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Measurement-based local quantum filters and their ability to transform quantum entanglement

DEBMALYA DAS[1], RITABRATA SENGUPTA[2] and ARVIND[1,*]

[1]Department of Physical Sciences, Indian Institute of Science Education and Research Mohali, Manauli 140 306, India
[2]Department of Mathematical Sciences, Indian Institute of Science Education and Research, Berhampur, Govt. ITI, Berhampur (Transit Campus), National Highway 59, Berhampur 760 010, India
*Corresponding author. E-mail: arvind@iisermohali.ac.in

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Abstract. We introduce local filters as a means to detect the entanglement of bound entangled states which do not yield to detection by witnesses based on positive maps which are not completely positive. We demonstrate how such non-detectable bound entangled states can be locally filtered into detectable bound entangled states. Specifically, we show that a bound entangled state in the orthogonal complement of the unextendible product bases (UPB), can be locally filtered into another bound entangled state that is detectable by the Choi map. We reinterpret these filters as local measurements on locally extended Hilbert spaces. We give explicit constructions of a measurement-based implementation of these filters for $2\otimes 2$ and $3\otimes 3$ systems. This provides us with a physical mechanism to implement such local filters.

Keywords. Entanglement; local filters; quantum measurement.

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1. Introduction

Ever since its introduction by Schrödinger [1,2] in the context of the EPR paradox [3], quantum entanglement has played a central role in quantum theory. While entanglement is responsible for non-classical correlations leading to the violation of Bell's inequalities [4,5], it also plays a key role in quantum computing where it is connected with the exponential advantage of quantum algorithms over their classical counterparts [6].

Studies of entanglement have led to a well-developed mathematical theory of entanglement where positive maps (P) which are not completely positive (CP) [7–14] and unextendible product bases (UPB) [15–17] play important roles [18–21]. These mathematical advances have led to the discovery of bound entangled states [22,23]: states from which one cannot distill EPR pairs although they are still provably non-separable.

Quantum states (pure or mixed) are represented by positive definite Hermitian operators $\rho \in \mathcal{B}(\mathcal{H})$ with unit trace. For the special case when the rank of ρ is one, it represents a pure state. For a bipartite composite system where states are defined on $\mathcal{B}(\mathcal{H}_{\rm A}\otimes\mathcal{H}_{\rm B})$, a state ρ is said to be separable if it can be written as a convex sum:

$$\rho = \sum_j p_j \rho_j^{\rm A}\otimes\rho_j^{\rm B}, \quad p_j > 0, \quad \sum_j p_j = 1, \tag{1}$$

where $\rho_j^{\rm A}$ and $\rho_j^{\rm B}$ are states in $\mathcal{B}(\mathcal{H}_{\rm A})$ and $\mathcal{B}(\mathcal{H}_{\rm B})$ respectively. A state is entangled, if it cannot be written in the above form. The fundamental problem of determining whether a given state ρ is separable or entangled remains open for general states of systems which are bigger than $2\otimes 2$ and $2\otimes 3$.

Allowed quantum evolutions are those P maps which are CP. The P maps which are not CP are quantum evolutions which are not physically allowed, because entangled states can lose their positivity when such a map is applied to one part of the system. Therefore, such maps act as entanglement witnesses. The partial transpose operation which is a particular entanglement witness, plays an important role in the identification of entangled states [24]. States which reveal their entanglement by acquiring one or more negative eigenvalues under partial transposition are called NPT (not positive under partial transpose) while the rest are called PPT (positive under partial transpose). NPT states are entangled while PPT states can be either entangled or separable. In $2 \otimes 2$ and $2 \otimes 3$ dimensional systems, it has been shown that a state is separable if and only if it is PPT [24,25]. On the other hand, in $3 \otimes 3$ dimensions and larger, there are entangled states which are PPT [26]. These states in general require other entanglement witnesses to implicate their entanglement and cannot be distilled to give EPR pairs. Such states (with non-distillable entanglement) are called PPT or bound entangled states.

Choi map and its generalizations have been used to detect the entanglement of certain classes of bound entangled states [27]. However, even for the $3 \otimes 3$ system, the detection of bound entangled states is far from complete. Local operations, including measurement and filtering, cannot alter the status of the state from PPT to NPT and therefore can be used to convert one NPT state to another NPT state or one PPT state to another PPT state [22]. Gisin showed that, starting with a mixed entangled state of a $2 \otimes 2$ system that does not violate Bell inequalities, one can set up a local filtration scheme based on measurements such that the filtered states violate Bell inequalities [28]. In this case, only NPT states were involved as there are no PPT entangled states for the $2 \otimes 2$ case. These results have been extended, used and experimentally validated by a number of researchers [29–33]. Our work is a generalization and extension of these results into the domain of PPT states of $3 \otimes 3$ systems. In this work, we introduce local filters which convert PPT entangled states not detectable by the Choi map, to states which are detectable by the Choi map. In particular, we are able to show that the PPT states obtained from the UPB can be converted into states detectable by the Choi map. Furthermore, we provide an explicit scheme for implementing our filtration protocol via local projective measurements involving local ancillas.

There are witnesses, other than the Choi maps that help in the detection of entanglement. Our motivation is that the inclusion of local filters in the scheme of entanglement detection via a Choi map leads to the detection of a larger class of states than would be possible without filters. Local filtration does not expose the whole set of entangled states but it certainly broadens the scope of entanglement witnesses and can be applied to any of the existing witnesses.

The material in this paper is arranged as follows: In §2 we describe our local filtration scheme. Two examples are taken up in §2.2 where such schemes are used to manipulate entangled states. Section 3 describes a measurement-based scheme to realize the local filters while §3.1 describes the general scheme. Section 3.2 describes the implementation of filters used by Gisin and §3.3 describes the implementation of filters on three-level systems. Section 4 offers some concluding remarks.

2. Local filtration and entanglement detection

2.1 *Local filters*

Local filters are local non-unitary operators represented by $L \otimes M$ where L, M are invertible operators acting in the state spaces of their respective systems. Given a bipartite quantum state $\rho \in \mathcal{B}(\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}})$, the filter acts on the state giving a new state

$$\rho^f = (L \otimes M)\rho(L \otimes M)^\dagger \tag{2}$$

which is a positive Hermitian operator belonging to the same space and its trace can be brought to one by dividing by an appropriate positive number. Thus, for every invertible set of operators L, M we have a filter.

Two important results about local filters are useful and therefore, we recapitulate them here. Let L, M be full rank operators. Then the map $\rho \mapsto (L \otimes M)\rho(L \otimes M)^\dagger$ does not change the Schmidt number of the state [34]. This result is important because it shows that entanglement is not created or destroyed by local operations. Further, it is worth noting that is a necessary condition for detection by an entanglement witness which is as follows: if the state is of Schmidt rank k then the corresponding map that detects the entanglement should be at most $k - 1$ positive but not k positive [34]. The strengthening that the filters can provide is only within this scope.

The classification of states into NPT and PPT is of central importance. Let us choose a standard basis $\{|j\rangle\}$, $j = 1, \ldots, n$ in an n-dimensional state space $\mathcal{H}$. Any density operator $\rho \in \mathcal{B}(\mathcal{H})$ can then be written as $\rho = \sum_{i,j} \rho_{ij} |i\rangle\langle j|$. Transpose operation is defined through its action on ρ

$$\rho \xrightarrow{T} \rho^T = \sum_{ij} \rho_{ji} |i\rangle\langle j|. \tag{3}$$

A bipartite state ρ is defined to be PPT if and only if $(\mathbf{1} \otimes T)\rho \geq 0$ where T is the transpose operation defined on $\mathcal{H}_{\mathrm{B}}$ as described in eq. (3). The important result that we need for our analysis is that the PPT or NPT character of a state is invariant under an invertible local filtration operation [22,23]. If the original state is entangled, the nature of its entanglement, (NPT or PPT), does not change when we apply a local filter. Therefore, for a given PPT entangled state, the filtered state is another PPT entangled state. It may turn out that even if the entanglement of the original state is not detectable by a given entanglement witness, the filtered state reveals its entanglement by the same witness. Thus, this allows us to generate new PPT entangled states from the old ones. This result can be easily extended to multipartite case where we consider different partitions of the multiparty system into bipartite systems and apply transpose operation. A multiparty state is PPT if it is PPT with respect to all possible bipartite partitions.

2.2 *Entanglement witnesses and local filters*

Two maps due to Choi [35] which are P maps that are not CP play an important role and are defined through their action on a 3×3 matrix as follows:

$$\phi\colon [[a_{ij}]] \mapsto \frac{1}{2}\begin{pmatrix} a_{11}+a_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22}+a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33}+a_{22} \end{pmatrix} \tag{4}$$

$$\psi\colon [[a_{ij}]] \mapsto \frac{1}{2}\begin{pmatrix} a_{11}+a_{22} & -a_{12} & -a_{13} \\ -a_{21} & a_{22}+a_{33} & -a_{23} \\ -a_{31} & -a_{32} & a_{33}+a_{11} \end{pmatrix}. \tag{5}$$

These maps provide us with important entanglement witnesses for PPT entangled states. Extensions of these maps have been used to unearth new PPT entangled states and to detect entanglement of PPT states formed out of UPB [19,20]. We recast these results in terms of local filters and show that the PPT entangled states described in [19] can be locally filtered into a state that is detectable by the Choi map. Similarly, we show that the PPT entangled states in the orthogonal complement of UPB can be locally filtered into states detectable by the Choi map.

Consider a density operator for the system $\mathcal{B}(\mathbb{C}^3 \otimes \mathbb{C}^3)$, defined by two parameters t and x.

$$\rho(x,t) = K\left(\begin{array}{ccc|ccc|ccc} 1+t & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\ 0 & t & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/t & 0 & 0 & 0 & x & 0 & 0 \\ \hline 0 & x & 0 & 1/t & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 1+t & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & t & 0 & x & 0 \\ \hline 0 & 0 & x & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 1/t & 0 \\ x & 0 & 0 & 0 & x & 0 & 0 & 0 & 1 \end{array}\right). \tag{6}$$

where $K = 1/(4 + \frac{3}{t} + 4t)$ is the normalization constant and this ρ is a unit trace density operator for $t > 0$ and $0 \le x \le 1$. The state is entangled for a range of values of x and t; however, it is not always detected by the Choi maps given in eqs (4) and (5). For instance, set $t = 1/20$, then for $0.6044 < x < 0.6554$, the state is not detectable by the Choi map. Consider a local filter

$$L_3 \otimes M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5/8 & 0 \\ 0 & 0 & 5/8 \end{pmatrix} \otimes I_3. \tag{7}$$

The filtered state after the application of this filter is obtained as

$$\rho^f(x,t) = (L_3 \otimes M_3)\rho(x,t)(L_3 \otimes M_3)^\dagger. \tag{8}$$

We now apply the Choi map given in eq. (4) on the first system via $\phi \otimes I_3$ to the filtered as well as non-filtered density operator to obtain

$$\rho(x,t) \xrightarrow{\phi\otimes I_3} \rho_{\text{Choi}}(x,t), \tag{9}$$

$$\rho^f(x,t) \xrightarrow{\phi\otimes I_3} \rho^f_{\text{Choi}}(x,t). \tag{10}$$

For the operator $\rho^f_{\text{Choi}}(x,t)$ for $t = 1/20$ and for $0.6044 < x < 0.6554$, the minimum eigenvalue turns out to be negative, indicating that the state is entangled. On the other hand, the minimum eigenvalue of the operator $\rho_{\text{Choi}}(x,t)$ which is obtained by the application of Choi map without filtering, is positive. This shows that the local filter defined in eq. (8) has converted the state $\rho(x,t)$ whose entanglement was not detectable via the Choi map into a state $\rho^f(x,t)$ whose entanglement is detectable via the Choi map.

An important example of bound entangled states is provided by the well-known UPB construction known as 'TILES' [15] for a $3 \otimes 3$ system

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle), \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle), \\ |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle, \\ |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle, \\ |\psi_4\rangle &= \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle). \end{aligned} \tag{11}$$

The mixed state

$$\rho^{\text{UPB}} = \frac{1}{4}\left(I_9 - \sum_{i=0}^{4}|\psi_i\rangle\langle\psi_i|\right) \tag{12}$$

provides an example of a PPT entangled state [15]. Choi maps, applied directly, cannot detect the entanglement of such states. Consider the local filter

$$L'_3 \otimes M'_3 = I_3 \otimes \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -(1/\sqrt{2}) & 0 & 1/\sqrt{2} \end{pmatrix}. \tag{13}$$

Applying this filter gives a new filtered state given by

$$\rho^{f\,\mathrm{UPB}} = (L'_3 \otimes M'_3)\rho^{\mathrm{UPB}}(L'_3 \otimes M'_3)^{\dagger}. \quad (14)$$

We now apply the second Choi map given in eq. (5) on the second system via $I_3 \otimes \psi$ to the filtered as well as non-filtered density operator to obtain

$$\rho^{\mathrm{UPB}} \xrightarrow{I_3\otimes\psi} \rho^{\mathrm{UPB}}_{\mathrm{Choi}} \quad (15)$$

$$\rho^{f\,\mathrm{UPB}} \xrightarrow{I_3\otimes\psi} \rho^{f\,\mathrm{UPB}}_{\mathrm{Choi}}. \quad (16)$$

The operator $\rho^{f\,\mathrm{UPB}}_{\mathrm{Choi}}$ has a negative eigenvalue which reveals the entanglement of ρ^{UPB} while $\rho^{\mathrm{UPB}}_{\mathrm{Choi}}$ does not have a negative eigenvalue. This shows that the entanglement of the state ρ^{UPB} is not directly revealed by the Choi map. However, it can be filtered into a state that is detected by the Choi map. This is directly related to the construction given in terms of the automorphisms in [20] and is much simpler than the construction given in [18].

3. Implementation of local filters

3.1 *General scheme*

We now turn to the question of the physical interpretation and implementation of the local quantum filtration process introduced in the previous section. A filter is defined through its action given in eq. (2) and comprises invertible operators L and M where L acts locally on $\mathcal{H}_{\mathrm{A}}$ (the Hilbert space of Alice) and M acts locally on $\mathcal{H}_{\mathrm{B}}$ (the Hilbert space of Bob). We choose the standard bases in $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$. Each of these operators has a singular valued decomposition given by $L = U_1 D_1 V_1$ and $M = U_2 D_2 V_2$. Here U_1, U_2, V_1, V_2 are unitary operators and D_1, D_2 are diagonal with real positive definite diagonal entries. The unitary operators correspond to Hamiltonian evolutions and can hence be physically realized in a straightforward way. We therefore focus here on the implementation of diagonal matrices D_1 and D_2.

Consider the implementation of D_1 on $\mathcal{H}_{\mathrm{A}}$. The diagonal matrix $D_1 = \mathrm{Diag}[d_1, d_2, \ldots, d_n]$ has diagonal entries d_j such that $0 < d_j \leq 1$ $(j = 1, \ldots, n)$. We now show that such a D_1 can be implemented by first extending the Hilbert space by adding one qubit as ancilla and then measuring an appropriate projection operator P. To achieve this, we first consider a set of n orthogonal but unnormalized vectors of the form $|u_j\rangle = \sqrt{d_j}\,|j\rangle$ in the n-dimensional system Hilbert space. We extend each of these vectors into a $2n$-dimensional Hilbert space to form a new set of vectors $\{|\xi_j\rangle\}$ given by $|\xi_j\rangle = \sqrt{d_j}\,|j\rangle + \sqrt{1-d_j}\,|j+n\rangle$. In addition to being mutually orthogonal, these vectors are also normalized. Thus, we have constructed an orthonormal set of n vectors in a $2n$-dimensional Hilbert space formed from the n-dimensional system and a two-dimensional ancilla. Corresponding to each of these vectors, we can construct a projection operator P_j given by

$$P_j = |\xi_j\rangle\langle\xi_j| = \left(\begin{array}{c|c} \eta_j & \delta_j \\ \hline \delta_j & \eta'_j \end{array}\right)_{2n\times 2n} \quad (17)$$

where the $n \times n$ matrices are given by

$$\eta_j = d_j|j\rangle\langle j|,$$

$$\eta'_j = (1-d_j)|j\rangle\langle j|,$$

$$\delta_j = \sqrt{d_j(1-d_j)}|j\rangle\langle j|. \quad (18)$$

The projection operator obtained by adding these mutually orthogonal projectors is given by

$$P = \sum_{j=1}^{n} P_j = \left(\begin{array}{c|c} D_1 & \Delta \\ \hline \Delta & D'_1 \end{array}\right)_{2n\times 2n}, \quad (19)$$

where $D_1 = \eta_1 + \cdots + \eta_n$ is the original operator that we wanted to implement, $D'_1 = \eta'_1 + \cdots + \eta'_n$ is a complementary operator obtained from D_1 and $\Delta = \delta_1 + \cdots + \delta_n$ represents the cross terms.

Now consider the system to be in an arbitrary state ρ_{A} and the one-qubit ancilla to be in the state $|0\rangle\langle 0|$. Consider a measurement of P on this composite system. If the outcome of the measurement is positive, we retain the state. The state after such a selection is given by the action of the projection operator P on the composite state:

$$P(|0\rangle\langle 0| \otimes \rho_{\mathrm{A}})P = \left(\begin{array}{c|c} D_1\rho_{\mathrm{A}}D_1 & D_1\rho_{\mathrm{A}}\Delta \\ \hline \Delta\rho_{\mathrm{A}}D_1 & \Delta\rho_{\mathrm{A}}\Delta \end{array}\right). \quad (20)$$

We further carry out a projective measurement of the projector operator $|0\rangle\langle 0|$ on the ancilla qubit and retain the state only if the outcome is positive. We now discard the ancilla which is anyway decoupled from the system. This completes the implementation of the map D_1 on the system density operator ρ_{A}. Thus, to realize the operator D_1 in the space of the system, we need to carry out two projective measurements: first measure the projector P in the combined space of the system and ancilla and retain the state if the answer is positive and then make another measurement of the projector $|0\rangle\langle 0|$ only in the ancilla space and retain the state if the outcome is positive. As mentioned earlier, to implement L we need to implement the unitaries U_1 and V_1 which can be accomplished via the standard Hamiltonian evolution. For the implementation of the map M on the second system in the Hilbert space $\mathcal{H}_{\mathrm{B}}$ we follow an analogous procedure. The non-unitary part represented by D_2 is implemented via two projective measurements after adding an ancilla

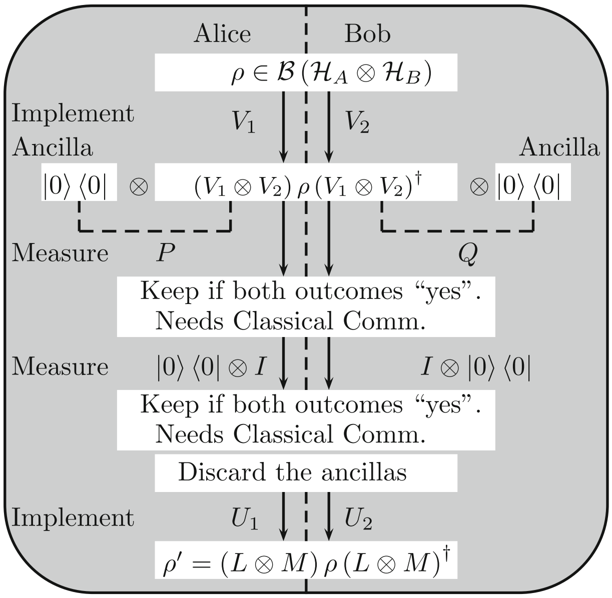

Figure 1. Schematic diagram for performing the local filtration via measurements.

qubit and the unitaries U_2 and V_2 involved in the singular decomposition of M are implemented via a Hamiltonian evolution.

In the actual implementation we require an ensemble with several copies of the shared state ρ between Alice and Bob. The protocol works as follows:

- *Step* 1: Alice implements the unitary V_1 on her part of the state and Bob implements the unitary V_2 on his part of the state.
- *Step* 2: Alice and Bob each attach a one-qubit ancilla prepared in a state $|0\rangle$, to their part of the state. Alice measures projector P corresponding to D_1 followed by a measurement of $|0\rangle\langle 0|$ on her ancilla qubit. Bob measures the projector Q corresponding to D_2 followed by a measurement of $|0\rangle\langle 0|$ on his ancilla qubit. They retain the state if all the four measurements give positive results. Otherwise they discard the state.
- *Step* 3: If they retain the state in the previous step, each one of them discards the ancilla qubits and then Alice implements U_1 on her part of the state and Bob implements U_2 on his part of the state. They repeat this process on all the copies of ρ to obtain the new filtered ensemble.

This protocol obviously requires classical communication between Alice and Bob because they need to know the outcome of the measurements that the other performs. The situation is schematically depicted in figure 1.

3.2 *Filtration of two-qubit systems*

An interesting example of quantum filtration was introduced by Gisin [28] for an entangled mixed state of two spin-$\frac{1}{2}$ particles not violating the Bell-CHSH inequality. In this scheme, by using a polarized beam splitter, one can convert such an input state to an output state which remains entangled but, this time, its entanglement can be detected by a Bell inequality violation. We recast this filtration scheme and connect it with our results.

Let us suppose that Alice and Bob share a $2\otimes 2$ system $\rho \in \mathcal{B}(\mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B})$ between themselves. Interpreting the Gisin filter in our formalism reveals that in that case $\rho \mapsto (L_2 \otimes M_2)\rho(L_2 \otimes M_2)^\dagger$, where the operators L_2 and M_2 are given by

$$L_2 = \begin{pmatrix} \kappa & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix}. \tag{21}$$

In the notation of ref. [28], $\kappa = \sqrt{\beta/\alpha}$, α and β are two real numbers such that $\alpha > \beta > 0$ and $\alpha^2 + \beta^2 = 1$ so that $0 < \kappa < 1$. Let us consider the implementation of the non-unitary operator L_2. As the operator L_2 is acting locally, we need to look for the map

$$\rho_\mathrm{A} \mapsto L_2 \rho_\mathrm{A} L_2.$$

As L_2 is a diagonal matrix ($L_2^\dagger = L_2$), we do not need to undertake a singular value decomposition. We directly introduce two mutually orthogonal but unnormalized vectors

$$|u_1\rangle = \begin{pmatrix} \sqrt{\kappa} \\ 0 \end{pmatrix}, \quad |u_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{22}$$

As discussed in §3.1, by adding a one-qubit ancilla these two-dimensional vectors can be extended into four-dimensional orthonormal vectors

$$|\xi_1\rangle = \begin{pmatrix} \sqrt{\kappa} \\ 0 \\ \sqrt{1-\kappa} \\ 0 \end{pmatrix}, \quad |\xi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{23}$$

Constructing the corresponding mutually orthogonal projectors results in

$$P_1 = \begin{pmatrix} \kappa & 0 & \sqrt{\kappa(1-\kappa)} & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\kappa(1-\kappa)} & 0 & 1-\kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

The required projector $P = P_1 + P_2$ is then given by

$$P = \left(\begin{array}{c|c} L_2 & \Delta_2 \\ \hline \Delta_2 & L_2' \end{array}\right). \tag{25}$$

Here L_2' and Δ_2 are defined by following the definitions below eq. (19).

Consider now the action of this projector on a state where the system is in an arbitrary state ρ_A and the ancilla qubit is in the state $\rho_a = |0\rangle\langle 0|$.

$$P(|0\rangle\langle 0| \otimes \rho_A)P = \left(\begin{array}{c|c} L_2\rho_A L_2 & L_2\rho_A\Delta_2 \\ \hline \Delta_2\rho_A L_2 & \Delta_2\rho_A\Delta_2 \end{array}\right). \tag{26}$$

This is the result of measurement of P on the composite system when the outcome is positive. To extract the top left block of the above matrix, we perform another projective measurement in the ancilla space and retain it if the outcome of measurement of $|0\rangle\langle 0|$ is positive. Bob does a similar exercise to implement M_2 in his laboratory and both Alice and Bob retain the state only when the outcomes of measurement of all four projectors are positive. This completes the protocol.

3.3 *The case of* $3 \otimes 3$ *systems*

In §2.2 we demonstrated the role of local filters in strengthening the entanglement detection capabilities of entanglement witnesses for $3 \otimes 3$ systems. Here we delineate the implementation of such a filtration process for these systems. We begin by discussing the filtration on a single three-level system $\rho_A \in \mathcal{B}(\mathcal{H}_A)$. We consider the following transformation $\rho_A \mapsto L_3\rho_A L_3^\dagger$ where $L = U_1 D_1 V_1$ is the singular value decomposition of L and

$$D_1 = \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix} \tag{27}$$

with $0 < d_1, d_2, d_3 \leq 1$.

We now introduce three mutually orthogonal and unnormalized vectors, in a three-dimensional Hilbert space

$$|u_1\rangle = \begin{pmatrix} \sqrt{d_1} \\ 0 \\ 0 \end{pmatrix},$$
$$|u_2\rangle = \begin{pmatrix} 0 \\ \sqrt{d_2} \\ 0 \end{pmatrix},$$
$$|u_3\rangle = \begin{pmatrix} 0 \\ 0 \\ \sqrt{d_3} \end{pmatrix}. \tag{28}$$

Now introducing a qubit as an ancilla and extending the above vectors in the composite Hilbert space of six dimensions, we obtain

$$|\xi_1\rangle = \begin{pmatrix} \sqrt{d_1} \\ 0 \\ 0 \\ \sqrt{1-d_1} \\ 0 \\ 0 \end{pmatrix}, \quad |\xi_2\rangle = \begin{pmatrix} 0 \\ \sqrt{d_2} \\ 0 \\ 0 \\ \sqrt{1-d_2} \\ 0 \end{pmatrix},$$
$$|\xi_3\rangle = \begin{pmatrix} 0 \\ 0 \\ \sqrt{d_3} \\ 0 \\ 0 \\ \sqrt{1-d_3} \end{pmatrix} \tag{29}$$

which are orthonormal.

The projection operators corresponding to these vectors are given by $P_j = |\xi_j\rangle\langle\xi_j|$ and can be written explicitly as

$$P_1 = \begin{pmatrix} d_1 & 0 & 0 & \sqrt{d_1(1-d_1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{d_1(1-d_1)} & 0 & 0 & 1-d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & \sqrt{d_2(1-d_2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{d_2(1-d_2)} & 0 & 0 & 1-d_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & \sqrt{d_3(1-d_3)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{d_3(1-d_3)} & 0 & 0 & 1-d_3 \end{pmatrix}. \tag{30}$$

The projector P corresponding to the operator L_3 is thus given by

$$P = P_1 + P_2 + P_3 = \left(\begin{array}{c|c} L_3 & \Delta_3 \\ \hline \Delta_3 & L_3' \end{array}\right). \tag{31}$$

For any operator $\rho_A \in \mathcal{B}(\mathbb{C}^3)$ we use a one-qubit ancillary system in the state $|0\rangle\langle 0|$ and measure the projector P. This leads us to an equation which is the same as eq. (26) with L_2 and Δ_2 replaced by L_3 and Δ_3. Similarly, we can do the analysis from Bob's point of view and arrive at the projector Q corresponding to D_2 ($M_3 = U_2 D_2 V_2$). Using this projector and a one qubit ancilla he sets up his measurements. Then they both follow the protocol steps 1–3 given in the last part of §3.1 to complete the filtration process and obtain the new joint density operator.

4. Conclusions

In this work, we discussed the role of local filters in transforming one PPT entangled state to another PPT entangled state. It may turn out that the entanglement of the new state is detectable by a P map which is not CP while the entanglement of the original state is not detectable by the map. It is in this sense that local filters can enhance the power of an entanglement witness in detecting entanglement. We give two concrete examples where this actually occurs. In the first example, a new class of bound entangled states becomes detectable by the Choi map and in the second example PPT entangled states in the orthogonal complement of UPB become detectable by the Choi map.

We then undertook the analysis of these filtration schemes as explicit local projective measurements coupled with local unitaries. It turns out that we need to add a one-qubit ancilla for both the parties involved in order to implement the non-unitary part of these filters as local measurements. We have constructed explicit projection operators corresponding to the filters that we have used.

We have transformed a given PPT entangled state to another PPT entangled state by using local filtration such that the final state reveals its entanglement through the entanglement witness constructed from the Choi map.

A point worth emphasizing is that these local filters do not change the NPT or PPT status of a state. Gisin exploited this fact to convert NPT states of two qubits which do not violate Bell's inequalities into the ones which violate Bell's inequalities. We have used these filters to convert one PPT entangled state into another PPT entangled state such that the PPT entanglement is detectable by a given entanglement witness.

The use of filters in detecting bound entanglement is not restricted to bipartite systems. Witnesses based on Choi map are also useful in the multipartite scenario. Recently, a multipartite witness has been constructed from a bipartite witness by a procedure whereby the bipartite witness is applied to all possible bipartite partitions of the multipartite system [36,37]. It is not very difficult to generalize our procedure for the above case and reformulate the above results in terms of local filters. The discussion on application of local filters in multiparty scenarios will be taken up elsewhere.

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