



Saha equation in Rindler space

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Abstract. The Saha equations for the photoionization process of hydrogen atoms and the creation of electron–positron pairs at high temperature are investigated in a reference frame undergoing a uniform accelerated motion. It is known as the Rindler space.

Keywords. Saha equation; uniformly accelerated frame; Rindler coordinates; photoionization; pair production.

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1. Introduction

The well-known Lorentz transformations are the space–time coordinate transformation between two inertial frame of references [1]. However, with the help of the principle of equivalence, one may obtain space–time transformations between a uniformly accelerated frame and an inertial frame and vice versa in the same manner as it is done in special theory of relativity [2–5]. In this scenario, the flat local geometry is called the Rindler space. For an illustration, let us consider two reference frames – the frame S is an inertial one, which is at rest or moving with uniform velocity, but there is a uniform gravitational field and the frame S' , which is undergoing a uniform accelerated motion with respect to this inertial frame, but in the absence of gravitational field. Then, in accordance with the principle of equivalence, these two frames are physically equivalent [6–8]. We may assume that the strong gravitational field is produced by a black-hole-like strong gravitating object. Therefore, the gravitational field may be approximated by a constant value within a small region at a distance x from the centre of the gravitating object. This is also called the local acceleration of the frame.

To investigate Saha equation in a uniformly accelerated frame of reference or in Rindler space, we assume the results obtained using the concepts of principle of equivalence [9,10]. The Lagrangian of the particle (in our study, it may be a hydrogen atom or a hydrogen ion or an electron) is derived from Hamilton's principle, which further gives the Hamiltonian of the particle in Rindler space from the standard relations of classical

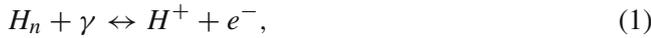
mechanics [9,10]. Then, considering a partially ionized hydrogen plasma which is a reactive mixture of neutral hydrogen atoms, hydrogen ions, electrons and photons or considering an electron–positron plasma composed of an interacting mixture of electrons, positrons and photons, we shall obtain the modified form of Saha equations for both the cases when observed from a uniformly accelerated frame of reference or in Rindler space. To the best of our knowledge, the study of Saha equation in Rindler space has not been reported earlier. We shall also compare our findings with the conventional results.

The article is organized in the following manner: In the next section we shall develop a formalism to investigate the photoionization of partially ionized hydrogen plasma in Rindler space. In §3 we shall study the electron–positron pair production from intense electromagnetic radiation at high temperature in an interacting electron–positron plasma in Rindler space. In the last section we give our conclusions.

2. Photoionization of hydrogen atoms

To study the Saha ionization process for hydrogen atoms, we consider a partially ionized hydrogen plasma. For the sake of mathematical simplicity, the system is assumed to have cylindrical geometry and the plasma is expanding with a constant acceleration g along the positive x -direction, which is also the symmetry axis of the cylinder. Then, according to the principle of equivalence, any ionization or deionization of the accelerated

particles taking place at some point is equivalent to their occurrence at rest frame in the presence of a gravitational field g . Now, it is well known that a dynamic dead-lock situation will be reached when the rate of ionization process and the rate of deionization process become just equal. We represent it by the reaction equation



where H_n indicates the n th excited state of hydrogen atom with $n = 1$ the ground state. Under equilibrium condition the chemical potentials of the components are related by the equation

$$\mu(H_n) = \mu(H^+) + \mu(e). \quad (2)$$

Since the number of photons is not conserved, $\mu(\gamma) = 0$. Defining

$$m' = m_0 \left(1 + \frac{gx}{c^2}\right)^{-1} \quad \text{and} \quad m'' = m_0 \left(1 + \frac{gx}{c^2}\right) \quad (3)$$

the single-particle energy may be written as

$$\varepsilon_p = m''c^2 + \frac{p^2}{2m'}. \quad (4)$$

From the standard results of text books on statistical mechanics [15–17] the number density of a component (except for photon) is given by

$$\begin{aligned} n &= \frac{N}{\Delta V} \\ &= \frac{4\pi g_d}{h^3} \int_0^\infty p^2 dp \exp \left[-\frac{1}{kT} \left(\frac{p^2}{2m'} + m''c^2 - \mu \right) \right], \end{aligned} \quad (5)$$

where g_d is the degeneracy of the species, $\Delta V = A \Delta x$ is a small volume element, Δx is a small length element in the x -direction at a distance x from the origin and A is the cross-sectional area of the cylinder. The length Δx is such that g is constant within ΔV . Evaluating the integral and rearranging, we have

$$\mu = m''c^2 - kT \ln \left(\frac{g_d n_Q}{n} \right). \quad (6)$$

This is the general expression for chemical potential for a particular species in terms of its concentration. Further,

$$n_Q = \left(\frac{2\pi m' kT}{h^2} \right)^{3/2} \quad (7)$$

is called the quantum concentration for the particular species in the mixture. Then, from eqs (2) and (6), we have

$$\begin{aligned} \frac{n(H^+)n(e)}{n(H_n)} &= \frac{n_{Q_e}}{g_n} \exp \left(-\frac{\Delta E_n}{kT} \right) \\ &= R_{g>0} \text{ (say)}, \end{aligned} \quad (8)$$

where $n(i)$ is the number density for the species i and $g_n = n^2$ is the degeneracy of the neutral hydrogen atom in the n th excited state (as the degeneracy of electrons has been considered separately in the derivation, the factor 2 does not appear in the expression for g_n) and

$$\Delta E_n = \Delta m''c^2 = \left(1 + \frac{gx}{c^2}\right) \Delta mc^2 \quad (9)$$

with

$$\Delta m = m(H_n) - m(H^+) - m(e)$$

is a measure of excitation energy, whereas the conventional form of Saha equation with $g = 0$ is given by

$$\begin{aligned} \frac{n(H^+)n(e)}{n(H_n)} &= \frac{n_{Q_e g=0}}{g_n} \exp \left(-\frac{\Delta mc^2}{kT} \right) \\ &= R_{g=0} \text{ (say)}. \end{aligned} \quad (10)$$

Hence, the ratio with $g > 0$ and $g = 0$ can be written as

$$\frac{R_{g>0}}{R_{g=0}} = \left(1 + \frac{gx}{c^2}\right)^{-3/2} \exp \left(-\frac{gx \Delta m}{kT} \right). \quad (11)$$

3. Pair production at high temperature

We next consider the electron–positron pair creation at high temperature ($kT > 2m_0$). The pair production process can be represented by the reaction equation



In this situation the chemical equilibrium condition is given by

$$\mu(e^+) + \mu(e^-) = 0 \quad (13)$$

with

$$\mu(\gamma) = 0$$

and

$$\mu(e^\mp) = m''(e)c^2 - kT \ln \left(\frac{g_e n_Q}{n_{e^\mp}} \right). \quad (14)$$

Then, we have

$$n_{e^-} n_{e^+} = g_e^2 n_Q^2 \exp \left(-\frac{2m''(e)c^2}{kT} \right). \quad (15)$$

Here $m'(e)$ and $m''(e)$ are given by eq. (3) with m_0 replaced by $m(e)$, the electron rest mass. Then, the ratio of the product of electron–positron concentration with $g > 0$ and $g = 0$ is given by

$$\frac{n_{e^-} n_{e^+ g>0}}{n_{e^-} n_{e^+ g=0}} = \left(1 + \frac{gx}{c^2}\right)^{-3} \exp \left(-\frac{2gx m(e)}{kT} \right). \quad (16)$$

4. Conclusion

It can be very easily shown that for very low value of gravitational field strength we get back the conventional results. On the other hand, in extreme conditions of very high gravitational field, the exponential terms in both eqs (11) and (16) become zero, making both the ratios equal to zero. Hence, we may conclude that strong gravitational field suppresses photoionization of hydrogen and also e^-e^+ pair creation. Therefore, photoionization of hydrogen is possible at the surface of a main-sequence star or a post main-sequence star or at the surface of a white dwarf and some spectroscopic measurement is in principle possible. But, for neutron stars, with very strong gravitational field, it is not likely to have hydrogen ions at the vicinity of its surface. Further, as there is no direct interaction of photons with classical gravitational field, the suppression of photoionization of hydrogen atoms or pair creation at the surface of strongly gravitating stellar objects is physically understandable. This is analogous to the suppression of e^-e^+ creation at the surface of bare quark stars. In this case, the surface field is dominated by strong force. Of course, the quantum field theoretic Schwinger mechanism of particle production at the surface of black holes in the presence of ultrastrong gravitational field – the Hawking radiation is quite different from what we are discussing here.

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