - T_1)) \\ \qquad + \alpha_2(\hat{\rho}u_1 - u_1u_3 - u_2(t - T_1)) \\ \qquad - 3(\hat{\beta}-\beta)e_2(t - T_2) - e_2 \\ \gamma_1\eta_3 = -(w_1w_2 - \hat{\beta}w_3(t - T_2)) \\ \qquad + \alpha_3(u_1u_2 - \hat{\beta}u_3(t - T_2)) \\ \qquad - (\hat{\beta}-\beta)e_1(t - T_2) - (\hat{\beta}-\beta)e_2(t - T_2) \\ \qquad - 3(\hat{\beta}-\beta)e_3(t - T_2) - e_3 \end{cases} \tag{24}$$

and the parameter update rule

$$\begin{cases} \dot{\hat{\sigma}} = e_1(w_2 - w_1) - \alpha_1e_1(u_2 - u_1) - (\hat{\sigma} - \sigma) \\ \dot{\hat{\rho}} = w_1e_2 - \alpha_2e_2u_1 - (\hat{\rho} - \rho) \\ \dot{\hat{\beta}} = 3e_1(t - T_2)e_1 + e_3(t - T_2)e_1 \\ \qquad + 3e_2(t - T_2)e_2 - w_3(t - T_2)e_3 \\ \qquad + \alpha_3u_3(t - T_2)e_3 \\ \qquad + e_1(t - T_2)e_3 + e_2(t - T_2)e_3 \\ \qquad + 3e_3(t - T_2)e_3 - (\hat{\beta} - \beta) \end{cases} \tag{25}$$

the master systems (16) *and* (17) *will achieve modified projective synchronization with the slave system* (18).

COROLLARY 2

For $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\gamma_1 = \gamma_2 = \gamma_3 = 1$ *and controllers*:

$$\begin{cases} \gamma_1\eta_1 = -\hat{\sigma}(w_2 - w_1) + \xi_1\hat{\sigma}(v_2 - v_1) \\ \qquad - 3(\hat{\beta} - \beta)e_1(t - T_2) \\ \qquad - (\hat{\beta} - \beta)e_3(t - T_2) - e_1 \\ \gamma_1\eta_2 = -(\hat{\rho}w_1 - w_1w_3 - w_2(t - T_1)) \\ \qquad + \xi_2(\hat{\rho}v_1 - v_1v_3 - v_2(t - T_1)) \\ \qquad - 3(\hat{\beta} - \beta)e_2(t - T_2) - e_2 \\ \gamma_1\eta_3 = -(w_1w_2 - \hat{\beta}w_3(t - T_2)) \\ \qquad + \xi_3(v_1v_2 - \hat{\beta}v_3(t - T_2)) \\ \qquad - (\hat{\beta} - \beta)e_1(t - T_2) - (\hat{\beta} - \beta)e_2(t - T_2) \\ \qquad - 3(\hat{\beta} - \beta)e_3(t - T_2) - e_3 \end{cases} \tag{26}$$

and the parameter update rule

$$\begin{cases} \dot{\hat{\sigma}} = e_1(w_2 - w_1) - \xi_1(v_2 - v_1)e_1 - (\hat{\sigma} - \sigma) \\ \dot{\hat{\rho}} = w_1e_2 - \xi_2e_2v_1 - (\hat{\rho} - \rho) \\ \dot{\hat{\beta}} = 3e_1(t - T_2)e_1 + e_3(t - T_2)e_1 \\ \quad + 3e_2(t - T_2)e_2 - w_3(t - T_2)e_3 \\ \quad + \xi_3v_3(t - T_2)e_3 + e_1(t - T_2)e_3 \\ \quad + e_2(t - T_2)e_3 + 3e_3(t - T_2)e_3 - (\hat{\beta} - \beta) \end{cases} \tag{27}$$

the master systems (16) *and* (17) *will achieve modified projective synchronization with the slave system* (18).

COROLLARY 3

For $\xi_1 = \xi_2 = \xi_3 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\gamma_1 = \gamma_2 = \gamma_3 = 1$ *and controllers*:

$$\begin{cases} \gamma_1\eta_1 = -\hat{\sigma}(w_2 - w_1) - 3(\hat{\beta} - \beta)e_1(t - T_2) \\ \qquad - (\hat{\beta} - \beta)e_3(t - T_2) - e_1 \\ \gamma_1\eta_2 = -(\hat{\rho}w_1 - w_1w_3 - w_2(t - T_1)) \\ \qquad - 3(\hat{\beta} - \beta)e_2(t - T_2) - e_2 \\ \gamma_1\eta_3 = -(w_1w_2 - \hat{\beta}w_3(t - T_2)) \\ \qquad - (\hat{\beta} - \beta)e_1(t - T_2) \\ \qquad - (\hat{\beta} - \beta)e_2(t - T_2) \\ \qquad - 3(\hat{\beta} - \beta)e_3(t - T_2) - e_3 \end{cases} \tag{28}$$

and the parameter update rule

$$\begin{cases} \dot{\hat{\sigma}} = e_1(w_2 - w_1) - (\hat{\sigma} - \sigma) \\ \dot{\hat{\rho}} = w_1e_2 - (\hat{\rho} - \rho) \\ \dot{\hat{\beta}} = 3e_1(t - T_2)e_1 + e_3(t - T_2)e_1 + 3e_2(t - T_2)e_2 \\ \quad - w_3(t - T_2)e_3 + e_1(t - T_2)e_3 + e_2(t - T_2)e_3 \\ \quad + 3e_3(t - T_2)e_3 - (\hat{\beta} - \beta) \end{cases} \tag{29}$$

then the equilibrium point of the response system (18) *becomes asymptotically stable.*

4.2 *Numerical simulations*

Numerical simulations are carried out in Matlab to verify the efficiency of the designed controllers. The parameter values are chosen so that system shows chaotic behaviour in the absence of controllers as shown in figure 1. The initial conditions of the master systems and slave system are chosen as $(u_1[t/;\ t \le 0],\ u_2[t/;\ t \le 0],\ u_3[t/;\ t \le 0]) = (-0.2, 0.1, -0.01)$, $(v_1[t/;\ t \le 0],\ v_2[t/;\ t \le 0],\ v_3[t/;\ t \le 0]) = (-0.1, 0.2, -0.02)$, $(w_1[t/;\ \dot{t} \le 0],\ w_2[t/;\ t \le 0],\ w_3[t/;\ t \le 0]) = (-0.4, 0.4, -0.04)$.

Case 1. Suppose that $\gamma_1 = \gamma_2 = \gamma_3 = 1, \alpha_1 = \alpha_2 = \alpha_3 = \xi_1 = \xi_2 = \xi_3 = -3$. The corresponding initial condition for the error system is obtained as $(e_1[t/;t \le 0], e_2[t/;t \le 0], e_3[t/;t \le 0]) = (-1.3,\ 1.3,\ -0.13)$. Also we choose the initial condition of parameter estimation function as $\hat{\sigma}[t/;\ t \le 0] = 1, \hat{\rho}[t/;\ t \le 0] = 2, \hat{\beta}[t/;\ t \le 0] = 3$. The convergence of error state variables in figure 2 shows that the projective combination synchronization

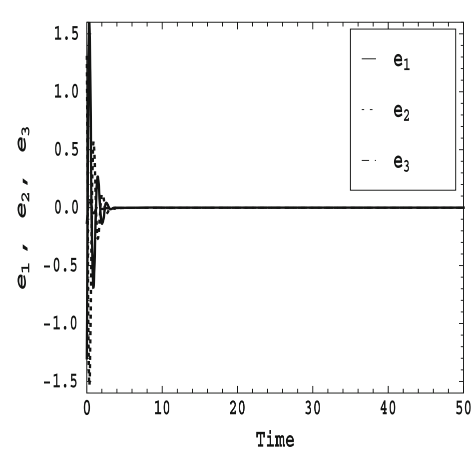

Figure 2. Error dynamics among the drive and the response system with controllers deactivated for $t > 0$, where $e_1 = w_1 + 3u_1 + 3v_1, e_2 = w_2 + 3u_2 + 3v_2$ and $e_3 = w_3 + 3u_3 + 3v_3$.

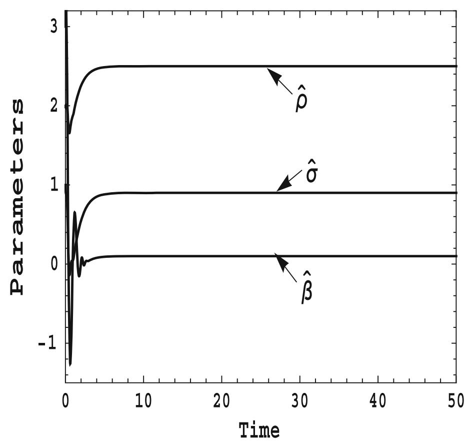

Figure 3. The estimated values of the unknown parameters $\hat{\sigma}$, $\hat{\rho}$ and $\hat{\beta}$ as combination synchronization occurs.

among time-delay chaotic systems (16), (17) and (18) is achieved when controllers are activated at $t > 0$. Figure 4 shows the trajectory of master and slave state variables when controllers are activated at $t > 0$. This again confirms projective combination synchronization among time-delay chaotic systems (16), (17) and (18). Also the estimated values of the unknown parameters tend to $\hat{\sigma} \to \sigma$, $\hat{\rho} \to \rho$, $\hat{\beta} \to \beta$ as displayed in figure 3.

Case 2. Suppose that $\gamma_1 = \gamma_2 = \gamma_3 = 0.5, \alpha_1 = \alpha_2 = \alpha_3 = 1.5, \xi_1 = \xi_2 = \xi_3 = 1$. The corresponding initial condition for the error system is obtained as $(e_1[t/; t \leq 0], e_2[t/; t \leq 0], e_3[t/; t \leq 0]) = (0.2, -0.15, 0.015)$. Also we choose the initial condition of parameter estimation function as $\hat{\sigma}[t/; t \leq 0] = 1, \hat{\rho}[t/; t \leq 0] = 2, \hat{\beta}[t/; t \leq 0] = 3$. The convergence of error state variables in figure 5 shows that the combination synchronization among time-delay chaotic systems (16), (17) and (18) is achieved when controllers are activated at $t > 0$. Figure 7 shows the trajectory of the master and the slave state variables when controllers are activated at $t > 0$. This again confirms combination synchronization among time-delay chaotic systems (16), (17) and (18). Also the estimated values of the unknown parameters tend to $\hat{\sigma} \to \sigma$, $\hat{\rho} \to \rho$, $\hat{\beta} \to \beta$ as displayed in figure 6.

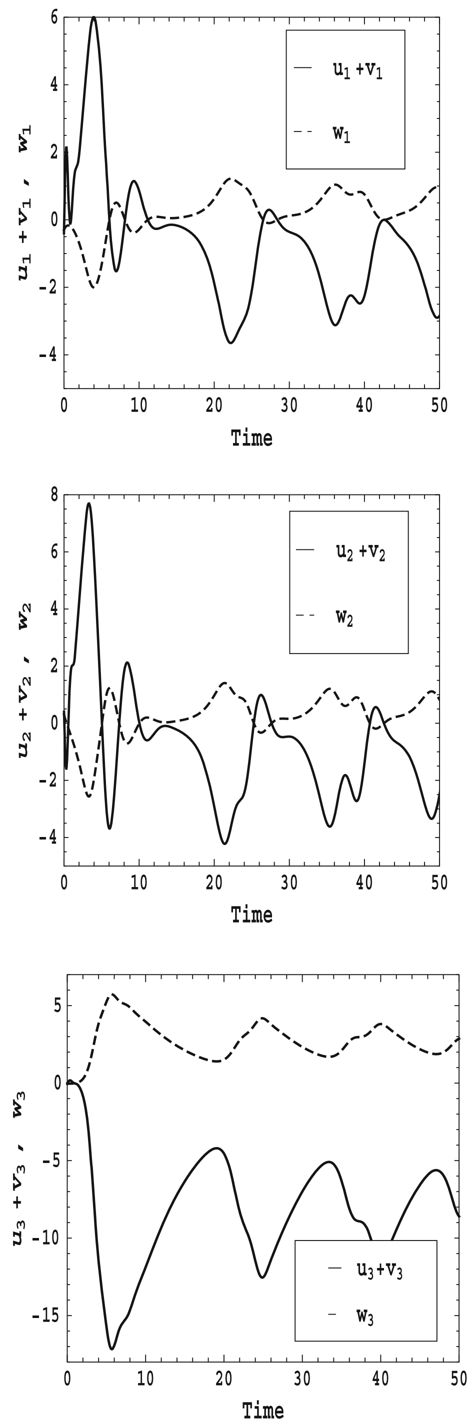

Figure 4. Dynamics of the drive and the response state variables with controllers activated for $t > 0$.

5. Conclusion

The combination synchronization of time-delay chaotic system using robust adaptive sliding mode control is accomplished. Also, the introduced method is applied on identical time-delay Lorenz chaotic system using robust adaptive sliding mode control. Finally, simulations are displayed to show the viability of the proposed methodology. Computational and analytical results are in excellent agreement. We have shown from the theoretical analysis that various controllers which are suitable for different types of synchronization scheme can be obtained from the general

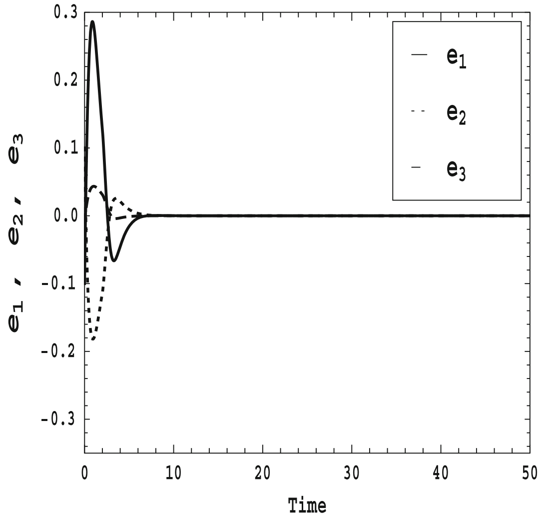

Figure 5. Error dynamics among the drive and the response system with controllers deactivated for $t > 0$, where $e_1 = 0.5w_1 - 1.5u_1 - v_1, e_2 = 0.5w_2 - 1.5u_2 - v_2$ and $e_3 = 0.5w_3 - 1.5u_3 - v_3$.

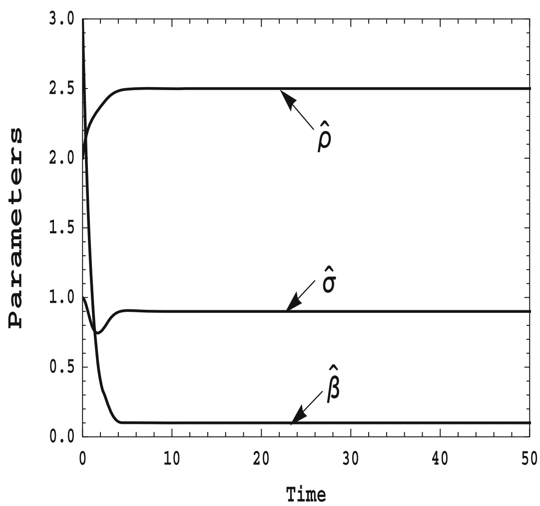

Figure 6. The estimated values of the unknown parameters $\hat{\sigma}$, $\hat{\rho}$ and $\hat{\beta}$ as combination synchronization occurs.

results. The typical synchronization between one master and one slave is a special case of combination synchronization. Combination synchronization of time-delay chaotic system using robust adaptive sliding mode control has many applications in secure communication, neural network and other important areas. Other synchronization techniques like combination–combination synchronization, compound synchronization and compound-combination synchronization of time-delay chaotic system are important issues to be discussed in future.

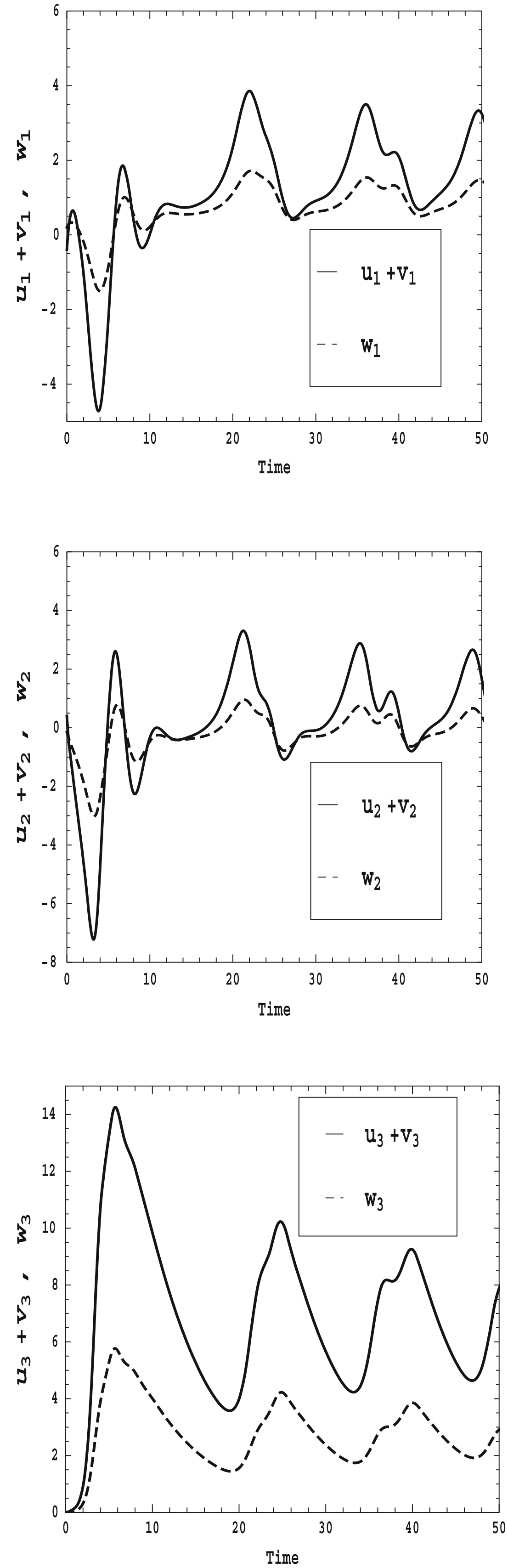

Figure 7. Dynamics of the drive and the response state variables with controllers activated for $t > 0$.

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