*Pramana – J. Phys.* (2017) 88: 77
DOI 10.1007/s12043-017-1381-4



# Dirac equation, hydrogen atom spectrum and the Lamb shift in dynamical non-commutative spaces

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MS received 4 May 2016; accepted 16 December 2016; published online 12 April 2017

**Abstract.** We derive the relativistic Hamiltonian of hydrogen atom in dynamical non-commutative spaces (DNCS or $\tau$-space). Using this Hamiltonian we calculate the energy shift of the ground state as well the $2P_{1/2}$, $2S_{1/2}$ levels. In all the cases, the energy shift depends on the dynamical non-commutative parameter $\tau$. Using the accuracy of the energy measurement, we obtain an upper bound for $\tau$. We also study the Lamb shift in DNCS. Both $2P_{1/2}$ and $2S_{1/2}$ levels receive corrections due to dynamical non-commutativity of space which is in contrast with the non-dynamical non-commutative spaces (NDNCS or $\theta$-space) in which the $2S_{1/2}$ level receives no correction.

**Keywords.** Dirac equation; hydrogen atom; Lamb shift; dynamical non-commutative spaces.

**PACS Nos 11.15.−q; 11.30.Er; 11.25.Sq**

## 1. Introduction

Recently, there have been notable studies on the formulation and possible experimental consequences of extensions of the usual physical theories in the non-commutative spaces. The study on non-commutative spaces is much important for understanding phenomena at short distances beyond the present test of different physical theories. Non-commutative geometry has great impact in diverse areas of modern physics such as cosmology, gravity, high energy and quantum physics. For a review of non-commutative field theories and non-commutative quantum mechanics, see [1–3]. The non-commutative space can be realized by the coordinate operators satisfying

$$[x_i,\ x_j] = i\theta_{ij}, \tag{1}$$

where $\theta_{ij}$ is an antisymmetric tensor. The authors called this new version of non-commutativity as dynamical non-commutative space or $\tau$-space.

In general, $\theta_{ij}$ can be a function of coordinates [4,5]. Recently, Fring *et al* [6] introduced a generalization of non-commutative space to a position-dependent space. The authors called this new version of non-commutativity as dynamical non-commutative space or $\theta$-space. The dynamical non-commtative variables in two dimensions satisfy the following Jacobi identities:

$$\begin{aligned} &[X, Y] = i\theta(1+\tau Y^2), && [X, P_x] = i\hbar(1+\tau Y^2),\\ &[X, P_y] = 2i\tau Y(\theta P_y + \hbar X), && [Y, P_y] = i\hbar(1+\tau Y^2),\\ &[P_y, P_x] = 0, && [Y, P_x] = 0. \end{aligned} \tag{2}$$

It should be noted that as $\tau \to 0$, we get the $\theta$-space commutation relations:

$$\begin{aligned} &[x_0,\ x_0] = i\theta, \quad [x_0,\ p_{x_0}] = i\hbar,\\ &[x_0,\ p_{y_0}] = 0, \quad [y_0,\ p_{y_0}] = i\hbar,\\ &[y_0,\ p_{x_0}] = 0, \quad [p_{x0},\ p_{y_0}] = 0. \end{aligned} \tag{3}$$

As mentioned in [6], the X-coordinate and the momentum $P_y$ are not Hermitian, but one can find a similarity transformation, i.e., a Dyson map $\eta^{-1}\text{O}\eta = 0 = 0^{\dagger}$ (with $\eta = ((1+\tau Y^2)^{-1/2})$ and convert the non-Hermitian variables into a Hermitian one. The new Hermitian variables $x$, $y$, $p_x$ and $p_y$ can be expressed in terms of non-commutative $\theta$-space as follows [6]:

$$\begin{aligned} x &= \eta X \eta^{-1} = (1+\tau y_0^2)^{-1/2}(1+\tau y_0^2)x_0(1+\tau y_0^2)^{1/2}\\ &= (1+\tau y_0^2)^{1/2} x_0 (1+\tau y_0^2)^{1/2}\\ y &= \eta Y \eta^{-1} = (1+\tau y_0^2)^{-1/2} y_0 (1+\tau y_0^2)^{1/2} \end{aligned}$$

$$p_x = \eta P_x \eta^{-1} = (1+\tau y_0^2)^{-1/2} p_{x_0} (1+\tau y_0^2)^{1/2} = p_{x_0}$$

$$p_y = \eta P_y \eta^{-1} = (1+\tau y_0^2)^{-1/2} (1+\tau y_0^2) p_{y_0} (1+\tau y_0^2)^{1/2}$$
$$= (1+\tau y_0^2)^{1/2} p_{y_0} (1+\tau y_0^2)^{1/2}. \quad (4)$$

These new Hermitian variables satisfy in the same commutation relations as (2)

$$[x,y] = i\theta(1+\tau y^2), \quad [x,p_x] = i\hbar(1+\tau y^2),$$
$$[p_x,p_y] = 0, \quad [x,p_y] = 2i\tau y(\theta p_y + \hbar x),$$
$$[y,p_x] = 0, \quad [y,p_y] = i\hbar(1+\tau y^2). \quad (5)$$

Using Bopp shift, one can express the non-commutative $\theta$-variables in terms of the standard (commutative) variables [7]:

$$x_{i_0} = x_{i_s} - \frac{\theta_{ij}}{2\hbar} p_{j_s}, \;\; p_{i_0} = p_{i_s}, \quad i,j = x,y, \quad (6)$$

where $\theta_{ij} = \epsilon_{ijk}\theta_k$, one can take $\theta_3 = \theta$ and the rest of the $\theta$-components to zero (which can be done by a rotation or a redefinition of coordinates).

It is worth mentioning that the subscript 0 indicates quantities in non-commutative $\theta$-space, while subscript $s$ denotes one in standard or commutative space.

The interesting point is that in the DNCS there is a minimum length for $X$ in a simultaneous $X$, $Y$ measurement [6]:

$$\Delta X_{\min} = \theta\sqrt{\tau}\sqrt{1+\tau\langle Y\rangle_\rho^2}. \quad (7)$$

But there is no non-vanishing minimal length for $Y$. This means that objects in DNCS are naturally of string type.

## 2. Dirac equation in the dynamical non-commutative space

Electronic bound states around charged impurities in two-dimensional systems can be described in terms of a two-dimensional hydrogen atom. In this section, we study the relativistic hydrogen atom in a two-dimensional dynamical non-commutative space. The general form of Dirac equation for the hydrogen atom is

$$H = \vec{\alpha}\cdot\vec{p} + m\beta + eA_0 = \alpha_1 P_1 + \alpha_2 P_2 + m\beta + eA_0, \quad (8)$$

where

$$A_0 = -\frac{e}{(X^2+Y^2)^{1/2}}. \quad (9)$$

Using relations (4), we rewrite the Hamiltonian in terms of $\theta$-space variables:

$$\alpha_1 p_{x_0} + \alpha_2((1+\tau y_0^2)^{1/2} p_{y_0} (1+\tau y_0^2)^{1/2})$$
$$+ m\beta - e^2((1+\tau y_0^2)^{1/2} x_0 (1+\tau y_0^2) x_0$$
$$\times (1+\tau y_0^2)^{1/2} + y_0^2)^{-1/2}. \quad (10)$$

Since $\tau$ and $\theta$ are small, the parentheses can be expanded to the first order using

$$(1+\tau y_0^2)^{1/2} = 1 + \frac{1}{2}\tau y_0^2.$$

Then we have

$$\alpha_1 p_{x_0} + \alpha_2\left(p_{y_0} + \frac{1}{2}\tau p_{y_0} y_0^2 + \frac{1}{2}\tau y_0^2 p_{y_0}\right) + m\beta$$
$$- e^2\left(\left(x_0^2 + \tau x_0 y_0^2 x_0 + \frac{1}{2}\tau x_0^2 y_0^2\right.\right.$$
$$\left.\left. + \frac{1}{2}\tau y_0^2 x_0^2\right) + y_0^2\right)^{-1/2}. \quad (11)$$

Using Bobb shift (6), we express the $\theta$-variables $x_0$, $y_0$, $p_{x_0}$ and $p_{y_0}$ in terms of the standard (commutative) space variables. Then eq. (11) reads as

$$\alpha_1 p_{x_s} + \alpha_2\left(p_{y_s} + \frac{1}{2}\tau p_{y_s}\left(y_s - \frac{\theta_{21}}{2\hbar}p_{x_s}\right)^2\right.$$
$$\left. + \frac{1}{2}\tau\left(y_s - \frac{\theta_{21}}{2\hbar}p_{x_s}\right)^2 p_{y_s}\right) + m\beta - e^2\left(\left(x_s - \frac{\theta_{12}}{2\hbar}p_{y_s}\right)^2\right.$$
$$+ \frac{1}{2}\tau\left(x_s - \frac{\theta_{12}}{2\hbar}p_{y_s}\right)^2\left(y_s - \frac{\theta_{21}}{2\hbar}p_{x_s}\right)^2$$
$$+ \tau\left(x_s - \frac{\theta_{12}}{2\hbar}p_{y_s}\right)\left(y_s - \frac{\theta_{21}}{2\hbar}p_{x_s}\right)^2\left(x_s - \frac{\theta_{12}}{2\hbar}p_{y_s}\right)$$
$$+ \frac{1}{2}\tau\left(y_s - \frac{\theta_{21}}{2\hbar}p_{x_s}\right)^2\left(x_s - \frac{\theta_{21}}{2\hbar}p_{y_s}\right)^2$$
$$\left. + \left(y_s - \frac{\theta_{21}}{2\hbar}p_{x_s}\right)^2\right)^{-1/2}. \quad (12)$$

So, to the first order in $\theta$, eq. (10) becomes

$$\alpha_1 p_{x_s} + \alpha_2\left(p_{y_s} + \frac{1}{2}\tau p_{y_s} y_s^2 + \frac{1}{2}\tau y_s^2 p_{y_s}\right) + m\beta$$
$$- e^2\left(x_s^2 - \frac{\theta}{\hbar}x_s p_{y_s} + \frac{1}{2}\tau x_s^2 y_s^2 + \tau x_s y_s^2 x_s\right.$$
$$\left. + \frac{1}{2}\tau y_s^2 x_s^2 + y_s^2 + \frac{\theta}{\hbar} y_s p_{x_s}\right)^{-1/2}.$$

Employing the fundamental relation $[x_i, p_j] = i\hbar\delta_{ij}$, we rearrange the term $\frac{1}{2}\tau p_{y_s} y_s^2$ as $\frac{1}{2}\tau p_{y_s} y_s^2 = \frac{1}{2}\tau y_s^2 p_{y_s} - i\hbar\tau y_s$.

Therefore, the Hamiltonian takes the following form:

$$H = \alpha_1 p_{x_s} + \alpha_2 p_{y_s} + m\beta + \alpha_2 \tau y_s^2 p_{y_s} - \alpha_2 i\hbar\tau y_s - \frac{e^2}{(x_s^2+y_s^2)^{1/2}} + \frac{e^2}{2(x_s^2+y_s^2)^{3/2}} \times \left(2\tau x_s^2 y_s^2 - \frac{\theta}{\hbar}L_z\right)$$

which can be written as

$$H = H^s + H^\tau + H^\theta,$$

where

$$H^\tau = \alpha_2\tau y_s^2 p_{y_s} - \alpha_2 i\hbar\tau y_s + \frac{e^2}{2(x_s^2+y_s^2)^{3/2}}(2\tau x_s^2 y_s^2),$$

$$H^\theta = -\frac{e^2}{2(x_s^2+y_s^2)^{3/2}}\frac{\theta}{\hbar}L_z \tag{13}$$

are the perturbation Hamiltonians and reflect the effects of dynamical and non-dynamical non-commutativity of space on the relativistic Hamiltonian of the hydrogen atom. The effects of $H^\theta$ on Dirac equation is studied in [8]. In this paper, we treat $H^\tau$ as a perturbative Hamiltonian and study its effect on Dirac equation. The energy shift of hydrogen atom due to $H^\tau$ described by Dirac equation is given by

$$\Delta E = \int_0^\infty\int_0^\pi\int_0^{2\pi} \psi_{jm}^\dagger H^\tau \psi_{jm} r^2 \mathrm{d}r \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi. \tag{14}$$

The unperturbed wave functions are [9,10]

$$\psi_{jm} = r^{-(n-1)/2}\begin{pmatrix} ig(r)\sqrt{\frac{j+m}{2j}}Y_{l,m-\frac{1}{2}} \\ ig(r)\sqrt{\frac{j-m}{2j}}Y_{l,m+\frac{1}{2}} \\ f(r)\sqrt{\frac{j+m}{2j}}Y_{l,m-\frac{1}{2}} \\ -f(r)\sqrt{\frac{j-m}{2j}}Y_{l,m+\frac{1}{2}} \end{pmatrix} \tag{15}$$

where $f(r)$ and $g(r)$ are given by

$$\left.\begin{matrix} g(r) \\ f(r) \end{matrix}\right\} = \frac{\pm(2\lambda)^{3/2}}{\Gamma(2\gamma_D+1)}\sqrt{\frac{(mc^2\pm E)\Gamma(2\gamma_D+n'+1)}{4mc^2\frac{(n'+\gamma_D)mc^2}{E}\left[\frac{(n'+\gamma_D)mc^2}{E}-k_D\right]n'!}} \times (2\lambda r)^{\gamma_D}e^{-\lambda r}\left\{\left[\frac{(n'+\gamma_D)mc^2}{E}-k_D\right]F(-n',\, 2\gamma_D+1,\, 2\lambda r) \mp n' F(1-n',\, 2\gamma_D+1,\, 2\lambda r)\right\} \tag{16}$$

and $F(a,\, c;\, x)$ is the hypergeometric function

$$F(a,\, c;\, x) = 1 + \frac{a}{c}x + \frac{a(a+1)x^2}{c(c+1)2!} + \cdots$$

$D$ is the dimension of space and

$$\gamma_D = \left(j + \frac{D-2}{2}\right)^2 - (Z\alpha)^2$$

$$\lambda = \frac{(m^2c^4 - E^2)^{1/2}}{\hbar c}, \quad j = l \pm 1/2,$$

$$n' = n - j - \frac{D-3}{2}.$$

Using the first-order perturbation theory we calculate the shift of energy for the ground state, and the result is as follows:

$$\Delta E_{\text{ground}} = (260.401\ \text{eV m}^2)\tau. \tag{17}$$

The accuracy of the energy measurement is $10^{-12}$ eV [11], and so we can put the following upper bound on the dynamical non-commutative parameter $\tau$:

$$\tau < \frac{10^{-12}\ \text{eV}}{260.401\ \text{eV m}^2} = \frac{10^{-12}}{260.401\ \text{m}^2}. \tag{18}$$

Using the relation 1 Fermi $\approx 5(\text{GeV})^{-1}$, one can get

$$\sqrt{\tau} \le 10^{-17}\ \text{eV} \tag{19}$$

which is consistent with the one obtained in [12].

It is worth mentioning that upper and lower bounds for the non-dynamical non-commutative parameter $\theta$ were obtained in [13] and [14] respectively.

## 3. The Lamb shift

According to the Dirac equation in commutative space, the energy states $2S_{1/2}$ and $2P_{1/2}$ are degenerate, i.e., have the same energy. $\theta$-non-commutativity has no effect on $2S_{1/2}$ level [8] and just causes energy correction to $2P_{1/2}$ level. But $\tau$-non-commutativity affects both $2P_{1/2}$ and $2S_{1/2}$ levels and make some corrections to their energies, and the situation is illustrated in figure 1.

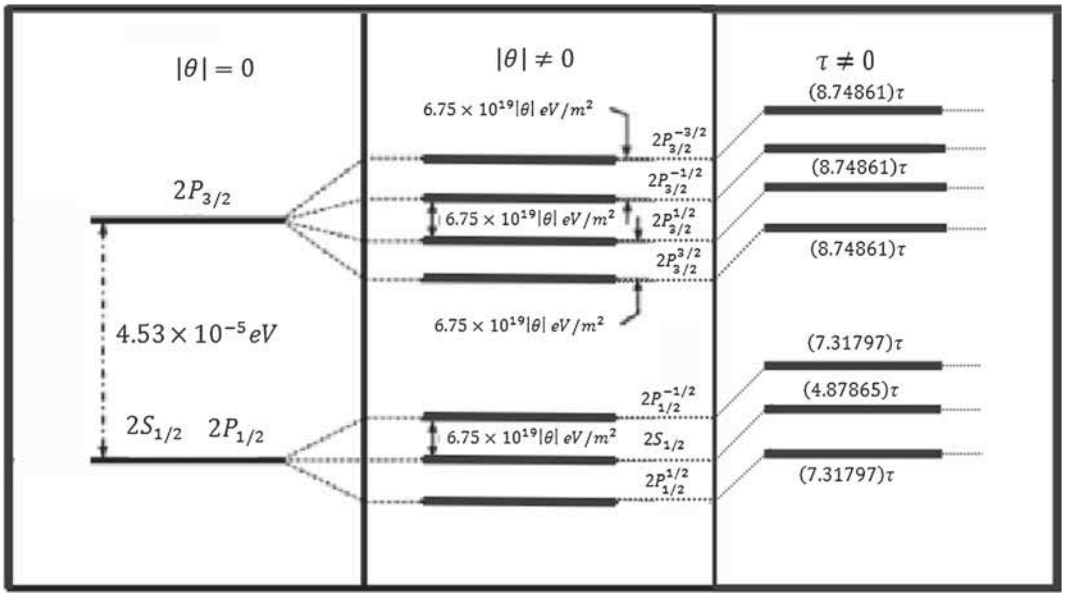


**Figure 1.** Splittings for energy levels due to non-dynamical and dynamical space non-commutativity. All numerical values of the coefficients of the first-order $\tau$-corrections are in units of $\mathrm{eV/m^2}$.

The energy shifts for the $2S_{1/2}$ and $2P_{1/2}$ levels, respectively are as follows:

$$\Delta E_{2S_{1/2}} = (4.87865\ \mathrm{eV\,m^2})\tau$$

$$\Delta E_{2P_{1/2}} = (7.31797\ \mathrm{eV\,m^2})\tau. \quad (20)$$

In ref. [8] it is shown that the $\theta$-non-commutativity in addition to the normal Lamb shift $2P_{1/2}^{1/2} \to 2S_{1/2}$ has a new channel $2P_{1/2}^{-1/2} \to 2S_{1/2}$ as well, but it has no effect on the $2S_{1/2}$ level. But, in a $\tau$-space, in addition to those two channels, the energy correction of $2S_{1/2}$ level, causes further enhancement in the transition width (rate). So, high accuracy energy measurement of the $2S_{1/2}$ level and the Lamb shift may be a good criterion to determine whether non-commutativity of space is dynamical or non-dynamical.

In ref. [15], it is shown that the energy correction of the ground state of a harmonic oscillator is also a good criterion for checking whether the non-commutativity of space is dynamical or non-dynamical. It may be also useful to mention that the non-commutative harmonic oscillator (in non-commutative $\theta$-space) has been studied in [16,17].

## 4. Concluding remarks

String theory provides the first way of putting quantum theory and general relativity together and is therefore a candidate for a the ory of everything. It is shown that in contrast with the non-dynamical non-commutative space, the objects in the dynamical non-commutative space studied here, are string-like, and so it seems that there is a stronger relation between DNCS and string theory than between NDNCS and string theory. On the other hand, in recent years there has been a growing interest in theoretical and experimental studies of non-Hermitian operators in physics and mathematics [18–21]. As mentioned in the 'Introduction', some operators in DNCS are non-Hermitian. This interface of DNCS and the theory of non-Hermitian operators on the one hand and DNCS and string theory on the other hand can lead to fundamental new insights in all three fields. So it is interesting to study fundamental phenomena in DNCS. In this paper, the Hamiltonian of the hydrogen atom described by Dirac equation in DNCS is derived and as a result, the energy corrections to the ground state as well the $2S_{1/2}$, $2P_{1/2}$ states have all been calculated. Using the energy measurement accuracy, an upper bound for the dynamical non-commutativity parameter $\tau$ has been obtained. Then it is shown that in contrast with the NDNCS, which has no effect on the $2S_{1/2}$ level, the DNCS affects both $2S_{1/2}$ and $2P_{1/2}$ levels, and subsequently causes energy shift and enhances the width of the transition. So, in future, high-precision spectroscopy data in combination with spectral retrieval techniques enable us to determine whether there exists any non-commutativity of space in nature and whether it is dynamical or non-dynamical.

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