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# Degenerate Hopf bifurcation in a self-exciting Faraday disc dynamo

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**Abstract.** In order to further understand a self-exciting Faraday disc dynamo (Hide *et al*, in *Proc. R. Soc. A* **452**, 1369 1996), showing chaotic attractors with very complicated topological structures, we present codimension one and two (degenerate) Hopf bifurcations and prove the existence of periodic solutions. In addition, numerical simulations are given for confirming the theoretical results.

**Keywords.** Chaotic attractor; Faraday disc dynamo; Lyapunov coefficient; degenerate Hopf bifurcations.

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## 1. Introduction

The study of chaos is very important in nonlinear theory. There is a long history of the study of related topics. Periodic and chaotic attractors can be classified into self-excited and hidden [1–8] attractors. More precisely, a self-excited attractor has a basin of attraction that is associated with an unstable equilibrium.

The study of periodic solutions [9–12] has attracted a number of researchers. Zoldi and Greenside [13] discovered many unstable periodic orbits of Kuramoto–Sivashinsky system, then found something around the statistical correspondence between the periodic orbits and chaos. Kato and Yamada [14] proved that an unstable periodic orbit will reproduce the classic structure of Couette turbulence. Ishiyama and Saiki [15] stated that the structure of a chaotic solution and a class of finite unstable periodic solutions are inseparable. Recently, self-exciting Faraday disk dynamo is also a topic of concern [16–20]. The lowest-order unstable periodic orbits can be obtained by using the method of close returns on a Poincaré section [18]. The values of these linking numbers between pairs of orbits play an irreplaceable role for chaos generation. Therefore, the existence of periodic orbits for the HSA dynamo is very important [20].

The research in the present paper contribute to understand analytically the degenerate Hopf bifurcations in the self-exciting Faraday disc homopolar dynamo (HSA dynamo). By using Lyapunov coefficients at the Hopf bifurcations, we study all possible bifurcations (generic and degenerate ones) which can happen at the equilibria in HSA dyanamo. In this way, the results presented in [20] are extended. More precisely, the Hopf surface is obtained in the space of parameters. In addition, the obatained first Lyapunov coefficient $l_1$ can show the possibility for codimension two bifurcations, then the second Lyapunov coefficient $l_2$ needs to be calculated.

The paper is organized as follows: In §2, the Faraday disc dynamo is explained, which comprises a disc and a coil, arranged in series with an electric motor. In §3, the outline of the Hopf bifurcation methods about codimension one and two Hopf bifurcations are given, in particular, how to calculate the Lyapunov coefficients related to the stability of the equilibrium. In §4, we obtain the main results about the existence of periodic orbits, described in Theorems 4.1–4.3. Finally, in §5, some concluding remarks are given.

## 2. The self-exciting Faraday disc dynamo

The Faraday disc dynamo with an electric motor [20]

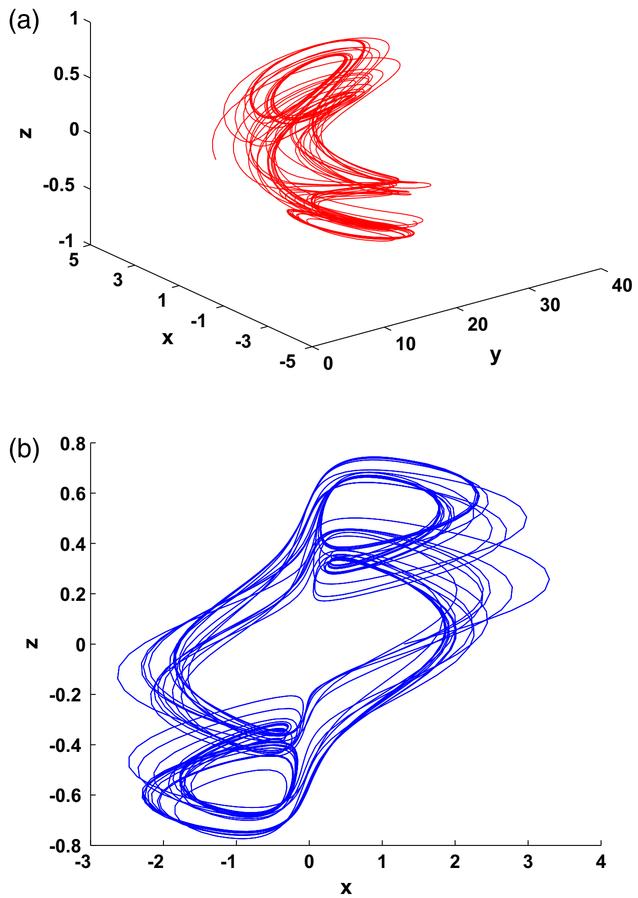


**Figure 1.** Phase diagram of system (2.1) with only stable equilibria vs. parameter $\alpha = 20, b = 1.5, k = 1, \lambda = 1.2$ with initial values (1.3, 6.0, 0) (**a**) in 3D space $x{-}y{-}z$ and (**b**) projected on the $x{-}z$ plane.

$$\begin{cases} \dot{x} = (\alpha - 1)x - xy - \beta\lambda z, \\ \dot{y} = k(\alpha x^2 - y), \\ \dot{z} = x - \lambda z, \end{cases} \tag{2.1}$$

where $\alpha$, $\lambda$, $k$ and $\beta$ are positive. Here, $x(t)$ is the current flowing through the dynamo, $y(t)$ is the angular rotation rate of the disk and $z(t)$ is the angular rotation rate of the motor. By choosing some parameter values $\alpha = 20$, $b = 1.5$, $k = 0.6$, $\lambda = 1.2$ [20], and the initial condition $(1, 10, 0.1)$, the system (2.1) can have a chaotic attractor as shown in figure 1.

System (2.1) is invariant under the transformation $(x, y, z) \to (-x, -y, z)$, namely the system has reflected symmetry around the $z$-axis. When $\alpha(-1 + \alpha - \beta) < 0$ or $\alpha = \beta + 1$, system (2.1) has only one equilibrium $E_0(0, 0, 0)$. When $a(-1 + a - b) > 0$, system (2.1) has three equilibria $E_0(0, 0, 0)$ and $E_{1,2}(x_0, y_0, z_0)$, where

$$x_0 = \sqrt{\frac{-1 + a - b}{a}},$$

$$y_0 = -1 + a - b, \qquad z_0 = \frac{1}{\lambda}\sqrt{\frac{-1 + a - b}{a}}.$$

The characteristic polynomial of system (2.1) at $E_0(0, 0, 0)$ is

$$\begin{aligned} &s^3 + s^2(1 - a + k + \lambda) \\ &\quad + s(k - ak + \lambda - a\lambda + b\lambda + k\lambda) \\ &\quad + k\lambda - ak\lambda + bk\lambda = 0. \end{aligned} \tag{2.2}$$

Suppose that the characteristic equation (2.2) has a pair of pure imaginary roots. When $\alpha = \alpha_0 = 1 + \lambda$, (2.2) yields

$$\lambda_1 = -k < 0, \quad \lambda_{2,3} = \pm\sqrt{\beta\lambda - \lambda^2}\, i,$$

where $\beta > \lambda$.

PROPOSITION 2.1

*If $\alpha = \alpha_0 = 1 + \lambda$, $\beta > \lambda$ and $k > 0$, then Jacobian matrix of system (2.1) at $E_0(0, 0, 0)$ has one negative real eigenvalue $-k$ and a pair of purely imaginary eigenvalues $\pm\sqrt{\beta\lambda - \lambda^2}\, i$.*

Now we observe that the characteristic polynomial of system (2.1) at points $E_{1,2}(x_0, y_0, z_0)$ is

$$\begin{aligned} &s^3 + s^2(-b + k + \lambda) \\ &\quad + s(-2k + 2ak - 3bk + k\lambda) - 2k\lambda \\ &\quad + 2ak\lambda - 2bk\lambda = 0. \end{aligned} \tag{2.3}$$

Again we assume that one of the roots of (2.3) is negative and the other two eigenvalues are purely imaginary, and so

$$\begin{aligned} \alpha &= \alpha_1 \\ &= -\frac{-2\beta - 3\beta^2 + 2k + 3\beta k + 2\beta\lambda - k\lambda - \lambda^2}{2(\beta - k)} > 0 \end{aligned}$$

must hold.

PROPOSITION 2.2

*If*

$$\begin{aligned} \alpha &= \alpha_1 \\ &= -\frac{-2\beta - 3\beta^2 + 2k + 3\beta k + 2\beta\lambda - k\lambda - \lambda^2}{2(\beta - k)} > 0, \end{aligned}$$

$$(\beta - \lambda)(\beta - k) < 0 \qquad \textit{and} \qquad \beta < k + \lambda,$$

*then Jacobian matrix of system* (2.1) *at* $E_{1,2}$ *has one negative real eigenvalue* $\beta - k - \lambda$ *and a pair of purely imaginary eigenvalues* $\pm\sqrt{(k(\beta-\lambda)\lambda)/(k-\beta)}\, i$.

## 3. Outline of the Hopf bifurcation methods

The projection method is described in Chapters 3 and 5 of [21], but based on the analysis of [12,22,23], for the first Lyapunov coefficient $l_1$, which shows the stability of a Hopf bifurcation. For the system

$$\dot{X} = f(X, \mu), \tag{3.1}$$

where $X \in R^3$ and $\mu \in R^4$, $f$ is a class of $C^\infty$ in $R^3 \times R^4$. Suppose that (3.1) has an equilibrium point $X = X_0$ at $\mu = \mu_0$, and marking the variable $X - X_0$ also by $X$, write

$$F(X) = f(X, \mu_0), \tag{3.2}$$

as

$$F(X) = AX + \frac{1}{2}B(X, X) + \frac{1}{6}C(X, X, X) + O(\|X\|^4),$$

where $A = f_x(0, \mu_0)$ and, for $i = 1, 2, 3$,

$$B(X, Y) = \sum_{j,k=1}^{3} \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k}\bigg|_{\xi=0} X_j Y_k,$$

$$C(X, Y, Z) = \sum_{j,k,l=1}^{3} \frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}\bigg|_{\xi=0} X_j Y_k Z_l.$$

Suppose that $A$ has a pair of complex eigenvalues $\lambda_{2,3} = \pm i w_0\ (w_0 > 0)$, and let $T^c$ be the generalized eigenspace of $A$ corresponding to $\lambda_{2,3}$. Let $p, q \in C^3$ be vectors such that

$$Aq = i w_0 q, \qquad A^T p = -i w_0 p, \qquad \langle p, q \rangle = 1,$$

where $A^T$ is the transpose of the matrix $A$. Any vector $y \in T^c$ can be represented as $y = wq + \bar{w}\bar{q}$, where $w = \langle p, y \rangle \in C$. The two-dimensional centre manifold at the eigenvalues $\lambda_{2,3}$ can be parametrized by $w$ and $\bar{w}$, by means of an immersion of the form $X = H(w, \bar{w})$, where $H: C^2 \to R^3$ has a Taylor expansion of the form

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2 \le j+k \le 3} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O(|w|^4),$$

with $h_{jk} \in C^3$ and $h_{jk} = \bar{h_{kj}}$. Substituting this expression into (3.2) we obtain the following differential equation:

$$H_w w' + H_w \bar{w}' = F(H(w, \bar{w})),$$

where $F$ is given by (3.2). Taking into account the coefficients of $F$, system (3.2), on the chart $w$ for a central manifold, can be written as follows:

$$\dot{w} = i w_0 w + \frac{1}{2} G_{21} w |w|^2 + O(|w|^4),$$

with $G_{21} \in C$. The 'first Lyapunov coefficient' can be given as

$$l_1 = \frac{1}{2} \mathrm{Re}\, G_{21},$$

where $G_{21} = \langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) \rangle$. Denoting $\mathcal{H}_{32}$ as

$$\begin{aligned}\mathcal{H}_{32} = {} & 6B(h_{11}, h_{21}) + B(\bar{h}_{20}, h_{30}) \\ & + 3B(\bar{h}_{21}, h_{20}) + 3B(q, h_{22}) \\ & + 2B(\bar{q}, h_{31}) + 6C(q, h_{11}, h_{11}) \\ & + 3C(q, \bar{h}_{20}, h_{20}) + 3C(q, q, \bar{h}_{21}) \\ & + 6C(q, \bar{q}, h_{21}) + 6C(\bar{q}, h_{20}, h_{11}) \\ & + C(\bar{q}, \bar{q}, h_{30}) + D(q, q, q, \bar{h}_{20}) \\ & + 6D(q, q, \bar{q}, \bar{h}_{11}) + 3D(q, \bar{q}, \bar{q}, h_{20}) \\ & + E(q, q, q, \bar{q}, \bar{q}) - 6G_{21}h_{21} - 3\bar{G}_{21}h_{21}\end{aligned}$$

and $G_{32} = \langle p, \mathcal{H}_{32} \rangle$, the second Lyapunov coefficient $l_2$ is given by

$$l_2 = \frac{1}{12} \mathrm{Re}\, G_{32}. \tag{3.3}$$

When $l_1 \neq 0$, the dynamic behaviour of system (3.1) based on the centre manifold, is orbitally topologically equivalent to the following complex normal form:

$$w' = (\eta + i w) w + l_1 w |w|^2,$$

where $w \in C$, $\eta$, $w$ and $l_1$ are real functions having derivatives of arbitrary higher order, which are continuations of 0, $w_0$ and the first Lyapunov coefficient at the Hopf point [23]. As $l_1 < 0$ ($l_1 > 0$) one family of stable (unstable) periodic orbits can be found on this family of manifolds, shrinking to an equilibrium point at the Hopf point.

A Hopf point of codimension two is a Hopf point when $l_1$ vanishes. It is called transversal if $\eta = 0$ and $l_1 = 0$

have transversal intersections, where $\eta = \eta(\mu)$ is the real part of the critical eigenvalues. When $l_2 \neq 0$, the dynamic behaviour of system (3.1) based on the centre manifold, is orbitally topologically equivalent to

$$w' = (\eta + iw)w + \tau w|w|^2 + l_2 w|w|^4,$$

where $\eta$ and $\tau$ are unfolding parameters.

## 4. Hopf bifurcation and statements of the main results in the self-exciting Faraday disc dynamo (2.1)

Now we apply the Hopf bifurcation theory to analyse the complex dynamical bifurcations.

### 4.1 *Hopf bifurcation at $E_0$*

**Theorem 4.1.** *Consider system (2.1) with $\alpha = \alpha_0 = 1 + \lambda$, $\beta > \lambda$ and $k > 0$. The first Lyapunov coefficient at the equilibrium $E_0$ is given by*

$$l_1 = -\frac{\beta\lambda(1+\lambda)(3k^2 + 8\beta\lambda - 2k\lambda - 8\lambda^2)}{2(k^2 + 4(\beta - \lambda)\lambda)}.$$

*If $l_1 > 0$ then the Hopf point at $E_0$ is unstable (weak repelling focus) and for each $\alpha < \alpha_0 = 1+\lambda$, but close to $\alpha_0$, an unstable limit cycle near the asymptotically stable equilibrium $E_0$ can be found; if $l_1 < 0$ then the Hopf point at $E_0$ is asymptotically stable (weak attractor focus) and for each $\alpha > \alpha_0 = 1 + \lambda$, but close to $\alpha_0$, a stable limit cycle near the unstable equilibrium $E_0$ can be found.*

*Proof.* Taking $\alpha$ as the Hopf bifurcation parameter, the transverse condition

$$\mathrm{Re}(s'(\alpha_0))|_{s=\sqrt{\beta\lambda-\lambda^2}\, i} = \frac{(\beta - \lambda)\lambda + k(1 + \lambda + \lambda^2)}{k^2 + (\beta - \lambda)\lambda} > 0 \quad (4.1)$$

is met. Accordingly, Hopf bifurcation at $E_0$ occurs. The value of the first Lyapunov coefficient $l_1$ shows the stability of the equilibrium point $E_0$ and the periodic orbits which appear. Using the mark in §3, the multilinear symmetric functions can be written as

$$B(x, y) = (-x_1y_2 - x_2y_1, 2k\alpha x_1y_1, 0),$$
$$C(x, y, z) = (0, 0, 0, 0).$$

In addition, one can also get

$$p = \left(-\frac{i\lambda + \sqrt{(\beta-\lambda)\lambda}}{2\lambda(-i\beta + i\lambda + \sqrt{(\beta-\lambda)\lambda})}, 0, \frac{\beta}{2\beta - 2\lambda + 2i\sqrt{(\beta-\lambda)\lambda}}\right),$$
$$q = \left(\lambda + i\sqrt{\lambda(\beta-\lambda)}, 0, 1\right),$$
$$h_{20} = \left(0, \frac{2k(1+\lambda)(\lambda + i\sqrt{(\beta-\lambda)\lambda})^2}{k + 2i\sqrt{(\beta-\lambda)\lambda}}, 0\right),$$
$$h_{11} = (0, 2\beta\lambda(1+\lambda), 0).$$

Then,

$$\mathrm{Re}[G_{21}] = -\frac{\beta\lambda(1+\lambda)(3k^2 + 8\beta\lambda - 2k\lambda - 8\lambda^2)}{k^2 + 4(\beta-\lambda)\lambda}$$

and

$$l_1 = -\frac{\beta\lambda(1+\lambda)(3k^2 + 8\beta\lambda - 2k\lambda - 8\lambda^2)}{2(k^2 + 4(\beta-\lambda)\lambda)}.$$

Therefore, Theorem 4.1 is proved. □

In order to study the relationship between the Hopf bifurcation and chaos, we justify the above theoretical analysis of the first Lyapunov coefficient for the Hopf bifurcation of system (2.1) with $\beta = 1.5$, $k = 0.6$, $\lambda = 1.2$. According to Theorem 3.1, we have $\alpha_0 = 2.2$ and $l_1 = -1.386 < 0$. Choosing initial condition $(0.1, 0.87, -0.09)$, a stable periodic solution should be obtained near the unstable equilibrium point $E_0$ for $\alpha = 2.3 > \alpha_0$, which is shown in figure 2.

Now we shall give the sign of the second Lyapunov coefficient $l_2$ when the first coefficient $l_1$ vanishes.

**Theorem 4.2.** *For system (2.1), if $\alpha = \alpha_0 = 1 + \lambda$, $\beta > \lambda$, $0 < k < \frac{2}{3}\lambda$ and $\beta = \dfrac{(3k^2 + 2k\lambda + 8\lambda^2)}{8\lambda}$, the second Lyapunov coefficients at $E_0$ is given by*

$$l_2|_{l_1=0} = \frac{3(k - 2\lambda)(1+\lambda)^2(3k + 4\lambda)^2(12k + 9k^2 + 8\lambda - 22k\lambda + 8\lambda^2)}{384(3k - 2\lambda)}.$$

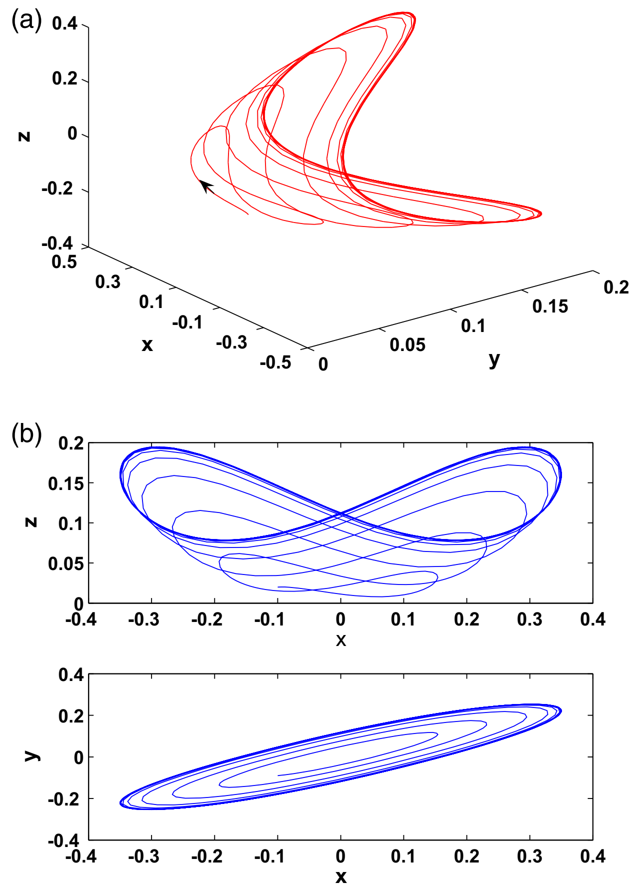


**Figure 2.** Periodic orbit with initial values $(-0.1, 0.02, -0.09)$ near unstable equilibrium $E_0(0, 0, 0)$ from Hopf bifurcation. (**a**) Periodic orbit from Hopf bifurcation and (**b**) projected on the $x$–$z$ plane and $x$–$y$ plane.

*Therefore, system* (2.1) *has a transversal Hopf point of codimension two at* $E_0$ *which is stable for* $l_2 < 0$ *and is unstable for* $l_2 > 0$ .

*Proof.* The second Lyapunov coefficient can be obtained for the parameters on the set $l_1 = 0$. For $\beta = (3k^2 + 2k\lambda + 8\lambda^2)/8\lambda$, one has

$$G_{21} = \frac{(k - 2\lambda)(1 + \lambda)(3k + 4\lambda)^2\sqrt{-6k^2 + 4k\lambda}}{8k(3k - 2\lambda)} i$$

and

$$h_{21} = \left(\frac{h_{2111}}{h_{2112}}, 0, \frac{h_{2131}}{h_{2132}}\right), \quad h_{30} = \left(\frac{h_{3011}}{h_{3012}}, 0, \frac{h_{3031}}{h_{3032}}\right),$$

$$h_{31} = \left(0, \frac{h_{3121}}{h_{3122}}, 0\right), \quad h_{22} = \left(0, \frac{h_{2221}}{h_{2222}}, 0\right),$$

where

$$\begin{aligned}
h_{2111} &= -(1 + \lambda)(-3k^2 + 2k\lambda + 8\lambda^2)^2 \\
&\quad \times (3k + i\sqrt{4k\lambda - 6k^2}) \\
&\quad \times (-12k^2 + k(-4\lambda - 3i\sqrt{4k\lambda - 6k^2}) \\
&\quad + 2\lambda(4\lambda - 3i\sqrt{-6k^2 + 4k\lambda})), \\
h_{2112} &= 8(-3k + 2\lambda)(-3k^2 + 2k\lambda \\
&\quad - 2i\sqrt{2}\lambda\sqrt{k(2\lambda - 3k)}) \\
&\quad \times (2k + i\sqrt{4k\lambda - 6k^2}) \\
&\quad \times (4k + i\sqrt{4k\lambda - 6k^2}), \\
h_{2131} &= -(1 + \lambda)(-3k^2 + 2k\lambda + 8\lambda^2) \\
&\quad \times (-3ik + \sqrt{4k\lambda - 6k^2}) \\
&\quad \times (-4i\lambda + \sqrt{4k\lambda - 6k^2}) \\
&\quad \times (-12ik^2 + k(-4i\lambda + 3\sqrt{4k\lambda - 6k^2}) \\
&\quad + 2\lambda(4i\lambda + 3\sqrt{4k\lambda - 6k^2})), \\
h_{2132} &= 4(2\lambda - 3k)(-3k^2 + 2k\lambda \\
&\quad - 2i\sqrt{2}\lambda\sqrt{k(2\lambda - 3k)}) \\
&\quad \times (4k + i\sqrt{4k\lambda - 6k^2}) \\
&\quad \times (-2ik + \sqrt{4k\lambda - 6k^2}), \\
h_{3011} &= 3(1 + \lambda)(-4i\lambda + \sqrt{4k\lambda - 6k^2})^3 \\
&\quad \times (-27ik^2 + 10ik\lambda + 6\sqrt{2}\lambda\sqrt{k(2\lambda - 3k)} \\
&\quad + 6k\sqrt{4k\lambda - 6k^2}), \\
h_{3012} &= 32(3k - 2\lambda)(-2ik + \sqrt{4k\lambda - 6k^2}) \\
&\quad \times (-4ik + 3\sqrt{4k\lambda - 6k^2}), \\
h_{3031} &= 3(1 + \lambda)(-4i\lambda + \sqrt{4k\lambda - 6k^2})^3, \\
h_{3032} &= 16(3k - 2\lambda)(-2ik + \sqrt{4k\lambda - 6k^2}), \\
h_{3121} &= k^2(1 + \lambda)^2(i\sqrt{2}\sqrt{k(2\lambda - 3k)} + 4\lambda) \\
&\quad \times (3k^2 - 2k\lambda - 8\Lambda^2) \\
&\quad \times (-26190ik^6 + 10665\sqrt{2}k^5\sqrt{k(2\lambda - 3k)} \\
&\quad + 8424ik^5\lambda + 306\sqrt{2}k^4\sqrt{k(2\lambda - 3k)}\lambda \\
&\quad + 99792ik^4\lambda^2 - 40572\sqrt{2}k^3\sqrt{k(2\lambda - 3k)}\lambda^2 \\
&\quad - 9088ik^3\lambda^3 \\
&\quad - 8840\sqrt{2}k^2\sqrt{k(2\lambda - 3k)}\lambda^3 - 72288ik^2\lambda^4 \\
&\quad + 23776\sqrt{2}k\sqrt{k(2\lambda - 3k)}\lambda^4 + 24448ik\lambda^5 \\
&\quad - 3072\sqrt{2}\sqrt{k(2\lambda - 3k)}\lambda^5), \\
h_{3122} &= 16(-2ik + \sqrt{2}\sqrt{k(2\lambda - 3k)})^2 \\
&\quad \times (-4ik + \sqrt{2}\sqrt{k(2\lambda - 3k)}) \\
&\quad \times (-4ik + 3\sqrt{2}\sqrt{k(2\lambda - 3k)}) \\
&\quad \times (3k - 2\lambda)(-3ik^2 + 2ik\lambda \\
&\quad + 2\sqrt{2}\sqrt{k(2\lambda - 3k)}\lambda), \\
h_{2221} &= -(k - 2\lambda)(1 + \lambda)^2(3k + 4\lambda)^2 \\
&\quad \times (3k^2 - 2k\lambda + 8\lambda^2), \\
h_{2222} &= 4k(3k - 2\lambda).
\end{aligned}$$

By the above theorem and calculation, one has

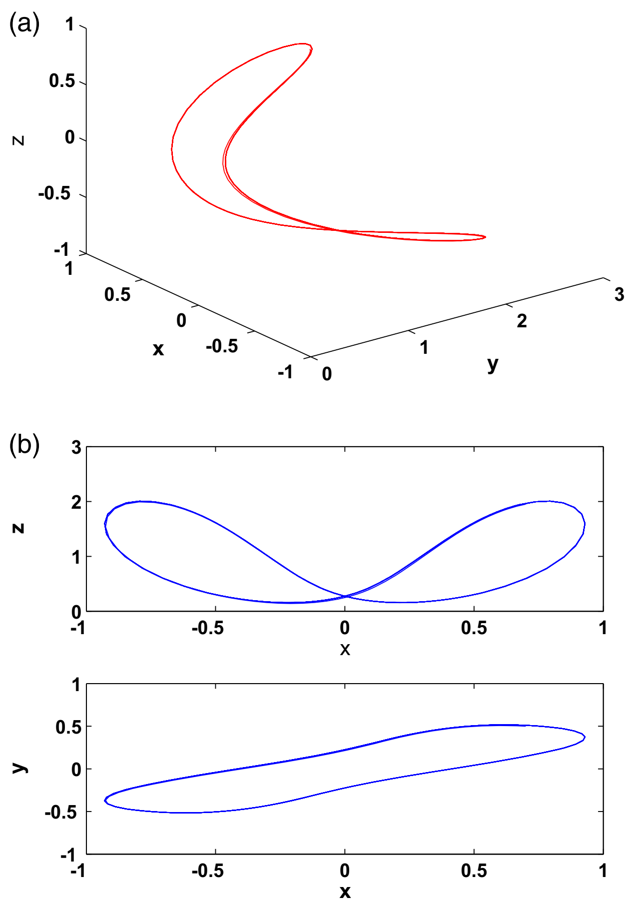


**Figure 3.** Periodic solution with initial values (0.7, 1.95, 0.5) near stable equilibrium $E_1(0.490022, 0.79, 0.408352)$ from Hopf bifurcation. (**a**) Projected on the $x$–$z$ plane and (**b**) projected on the $x$–$y$ plane.

$$\mathrm{Re}[G_{32}] = \frac{3(k-2\lambda)(1+\lambda)^2(3k+4\lambda)^2(12k+9k^2+8\lambda-22k\lambda+8\lambda^2)}{32(3k-2\lambda)}.$$

Therefore, we can arrive at Theorem 4.2. □

### 4.2 *Hopf bifurcation at $E_{1,2}$*

In this section, we only consider Hopf bifurcation at $E_1$ because of the symmetry. Taking $\alpha$ as the Hopf bifurcation parameter, the transverse condition

$$\mathrm{Re}(s'(\alpha_1))|_{\lambda=\sqrt{k(\beta-\lambda)\lambda k-\beta} i} = \frac{(\beta-k)^2 k}{\beta^3 - k^2(k+2\lambda) - \beta^2(3k+2\lambda) + \beta(3k^2+3k\lambda+\lambda^2)} \neq 0 \quad (4.2)$$

is also satisfied.

Here, we set $\beta = 1.5$, $k = 1.8$, $\lambda = 1.2$ and take $\alpha$ as the Hopf bifurcation parameter, the transverse condition

$\mathrm{Re}(s'(\alpha_1))|_{\lambda=\sqrt{k(\beta-\lambda)\lambda k-\beta} i} = -0.1224 < 0.$

is also satisfied. From §3, we have

$$q = (1.2 + 1.46969i, 4.4964 + 0.917824i, 1),$$
$$p = (-0.0340136 + 0.374922I, 0.0544654 - 0.0555885i, 0.295918 - 0.199958i).$$

The complex vectors $h_{11}$ and $h_{20}$ are

$$h_{11} = (-16.4868, -28.08, -13.739),$$
$$h_{20} = (-6.04062 + 10.1239i, 17.8691 + 25.3589i, 2.23306 + 2.9667i).$$

The complex coefficient $G_{21}$ defined in §3 is

$$G_{21} = 13.1283 - 105.753i.$$

We then have the following theorem:

**Theorem 4.3.** *Consider system (2.1) with $\beta = 1.5$, $k = 1.8$, $\lambda = 1.2$. The first Lyapunov coefficient associated with the equilibria $E_{1,2}$ is given by*

$$l_1|_{\alpha=\alpha_1=3.25} = 6.56416 > 0.$$

*Then the equilibria $E_{1,2}$ undergo a transversal Hopf bifurcation when $\alpha = \alpha_1 = 3.25$. More specifically, when $\alpha = 3.29 > \alpha_1$, but near to $\alpha_1$, there exist two unstable limit cycles around the asymptotically stable equilibria $E_{1,2}$ (see figure 3).*

## 5. Conclusion

In this paper, the parameter conditions in which the self-exciting Faraday disc dynamo presents Hopf bifurcations are obtained. Then we make a further extension of the analysis to the degenerate cases theoretically. The values of the first and second Lyapunov coefficients, which make the determination of the Lyapunov stability at the equilibria possible, allow the self-exciting Faraday disc dynamo to exhibit Hopf bifurcation and show periodic orbits in the parameter region. We are convinced that the complex dynamical behaviours of the self-exciting Faraday disc dynamo deserve further study and are very desirable for real-life applications in the near future.

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