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Algebraic resolution of the Burgers equation with a forcing term

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Abstract. We introduce an inhomogeneous term, $f(t, x)$, into the right-hand side of the usual Burgers equation and examine the resulting equation for those functions which admit at least one Lie point symmetry. For those functions $f(t, x)$ which depend nontrivially on both t and x, we find that there is just one symmetry. If f is a function of only x, there are three symmetries with the algebra $sl(2, R)$. When f is a function of only t, there are five symmetries with the algebra $sl(2, R) \oplus_s 2A_1$. In all the cases, the Burgers equation is reduced to the equation for a linear oscillator with nonconstant coefficient.

Keywords. Lie algebra; Burgers equation; symmetry reduction.

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1. Introduction

Burgers equation is

$$u_t + uu_x + u_{xx} = 0, \tag{1}$$

where $u(t, x)$ represents the velocity with t and x representing time and space, respectively. It is used to describe wave processes in acoustics and hydrodynamics, and also used as a model in fields as wide as continuous stochastic processes, dispersive water waves, gas dynamics, heat conduction, longitudinal elastic waves in an isotropic solid, number theory, shock waves and turbulence. It includes nonlinearity and dissipation together in the simplest possible way and may be thought of as a nonlinear version of the heat equation.

The solutions can be obtained using the Hopf–Cole transformation [1,2] $w(x, t) = 2u_x/u$, where $w = w(x, t)$ is a solution of the linear heat equation, $w_t = w_{xx}$.

The equation was known to Forsyth [3] and had been discussed by Bateman [4]. Due to the extensive work of Burgers [5] it is known as Burgers equation.

We consider the inhomogeneous Burgers equation

$$u_t + uu_x + u_{xx} = f(t, x). \tag{2}$$

Burgers equation is a Navier–Stokes equation for an incompressible fluid without pressure. It is one of the most popular models of hydrodynamics and has many applications. The forced Burgers equation, $u_t + uu_x + u_{xx} = f(x, t)$, where $f(x, t)$ is an external forcing term which is assumed to be smooth, for a one-dimensional velocity field, $u(t, x)$ has served as a simple model for investigating a variety of interesting issues that arise in fluid turbulence. The forcing term is responsible for the constant pumping of energy into the system. The energy also dissipates through different mechanisms of dissipation. The literature devoted to eq. (2) is enormous and it is still the subject of many studies [6–10]. Burgers equation is a natural generalization of the homogeneous equation corresponding to autonomous motions. The fluid is stirred randomly and steadily on scales of large length.

Equation (2) is an effective model of the dynamics of diverse physical nature [11–13] and models the time evolution of the profile for an interface in different reaction-diffusion systems. It gives an example of a flow with strong intermittency, created by large-scale structures.

Burgers equation has interesting applications in a wide range of nonequilibrium statistical physics. For instance, it arises in cosmology, in which it is known as the adhesion model [14], vehicular traffic [13], the study of directed polymers in random media [15], the evolution of weak one-dimensional acoustic perturbations in the reference frame moving with the velocity of sound [14], vortex lines in superconductors [16], the simulation of large eddies, ballistic deposition, the dynamics of a positive random walk on a growing surface, charge-density waves [17] etc. This problem is also related to interesting problems in condensed matter physics, such as the elastic lines in random media and growth problems [18]. In this respect, it is interesting to note that Burgers equation also appears naturally in the renormalization-group study of manifolds in random media [19]. Therefore, the phenomenology of Burgers equation might be directly relevant to experimental studies of pinned Bloch walls [20], dislocations in disordered solids and kinetic roughening of interfaces in epitaxial growth [21], formation of large-scale structures in the Universe [22,23], constructive quantum field theory [24] and provides a useful benchmark setting to test new analytical and numerical methods for real-world turbulence [25,26].

Its higher-dimensional analogue is the differentiated version of the well-known KPZ equation describing, among other things, kinetic roughening of growing surfaces [27]. Most recently, (2) has received a great deal of interest as the testing ground for field-theoretic techniques in hydrodynamics [9,28].

However, it has been attacked recently by various methods such as the operator product expansion [28], direct probabilistic method [29], instantons calculus method [30], the replica method [31], the exact soliton-like solution [32], explicit solution with the initial condition under distributional force [7] and stationary solution under boundary control [33].

From an analytic point of view, the inhomogeneous Burgers equation is little studied. The complete analytic solution is closely dependent on the form of the forcing term. The Hopf–Cole transformation can linearize the forced Burgers equation and then an integral of the forcing term appears as a coefficient of the linear heat equation. In addition, this linearized equation admits a forced closed-form solution in terms of Wiener functional integral method. In this sense, general solutions are possible for the initial-value problem of the forced Burgers equation. The initial-value problem of the Burgers equation with forcing, either deterministic or random, admits closed-form solutions.

2. Symmetry calculation of the inhomogeneous equation

We consider Burgers equation (1), with the addition of a forcing term so that we have the nonhomogeneous equation

$$u_t + uu_x + u_{xx} = f(t,\ x), \tag{3}$$

where $f(t,\ x)$ is a so-far-arbitrary smooth function of the variables t and x.

When we apply Sym [34–37] to (3), we obtain a symmetry of the form

$$\Gamma = a\partial_t + \left(\frac{1}{2}\dot{a}x + d\right)\partial_x + \left(\frac{1}{2}\ddot{a}x + \dot{d} - \frac{1}{2}\dot{a}u\right)\partial_u \tag{4}$$

provided $f(t,\ x)$ satisfies the linear partial differential equation

$$-\frac{3}{2}f\dot{a} + \ddot{d} + \frac{1}{2}\dddot{a}x - af_t - \left(d + \frac{1}{2}\dot{a}x\right)f_x = 0, \tag{5}$$

where a and d are arbitrary functions of t.

As (5) is a first-order partial differential equation, the equation of the characteristic curve may be expressed invariantly by the Lagrange–Charpit method. Equation (5) can be rewritten in the Lagrange form,

$$\frac{\mathrm{d}t}{a} = \frac{\mathrm{d}x}{(1/2)\dot{a}x + d} = \frac{\mathrm{d}f}{-(3/2)\dot{a}f + (1/2)x\dddot{a} + \ddot{d}}. \tag{6}$$

The characteristic system is

$$\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\dot{a}}{2a}x - \frac{\mathrm{d}}{a} = 0 \tag{7}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{3\dot{a}}{2a}f - \frac{\dddot{a}}{2a}x - \frac{\ddot{d}}{a} = 0. \tag{8}$$

The first equation gives a characteristic $z = (x - \alpha)/\rho =$ constant, where $a = \rho^2$ and $\alpha = \rho \int d/\rho^3\ \mathrm{d}t$.

The second equation becomes

$$\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{3\dot{a}}{2a}f - \frac{1}{2}(a^{1/2}u + \alpha)\frac{\dddot{a}}{a} - \frac{\ddot{d}}{a} = 0 \tag{9}$$

and gives the characteristic

$$H(z) = f\rho^3 - \rho^2\ddot{\rho}x - (\ddot{\alpha}\rho - \alpha\ddot{\rho})\rho^2 = \text{constant}.$$

Then, the solution of eq. (5) is

$$f(x,t) = \ddot{\rho}\left(\frac{x-\alpha}{\rho}\right) + \ddot{\alpha} - \frac{1}{\rho^3}H\left(\frac{x-\alpha}{\rho}\right), \tag{10}$$

where H is an arbitrary function of its argument.

3. Reduction to an ordinary differential equation

The invariant surface condition for (4) is

$$Q = \frac{1}{2}\ddot{a}f - \frac{1}{2}\dot{a}u + \dot{d} - au_t - \left(\frac{1}{2}\dot{a}x + d\right)u_x = 0 \quad (11)$$

which is solved by integrating the characteristic equations

$$\frac{\mathrm{d}t}{a} = \frac{\mathrm{d}x}{(1/2)\dot{a}x + d} = \frac{\mathrm{d}u}{(1/2)\ddot{a}f - (1/2)\dot{a}u + \dot{d}}. \quad (12)$$

The quadrature yields $z = (x-\alpha)/\rho$ and $w = \rho u - \rho\dot{\rho}z - \rho\dot{\alpha}$, where z is independent of u. Every invariant solution is of the form $w = w(z)$ which is equivalent to

$$\rho u - \rho\dot{\rho}z - \rho\dot{\alpha} = w(z) \quad (13)$$

$$u = \dot{\rho}z + \dot{\alpha} + \frac{w(z)}{\rho}. \quad (14)$$

When one substitutes the value of u into eq. (3), we have

$$\frac{1}{\rho^3}w\dot{w} + \frac{1}{\rho^3}\ddot{w} = -\frac{1}{\rho^3}H \quad (15)$$

which may be integrated immediately to give

$$\dot{w} + \frac{1}{2}w^2 + \mathcal{H} = 0, \quad (16)$$

where $\mathcal{H}'(z) = H(z)$. This is a Riccati equation, which can always be extended to a second-order linear ordinary differential equation. The transformation $w = (2\dot{P})/P$ leads to the second-order linear homogeneous equation,

$$\ddot{P} + \frac{1}{2}\mathcal{H}P = 0. \quad (17)$$

Equation (17) is the equation for a time-dependent oscillator and as such has a long and notable history. It became an important problem in physics when Lorentz proposed an adiabatic invariant for the slowly-lengthening pendulum at the Solvay Conference in 1911. This equation, (17), became of further importance some fifty years later during the study of the motion of a charged particle in an electromagnetic field, such as one finds in plasma devices. In 1966, Lewis [38–41] derived an invariant for (17) using Kruskal's asymptotic method [42]. In 1977, Leach [43,44] showed how to obtain the invariant using a time-dependent canonical transformation. This technique has subsequently found applications in several fields, including classical and quantum mechanics and optics [45].

There are a few instances for which the solution of (17) is known in closed form or as a special function. A good source is the compendium of Polyanin and Zaitsev [46].

Example 1. $\ddot{P} + aP = 0$. *(Free Oscillations.)* $a =$ *constant.*

Solution:

$$P = \begin{cases} C_1\sinh(z\sqrt{|a|}) + C_2\cosh(z\sqrt{|a|}) & \text{if } a < 0. \\ C_1 + C_2 z & \text{if } a = 0. \\ C_1\sin(z\sqrt{a}) + C_2\cos(z\sqrt{a}) & \text{if } a > 0. \end{cases}$$

C_1 and C_2 are constants of integration.

Example 2. $\ddot{P} - (az^2 + b)P = 0$ *(Weber equation)*

Case 1. The transformation $t = z^2\sqrt{a},\ u = \exp\left[t/2\right]P$ leads to the degenerate hypergeometric equation

$$t\ddot{u} + \left(\frac{1}{2} - t\right)\dot{u} - \frac{1}{4}\left(\frac{b}{\sqrt{a}} + 1\right)u = 0. \quad (18)$$

Case 2. For $a = k^2 > 0,\ b = -(2n+1)k$, where $n = 1, 2, \ldots$, there is a solution of the form

$$P = \exp\left[-\frac{1}{2}kz^2\right]H_n(\sqrt{k}z), \quad k > 0, \quad (19)$$

where

$$H_n(z) = (-1)^n \exp[z^2]\frac{\mathrm{d}^n}{\mathrm{d}z^n}(\exp[-z^2])$$

is the Hermite polynomial of order n.

Example 3. $\ddot{P} - az^nP = 0$.

Case 1. For $n = -2$ this is an equation of Euler type,

$$z^2\ddot{P} + az\dot{P} + bP = 0, \quad (20)$$

while for $n = -4$ the equation is

$$z^4\ddot{P} + aP = 0. \quad (21)$$

In both cases the solution is expressed in terms of elementary function.

Case 2. For any n the solution is expressed in terms of Bessel functions and modified Bessel functions of the first or second kind:

$$P = \begin{cases} C_1\sqrt{z}J_{1/2q}\left(\frac{\sqrt{-a}}{q}z^q\right) + C_2\sqrt{z}Y_{1/2q} \\ \quad \times\left(\frac{\sqrt{-a}}{q}z^q\right) \quad \text{if } a < 0, \\ C_1\sqrt{z}I_{1/2q}\left(\frac{\sqrt{a}}{q}z^q\right) + C_2\sqrt{z}K_{\frac{1}{2q}} \\ \quad \times\left(\frac{\sqrt{a}}{q}z^q\right) \quad \text{if } a > 0, \end{cases}$$

where $q = (1/2)(n+2)$ and C_1 and C_2 are constants of integration.

4. The case $f(t)$

We consider the partial differential equation of the form

$$u_t + uu_x + u_{xx} = f(t). \tag{22}$$

When we apply Sym to (22), we obtain a symmetry of the form

$$\Gamma = a\partial_t + \left(b + \frac{x\dot{a}}{2}\right)\partial_x + \left(\frac{x\ddot{a}}{2} + \dot{b} - \frac{\dot{a}u}{2}\right)\partial_u \tag{23}$$

with the constraints

$$\dddot{a} = 0$$
$$\ddot{b} = \frac{3}{2}f\dot{a} + a\dot{f},$$

that is,

$$a = A_0 + A_1 t + A_2 t^2 \tag{24}$$
$$b = B_0 + B_1 t + \int\int\left(\frac{3}{2}f\dot{a} + a\dot{f}\right)\mathrm{d}t\mathrm{d}t. \tag{25}$$

We have the set of symmetries

$$\begin{aligned}
\Gamma_{B_0} &= \partial_x\\
\Gamma_{B_1} &= t\partial_x + \partial_u\\
\Gamma_{A_0} &= \partial_t + \int f\mathrm{d}t\partial_x + f\partial_u\\
\Gamma_{A_1} &= 2t\partial_t + \left(x + 2\int tf\mathrm{d}t + \int\int f\mathrm{d}t\mathrm{d}t\right)\partial_x\\
&\quad + \left(-u + 2tf + \int f\mathrm{d}t\right)\partial_u\\
\Gamma_{A_2} &= t^2\partial_t + \left(\int\int ft\mathrm{d}t\mathrm{d}t + \int t^2 f\mathrm{d}t + tx\right)\partial_x\\
&\quad + \left(x + t^2 f + \int tf\mathrm{d}t - tu\right)\partial_u.
\end{aligned}$$

Let $f(t) = \ddot{g}(t)$. Then the symmetries can be rewritten as

$$\begin{aligned}
\Gamma_{B_0} &= \partial_x\\
\Gamma_{B_1} &= t\partial_x + \partial_u\\
\Gamma_{A_0} &= \partial_t + \dot{g}\partial_x + \ddot{g}\partial_u\\
\Gamma_{A_1} &= 2t\partial_t + (x + 2t\dot{g} - g)\partial_x + (-u + 2t\ddot{g} + \dot{g})\partial_u\\
\Gamma_{A_2} &= t^2\partial_t + (tx - tg + t^2\dot{g})\partial_x\\
&\quad + (x - tu - g + t\dot{g} + t^2\ddot{g})\partial_u.
\end{aligned}$$

Here, Γ_{A_0}, Γ_{A_1} and Γ_{A_2} form the algebra $sl(2, R)$ and Γ_{B_0} and Γ_{B_1} form $2A_1$. The algebra of the above symmetries is $sl(2, r) \oplus_s 2A_1$.

Reduction and solutions of eq. (22) are given in table 1, where $f(t) = \ddot{g}(t)$.

5. The case $f(x)$

We now consider the partial differential equation

$$u_t + uu_x + u_{xx} = f(x). \tag{26}$$

When we apply Sym to (26), we obtain a symmetry of the form

$$a\partial_t + \left(\frac{1}{2}\dot{a}x + b\right)\partial_x + \left(\frac{1}{2}\ddot{a}x + \dot{b} - \frac{1}{2}\dot{a}u\right)\partial_u, \tag{27}$$

where a and b are functions of t, provided $f(x)$ satisfies the linear differential equation

$$\frac{1}{2}\dddot{a}x - \dot{f}b - \frac{3}{2}f\dot{a} + \ddot{b} - \frac{1}{2}\dot{a}\dot{f}x = 0. \tag{28}$$

Table 1. Reduction and solutions of equation $u_t + uu_x + u_{xx} = f(t)$ where $f(t) = \ddot{g}(t)$.

	Symmetry	Similarity variable	Reduced equation	Solution
Γ_{B_0}	∂x	$z=t,\ w(z)=u$	$F'(z)=g''(z)$	$u=g'(t)$
Γ_{B_1}	$t\partial x+\partial u$	$z=t,\ w(z)=\frac{x}{t}-u$	$F'(z)+\frac{F(z)}{z} + g''(z) = 0$	$u=\frac{x}{t}-\frac{c_1}{t}-\frac{g(t)-tg'(t)}{t}$
Γ_{A_0}	$\partial t+g'(t)\partial x + g''(t)\partial u$	$z=g(t)-x$, $w(z)=g'(t)-u$	$F''(z)+F(z)F'(z) =0$	$u=-\sqrt{2c_1}\tanh[\frac{1}{2}(\sqrt{2c_1}(g(t) - x)+\sqrt{2c_1}c_2)]+g'(t)$
Γ_{A_1}	$2t\partial t+(x+2tg'(t) - g(t))\partial x + (2tg''(t) + g'(t)-u)\partial u$	$z=\frac{x-g(t)}{\sqrt{t}}$, $w(z)=\sqrt{t}(u-g'(t))$	$2F''(z)+(2F(z) - z)F'(z)-F(z)=0$	$u=\frac{2e^{\frac{(x-g(t))^2}{4t}}}{\left(2c_1+\sqrt{\pi}\,Erfi\left(\frac{x-g(t)}{2\sqrt{t}}\right)\right)\sqrt{t}} + g'(t)$
Γ_{A_2}	$t^2\partial t+(tx+t^2g'(t) - tg(t))\partial x + (-tu + t^2g''(t) + tg'(t)-g(t)+x)\partial u$	$z=\frac{x-g(t)}{t}$, $w(z)=tu-x+g(t) - tg'(t)$	$F''(z)+F(z)F'(z)=0$	$u=\frac{1}{t}\sqrt{2c_1}\tanh\left[\frac{1}{2}\left(\sqrt{2c_1}\left(\frac{x-g(t)}{t}\right) + \sqrt{2c_1}c_2\right)\right]+\frac{x-g(t)}{t}+g'(t)$

Assume that $a(t) = Ae^{kt}$ and $b(t) = Be^{kt}$, where A, B and α are constants. Then eq. (28) becomes

$$\frac{1}{2}k^3Ax + Bk^2 - \frac{1}{2}Akx\dot{f} - \frac{3}{2}Akf - B\dot{f} = 0 \tag{29}$$

$$\left(\frac{1}{2}Akx + B\right)\dot{f} + \frac{3}{2}Akf = k^2\left(\frac{1}{2}Akx + B\right)$$

$$\frac{d}{dx}\left\{\left(\frac{1}{2}Akx + B\right)^3 f\right\} = k^2\left(\frac{1}{2}Akx + B\right)^3$$

$$\left(\frac{1}{2}Akx + B\right)^3 f = \frac{2k^2}{4Ak}\left(\frac{1}{2}Akx + B\right)^4 + c.$$

The solution of eq. (28) is

$$f = \frac{c}{((1/2)Akx + B)^3} + \frac{k}{2A}\left(\frac{1}{2}Akx + B\right) \tag{30}$$

which can be written as

$$f = \frac{c}{(k^3A^3/8)\,(x + (2B/kA))^3} + \frac{k^2}{4}\left(x + \frac{2B}{kA}\right). \tag{31}$$

Let

$$m_1 = \frac{8c}{k^3A^3},\ m_2 = \frac{2B}{kA} \text{ and } m_3 = \frac{k^2}{4}.$$

Then $k = \pm 2\sqrt{m_3}$. Equation (26) can be written as

$$u_t + uu_x + u_{xx} = \frac{m_1}{(x + m_2)^3} + \frac{b^2}{4}(x + m_2) \tag{32}$$

and admits a symmetry of the form

$$\Gamma = e^{kt}\left\{\partial_t + \frac{1}{2}k(x+m_2)\partial_x + \left[\frac{1}{2}k^2(x+m_2) - \frac{1}{2}ku\right]\partial_u\right\}. \tag{33}$$

The characteristics corresponding to the generator (33) are found by solving

$$\frac{dt}{1} = \frac{dx}{(1/2)k(x + m_2)} = \frac{du}{(1/2)k^2(x + m_2) - (1/2)ku}. \tag{34}$$

The first characteristic for eq. (34) is

$$z = (x + m_2)\exp\left[-\frac{1}{2}kt\right].$$

The second characteristic can be obtained by solving the linear first-order equation

$$\frac{du}{dt} + \frac{1}{2}ku - \frac{1}{2}k^2 z e^{\frac{1}{2}kt} = 0. \tag{35}$$

The solution of (35) is

$$w(z) = ue^{\frac{1}{2}kt} - \frac{1}{2}k^2 z\frac{1}{k}e^{kt} \tag{36}$$

so that

$$u = e^{-\frac{1}{2}kt}w(z) + \frac{1}{2}k(x + m_2). \tag{37}$$

When one substitutes the value of u into eq. (32), one has

$$\ddot{w} + w\dot{w} = \frac{m_1}{z^3}. \tag{38}$$

We integrate this with respect to z and have

$$\dot{w} + \frac{1}{2}w^2 = -\frac{m_1}{2z^2} + 2j. \tag{39}$$

This is a Riccati equation which can always be transformed into a second-order linear ordinary differential equation. The transformation $w = (2\dot{v})/v$ leads to the second-order linear homogeneous equation.

$$\ddot{v} + \left(\frac{m_1}{4z^2} - j\right)v = 0. \tag{40}$$

The solution of eq. (40) is

$$v = c_1\sqrt{z}J\left(\frac{\sqrt{1 - m_1}}{2}, -i\sqrt{j}z\right) + c_2\sqrt{z}Y\left(\frac{\sqrt{1 - m_1}}{2}, -i\sqrt{j}z\right), \tag{41}$$

where J and Y are Bessel functions of the first and second kind respectively.

6. Conclusion

The analysis of the standard form of Burgers equation and its potential form is well known. What we have done in this paper is to introduce an inhomogeneous term, the function $f(t, x)$ in (3), and then to demand that the form of this function be such that there exists at least one symmetry for the whole equation. We found that, if $f(t, x)$ is given by

$$f(t, x) = \frac{\ddot{\rho}}{\rho}(x - \alpha) + \ddot{\alpha} - \frac{1}{\rho^3}H\left(\frac{x - \alpha}{\rho}\right), \tag{42}$$

where ρ and α are arbitrary functions of t and H is an arbitrary function of its argument, there exists a symmetry

$$\Gamma = \rho^2\partial_t + \rho\,(\dot{\rho}x + \rho\dot{\alpha} - \dot{\rho}\alpha)\partial_x + \{(\rho\ddot{\rho} - \dot{\rho}^2)x + \dot{\rho}(\rho\dot{\alpha} - \dot{\rho}\alpha) + \rho(\rho\ddot{\alpha} - \ddot{\rho}\alpha) - \rho\dot{\rho}u\}\partial_u. \tag{43}$$

In the special cases that $f(t, x)$ is a function of just t or just x we obtained additional symmetries. We find five symmetries of eq. (22) in the former case. The first two symmetries form the algebra $2A_1$ and the other three symmetries are $sl(2, R)$. In the latter case for eq. (26) we have three symmetries which admit an algebra of the form $sl(2, R)$.

For all these cases the given equation is reduced to the equation for a linear oscillator with nonconstant coefficient.

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References

[1] D Julian *et al*, *Cole Quart. Appl. Math.* **9**(**3**), 225 (1951)
[2] Eberhard Hopf, *Commun. Pure Appl. Math.* **3**(**3**), 201 (1950)
[3] A R Forsyth, *Theory of Differential Equations...: (vol. V–VI) Partial differential equations. 1906.* Theory of Differential Equations. University Press (1902)
[4] Harry Bateman, *Mon. Weather Rev.* **43**(**4**), 163 (1915)
[5] Johannes Martinus Burgers, *Adv. Appl. Mech.* **1**, 171 (1948)
[6] M J Ablowitz and S De Lillo, *Phys. Lett. A* **156**(**9**), 483 (1991)
[7] M J Ablowitz and S De Lillo, *Physica D* **92**(**3**), 245 (1996)
[8] Jérémie Bec and Konstantin Khanin, *J. Stat. Phys.* **113**(**5**), 741 (2003)
[9] Alexei Chekhlov and Victor Yakhot, *Phys. Rev. E* **51**(**4**), R2739 (1995)
[10] Simon Hood, *J. Math. Phys.* **36**(**4**), 1971 (1995)
[11] C H Su and C S Gardner, *J. Math. Phys.* **10**(**3**), 536 (1969)
[12] M B Vinogradova, O V Rudenko and A P Sukhorukov, *Wave theory* (Izd. Nauka, Moscow, 1979)
[13] G B Whitham, *Linear and nonlinear waves* (John Wiley & Sons Inc., New York, 1974)
[14] J M Burgers, *The nonlinear diffusion equation. Asymptotic solutions and statistical problems* (D Reidel, Dordrecht, 1974)
[15] F Calogero and S De Lillo, *Nonlinearity* **2**(**1**), 37 (1989)
[16] G Blatter, M V Feigel'Man, V B Geshkenbein, A I Larkin and V M Vinokur, *Rev. Mod. Phys.* **66**(**4**), 1125 (1994)
[17] M V Feigel'Man, *Sov. J. Exp. Theor. Phys.* **52**, 555 (1980)
[18] M Kardar, G Parisi and Y-C Zhang, *Phys. Rev. Lett.* **56**(**9**), 889 (1986)
[19] Leon Balents, Jean-Philippe Bouchaud and Marc Mézard, *J. de Phys. I* **6**(**8**), 1007 (1996)
[20] R Vergne, J C Cotillard and J L Porteseil, *Revue de Physique Appliquée* **16**(**9**), 449 (1981)
[21] Albert-László Barabási, *Comput. Mater. Sci.* **6**(**2**), 127 (1996)
[22] Sergei Nikolayevich Gurbatov, A I Saichev and S F Shandarin, *Mon. Not. R. Astron. Soc.* **236**(**2**), 385 (1989)
[23] S F Shandarin and Ya B Zeldovich, *Rev. Mod. Phys.* **61**(**2**), 185 (1989)
[24] L Bertini, N Cancrini and G Jona-Lasinio, *Commun. Math. Phys.* **165**(**2**), 211 (1994)
[25] U Frisch and J Bec, Burgulence, in: *New trends in turbulence: Turbulence, nouveaux aspects* (Springer, 2001) pp. 341–383
[26] Jérémie Bec and Konstantin Khanin, *Phys. Rep.* **447**(**1**), 1 (2007)
[27] Yuri Kifer, *Prob. Theory Related Fields* **108**(**1**), 29 (1997)
[28] Alexander M Polyakov, *Phys. Rev. E* **52**(**6**), 6183 (1995)
[29] Toshiyuki Gotoh and Robert H Kraichnan, *Phys. Fluids A* **5**(**2**), 445 (1993)
[30] Victor Gurarie and Alexander Migdal, *Phys. Rev. E* **54**(**5**), 4908 (1996)
[31] Jean-Philippe Bouchaud, Marc Mézard and Giorgio Parisi, *Phys. Rev. E* **52**(**4**), 3656 (1995)
[32] Yi-Tian Gao, Xiao-Ge Xu and Tian Bo, *Int. J. Mod. Phys. C* **14**(**9**), 1207 (2003)
[33] A Balogh, D S Gilliam and Victor I Shubov, *Math. Comput. Model.* **33**(**1**), 21 (2001)
[34] K Andriopoulos, S Dimas, P G L Leach and D Tsoubelis, *J. Comput. Appl. Math.* **230**(**1**), 224 (2009)
[35] S Dimas, *Partial differential equations, algebraic computing and nonlinear systems* Ph.D. thesis (University of Patras, Greece, 2008)
[36] Stelios Dimas and Dimitri Tsoubelis, Sym: A new symmetry-finding package for mathematica, in: *Proceedings of the 10th International Conference in Modern Group Analysis* (2004) pp. 64–70
[37] Stelios Dimas and Dimitri Tsoubelis, A new mathematica-based program for solving overdetermined systems of pdes, in: *8th International Mathematica Symposium* (2006)
[38] H Ralph Lewis Jr, *Phys. Rev. Lett.* **18**(**15**), 636 (1967)
[39] H Ralph Lewis Jr, *J. Math. Phys.* **9**(**11**), 1976 (1968)
[40] H Ralph Lewis Jr, *Phys. Rev.* **172**(**5**), 1313 (1968)
[41] H Ralph Lewis Jr and W B Riesenfeld, *J. Math. Phys.* **10**(**8**), 1458 (1969)
[42] Martin Kruskal, *J. Math. Phys.* **3**(**4**), 806 (1962)
[43] P G L Leach, *J. Austral. Math. Soc. Ser. B* **20**(**1**), 97 (1977)
[44] P G L Leach, *J. Math. Phys.* **18**(**8**), 1608 (1977)
[45] M Sebawe Abdalla and P G L Leach, *J. Phys. A: Math. Gen.* **36**(**49**), 12205 (2003)
[46] Andrei D Polyanin and Valentin F Zaitsev, *Handbook of nonlinear partial differential equations* (CRC Press, 2003)